Cooperative Games
Lecture 3: The core

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The Bondareva Shapley theorem:
a characterization of games with non-empty core.

The theorem was proven independently by
O. Bondareva (1963) and L. Shapley (1967).

Notations:
- \( \nu(N) \) is the set of all coalition functions on \( 2^N \).
- \( \nu(C) \) is a function \( 2^N \rightarrow \mathbb{R} \) for all \( C \subseteq N \).

The theorem states that the non-empty core
is characterized by the following linear program:

\[
\begin{align*}
\max & \quad \sum_{C \subseteq N} \nu(C) x_C \\
\text{s.t.} & \quad \sum_{i \in C} x_i = \chi_i(C), \quad \forall i \in N \\
& \quad \sum_{C \subseteq N} x_C = 1 \\
& \quad x_C \geq 0, \quad \forall C \subseteq N.
\end{align*}
\]

Example: maximize \( 8x_1 + 10x_2 + 5x_3 \)
subject to
\( \begin{align*}
3x_1 + 4x_2 + 2x_3 & \leq 7 \\
1x_1 + 2x_2 + 3x_3 & \leq 8
\end{align*} \)

The dual of this LP can be found by:

\[
\begin{align*}
\min & \quad \sum_{i \in N} \sum_{C \subseteq N} \nu(C) y_C \\
\text{s.t.} & \quad \sum_{C \subseteq N} y_C \chi_i(C) = c_i, \quad \forall i \in N \\
& \quad y_C \geq 0, \quad \forall C \subseteq N.
\end{align*}
\]

Linear Programming and the core

We consider the following linear programming problem:

\[
\begin{align*}
\text{(LP)} \quad & \max \sum_{C \subseteq N} \nu(C) x_C \\
\text{s.t.} & \quad \sum_{i \in C} x_i = \chi_i(C), \quad \forall i \in N \\
& \quad \sum_{C \subseteq N} x_C = 1 \\
& \quad x_C \geq 0, \quad \forall C \subseteq N.
\end{align*}
\]

A feasible solution is a solution that satisfies the constraints.

Example: maximize \( 8x_1 + 10x_2 + 5x_3 \)
subject to
\( \begin{align*}
3x_1 + 4x_2 + 2x_3 & \leq 7 \\
1x_1 + 2x_2 + 3x_3 & \leq 8
\end{align*} \)

\( (0,1,1) \) is feasible, with objective function value 15.
\( (1,0,0) \) is feasible, with objective function value 18.

The dual of a LP finding an upper bound to the objective function of the LP:
\( \begin{align*}
(1) & \quad x_1 + 2x_2 + 3x_3 \leq 19 \\
(2) & \quad 2x_1 + 2x_2 + 3x_3 \leq 18
\end{align*} \)

The coefficients are as large as in the objective function, so the bound is an upper bound for the objective function.

Hence, the solution cannot be better than 18, and we found one: problem solved! ✓

Today

- Characterize the set of games with non-empty core
  (Bondareva Shapley theorem), and we will informally introduce linear programming.
- Application of the Bondareva Shapley theorem to market games.

Linear programming

A linear program has the following form:

\[
\begin{align*}
\text{max} & \quad c^T x \\
\text{subject to} & \quad Ax \leq b, \quad x \geq 0
\end{align*}
\]

where:
- \( x \) is a vector of variables
- \( c \) is the objective function
- \( A \) is a \( m \times n \) matrix
- \( b \) is a vector of size \( n \)
- \( A \) and \( b \) represent the linear constraints

Example:
\( \text{maximize } 8x_1 + 10x_2 + 5x_3 \)
subject to
\( \begin{align*}
3x_1 + 4x_2 + 2x_3 & \leq 7 \\
1x_1 + 2x_2 + 3x_3 & \leq 8
\end{align*} \)

\[A = \begin{pmatrix} 3 & 4 & 2 \\ 1 & 2 & 3 \end{pmatrix}, b = \begin{pmatrix} 7 \\ 8 \end{pmatrix}, c = \begin{pmatrix} 8 \\ 9 \end{pmatrix}\]

Primal

\[
\begin{align*}
\text{max} & \quad c^T x \\
\text{subject to} & \quad Ax \leq b, \quad x \geq 0
\end{align*}
\]

Dual

\[
\begin{align*}
\text{min} & \quad y^T b \\
\text{subject to} & \quad y^T A \geq c^T, \quad y \geq 0
\end{align*}
\]

Theorem (Duality theorem)

When the primal and the dual are feasible, they have optimal solutions with equal value of their objective function.

Let \( C \subseteq N \). The characteristic vector \( \chi_C \) of \( C \) is the member of \( \mathbb{R}^N \) defined by \( \chi_C(i) = 1 \) if \( i \in C \).

A map \( \lambda: 2^N \setminus \emptyset \rightarrow \mathbb{R} \) is positive if \( \lambda(C) > 0 \) for all \( C \subseteq N \).

A map \( \lambda \) is balanced if and only if \( \sum_{C \subseteq N} \chi_C(C) = \chi_N \).

A map is balanced when the amount received over all the coalitions containing an agent \( i \) sums up to 1.

