If agents desire the kind of stability offered by the core, they will be unable to reach an agreement. They have no choice but to relax their stability requirements.

We would like a solution that allows agents to always reach an agreement, while guaranteeing some stability.

The bargaining set is one such solution.
A second solution concept:

The bargaining set.


Let \((N, v, S)\) be a game with coalition structure and \(x\) an imputation.

The bargaining set models stability in the following sense:

Any **argument** from an agent \(i\) against a payoff distribution \(x\) is of the following form:

\[
\text{I get too little in the imputation } x, \text{ and agent } j \text{ gets too much! I can form a coalition that excludes } j \text{ in which some members benefit and all members are at least as well off as in } x.
\]

The argument is **ineffective** for the bargaining set if agent \(j\) can answer the following:

\[
\text{I can form a coalition that excludes agent } i \text{ in which all agents are at least as well off as in } x, \text{ and as well off as in the payoff proposed by } i \text{ for those who were offered to join } i \text{ in the argument.}
\]
Definition (Objection)

Let \((N, v, S)\) be a game with coalition structure, \(x \in X_{(N,v,S)}\) (the set of all feasible payoff vectors for \((N,v,S))\), \(C \in S\) be a coalition, and \(i\) and \(j\) two distinct members of \(C\) \(((i,j) \in C^2, i \neq j)\).

An **objection of \(i\) against \(j\)** is a pair \((P, y)\) where

- \(P \subseteq N\) is a coalition such that \(i \in P\) and \(j \notin P\).
- \(y \in \mathbb{R}^p\) where \(p\) is the size of \(P\)
- \(y(P) \leq v(P)\) (\(y\) is a feasible payoff distribution for the agents in \(P\))
- \(\forall k \in P, y_k \geq x_k\) and \(y_i > x_i\) (agent \(i\) strictly benefits from \(y\), and the other members of \(P\) do not do worse in \(y\) than in \(x\).)

An objection \((P, y)\) of \(i\) against \(j\) is a **potential threat** by coalition \(P\), which contains \(i\) but not \(j\), to deviate from \(x\). The goal is not to change \(S\), but to obtain a side payment from \(j\) to \(i\), i.e., to modify \(x\) within \(X_{(N,v,S)}\).
Definition (Counter-objection)

An **counter-objection** to \((P, y)\) is a pair \((Q, z)\) where

- \(Q \subseteq N\) is a coalition such that \(j \in Q\) and \(i \notin Q\).
- \(z \in \mathbb{R}^q\) where \(q\) is the size of \(Q\)
- \(z(Q) \leq v(Q)\) (\(z\) is a feasible payoff distribution for the agents in \(Q\))
- \(\forall k \in Q, z_k \geq x_k\) (the members of \(Q\) get at least the value in \(x\))
- \(\forall k \in Q \cap P, z_k \geq y_k\) (the members of \(Q\) which are also members of \(P\) get at least the value promised in the objection)

In a counter-objection, agent \(j\) must show that it can protect its payoff \(x_j\) in spite of the existing objection of \(i\).
**Definition** (Stability)

Let $(N,v,S)$ a game with coalition structure. A vector $x \in X_{(N,v,S)}$ is **stable** iff for each objection at $x$ there is a counter-objection.

**Definition** (Pre-bargaining set)

The **pre-bargaining set** ($preBS$) is the set of all stable members of $X_{(N,v,S)}$.

**Lemma**

Let $(N,v,S)$ a game with coalition structure, we have $Core(N,v,S) \subseteq preBS(N,v,S)$.

This is true since, if $x \in Core(N,v,S)$, no agent $i$ has any objection against any other agent $j$. 
Example

Let \((N, v)\) be a 7-player simple majority game, i.e.

\[
v(C) = \begin{cases} 
1 & \text{if } |C| \geq 4 \\
0 & \text{otherwise}
\end{cases}
\]

Let us consider \(x = \langle -\frac{1}{5}, \frac{1}{5}, \ldots, \frac{1}{5} \rangle\). It is clear that \(x(N) = 1\).

Let us prove that \(x\) is in the pre-bargaining set of the game \((N, v, \{N\})\).

Objections within members of \(\{2,3,4,5,6,7\}\) will have a counter-objection by symmetry. ✔
Example

Let \((N, v)\) be a 7-player simple majority game, i.e.
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Let us prove that \(x\) is in the pre-bargaining set of the game \((N, v, \{N\})\).

Objections within members of \(\{2, 3, 4, 5, 6, 7\}\) will have a counter-objection by symmetry.