Example: \( n = 3 \), \( \lambda(C) = \begin{cases} 1 & \text{if } |C| = 2 \\ 0 & \text{otherwise} \end{cases} \)

<table>
<thead>
<tr>
<th>( C )</th>
<th>1</th>
<th>2</th>
<th>3</th>
</tr>
</thead>
<tbody>
<tr>
<td>{1,2}</td>
<td>1</td>
<td>0</td>
<td>0</td>
</tr>
<tr>
<td>{1,3}</td>
<td>1</td>
<td>0</td>
<td>0</td>
</tr>
<tr>
<td>{2,3}</td>
<td>0</td>
<td>1</td>
<td>1</td>
</tr>
</tbody>
</table>

Each of the column sums up to 1.
We consider the following linear programming problem:

\[(LP)\begin{align*}
\text{min } & \{ x \} \\
\text{subject to } & x \geq \{ y \} \quad \forall \{ C \} \subseteq N, \quad S \neq \emptyset \\
& x \in \mathcal{V} \quad \text{if the value of (LP) is } \{ v \}.
\end{align*}\]

The dual of \((LP)\):

\[(DLP)\begin{align*}
\max & \{ \sum_{C \subseteq N} y_C \} \\
\text{subject to } & \sum_{C \subseteq N} y_C x_C = \chi_i \quad \forall i, \quad y_C \geq 0 \quad \forall \{ C \} \subseteq N, \quad C \neq \emptyset.
\end{align*}\]

It follows from the duality theorem of linear programming:

\([N,v]\) has a non empty core iff \(\{ v \} \geq \sum_{C \subseteq N} y_C \{ v \} \) for all feasible \(y\).

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**Characterization of games with non-empty core**

**Definition (Balanced game)**

A game is balanced iff for each balanced map \(\lambda\) we have \(\sum_{C \subseteq N} \lambda_C \{ v \} \leq \{ v \}\).

**Theorem (Bondareva Shapley)**

A TU game has a non-empty core iff it is balanced.

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**Coalition Structure**

**Definition (Coalition Structure)**

A coalition structure (CS) is a partition of the grand coalition into coalitions:

\(\mathcal{S} = \{ \emptyset, C_1, C_2, \ldots, C_n \}\) where \(\emptyset \subseteq C_1 \subseteq \ldots \subseteq C_n = N\) and \(i \neq j \Rightarrow C_i \cap C_j = \emptyset\).

We note \(\mathcal{S}_f\) the set of all coalition structures over the set \(N\).

\[\text{ex. } \{1,3,4,2,7,5,6,8\}\text{ is a coalition structure for } n = 8\text{ agents.}\]

We will study three solution concepts: the bargaining set, the nucleolus and the kernel. They form the “bargaining set family” and we will see later why. In addition, the definition of each of these solution concepts uses a CS.

We start by defining a game with coalition structure, and see how we can define the core of such a game. Then, we’ll start studying the bargaining set family.

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**Game with Coalition Structure**

**Definition (TU game)**

A TU game is a pair \([N,v]\) where \(N\) is a set of agents and \(v\) is a valuation function.

**Definition (Game with Coalition Structures)**

A TU-game with coalition structure \([N,v,S]\) consists of:

- A TU game \([N,v]\) and a CS \(S \subseteq \mathcal{P} N\).
- We assume that the players agreed upon the formation of \(S\) and only the payoff distribution choice is left open.
- The CS may model affinities among agents, may be due to external causes (e.g. affinities based on locations).
- The agents may refer to the value of coalitions with agents outside their coalition (i.e. opportunities they would have outside of their coalition).
- \([N,v]\) and \([N,v,S]\) represent the same game.

**Definition (Superadditive cover)**

The superadditive cover of \([N,v]\) is the game \([N,\tilde{v}]\) defined by:

\[\tilde{v}(\{ C \}) = \max\{ \sum_{T \subseteq C} v(T) \} \quad \forall \{ C \} \subseteq N \quad \tilde{v}(\emptyset) = 0\]

We have \(\tilde{v}(N) = \max\{ \sum_{T \subseteq N} v(T) \}\), i.e., \(\tilde{v}(N)\) is the maximum value that can be produced by \(N\). We call it the value of the optimal coalition structure.

The superadditive cover is a superadditive game (why?).

**Theorem**

Let \([N,v,S]\) be a game with coalition structure. Then

a) \(\text{Core}(N,v,S) \neq \emptyset\) iff \(\text{Core}(N,\tilde{v}) \neq \emptyset\).

b) if \(\text{Core}(N,v,S) \neq \emptyset\), then \(\text{Core}(N,v,S) = \text{Core}(N,\tilde{v})\).

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**Definition (Substitutes)**

Let \((N,v,S)\) be a game with coalition structure, let \(i\) and \(j\) be substitutes, and let \(x^i \in \text{Core}(N,v,S)\). If \(i\) and \(j\) belong to different members of \(S\), then \(x^i = x^j\).
Summary

- We introduced a stability solution concept: the core.
- We looked at examples:
  - individual games: some games have an empty core.
  - classes of games have a non-empty core: e.g. convex games and minimum cost spanning tree games.
- We look at a characterization of games with non-empty core: the Shapley Bondareva theorem, which relies on a result from linear programming.
- We apply the Bondareva-Shapley to market games.
- We considered the core of games with coalition structures.

Coming next

- Bargaining sets.