Let us consider the objections \((P, y)\) of 1 against another member of \(\{2, 3, 4, 5, 6, 7\}\). Since the players \(\{2, \ldots, 7\}\) play symmetric roles, we consider an objection \((P, y)\) of 1 against 7 using successively as \(P\) \(\{1, 2, 3, 4, 5, 6\}\), \(\{1, 2, 3, 4, 5\}\), \(\{1, 2, 3, 4\}\), \(\{1, 2, 3\}\), \(\{1, 2\}\) and \(\{1\}\). We will look for a counter-objection of player 7 using \((Q, z)\).
\( P = \{1, 2, 3, 4, 5, 6\} \). We need to find the payoff vector \( y \in \mathbb{R}^6 \) so that \((P, y)\) is an objection.

\[ y = \langle \alpha_1, 15 + \alpha_2, 15 + \alpha_3, \ldots, 15 + \alpha_6 \rangle. \]

The conditions for \((P, y)\) to be an objection are the following:

- Each agent is as well off as in \( x \):
  \[ \alpha > -15, \alpha_i \geq 0 \]

- \( y \) is feasible for coalition \( P \):
  \[ \sum_{i=1}^6 (15 + \alpha_i) + \alpha \leq 1. \]

w.l.o.g \( 0 \leq \alpha_2 \leq \alpha_3 \leq \alpha_4 \leq \alpha_5 \leq \alpha_6 \).

Then
\[ \sum_{i=1}^6 (15 + \alpha_i) + \alpha = 55 + \sum_{i=2}^6 \alpha_i \leq 1 \]
\[ \sum_{i=1}^6 \alpha_i \leq -\alpha < 15. \]

We need to find a counter-objection for \((P, y)\).

Claim: we can choose \( Q = \{2, 3, 4, 7\} \) and
\[ z = \langle 15 + \alpha_2, 15 + \alpha_3, 15 + \alpha_4, 15 + \alpha_5 \rangle \]
\( z(Q) = 15 + \alpha_2 + 15 + \alpha_3 + 15 + \alpha_4 + 15 + \alpha_5 = 45 + \sum_{i=2}^5 \alpha_i \leq 1 \) since
\[ \sum_{i=2}^5 \alpha_i \leq \sum_{i=1}^6 \alpha_i < 15 \]
so \( z \) is feasible.

It is clear that \( \forall i \in Q, z_i \geq x_i \) and that \( \forall i \in Q \cap P, z_i \geq y_i \)

Hence, \((Q, z)\) is a counter-objection.
\( P = \{1,2,3,4,5,6\} \). We need to find the payoff vector \( y \in \mathbb{R}^6 \) so that \((P,y)\) is an objection.  

\[ y = \langle \alpha, \frac{1}{5} + \alpha_2, \frac{1}{5} + \alpha_3, \ldots, \frac{1}{5} + \alpha_6 \rangle. \]
$P = \{1, 2, 3, 4, 5, 6\}$. We need to find the payoff vector $y \in \mathbb{R}^6$ so that $(P, y)$ is an objection. $y = \langle \alpha, \frac{1}{5} + \alpha_2, \frac{1}{5} + \alpha_3, \ldots, \frac{1}{5} + \alpha_6 \rangle$.

The conditions for $(P, y)$ to be an objection are the following:

- each agent is as well off as in $x$: $\alpha > -\frac{1}{5}$, $\alpha_i \geq 0$
- $y$ is feasible for coalition $P$: $\sum_{i=2}^{6} \left( \alpha_i + \frac{1}{5} \right) + \alpha \leq 1$. 
\( P = \{1, 2, 3, 4, 5, 6\} \). We need to find the payoff vector \( y \in \mathbb{R}^6 \) so that \((P, y)\) is an objection. \( y = \langle \alpha, \frac{1}{5} + \alpha_2, \frac{1}{5} + \alpha_3, \ldots, \frac{1}{5} + \alpha_6 \rangle \).

The conditions for \((P, y)\) to be an objection are the following:

- each agent is as well off as in \( x \): \( \alpha \succ -\frac{1}{5}, \alpha_i \geq 0 \)
- \( y \) is feasible for coalition \( P \): \( \sum_{i=2}^{6} \left( \alpha_i + \frac{1}{5} \right) + \alpha \leq 1 \).

w.l.o.g 0 \leq \alpha_2 \leq \alpha_3 \leq \alpha_4 \leq \alpha_5 \leq \alpha_6.
\( P = \{1, 2, 3, 4, 5, 6\} \). We need to find the payoff vector \( y \in \mathbb{R}^6 \) so that \((P, y)\) is an objection. \( y = \langle \alpha, \frac{1}{5} + \alpha_2, \frac{1}{5} + \alpha_3, \ldots, \frac{1}{5} + \alpha_6 \rangle \).

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- each agent is as well off as in \( x \): \( \alpha > -\frac{1}{5}, \alpha_i \geq 0 \)
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Then \( \sum_{i=2}^{6} \left( \frac{1}{5} + \alpha_i \right) + \alpha = \frac{5}{5} + \sum_{i=2}^{6} \alpha_i + \alpha = 1 + \sum_{i=2}^{6} \alpha_i + \alpha \leq 1 \).
$P = \{1, 2, 3, 4, 5, 6\}$. We need to find the payoff vector $y \in \mathbb{R}^6$ so that $(P, y)$ is an objection. $y = \langle \alpha, \frac{1}{5} + \alpha_2, \frac{1}{5} + \alpha_3, \ldots, \frac{1}{5} + \alpha_6 \rangle$.

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Then $\sum_{i=2}^{6} \left( \frac{1}{5} + \alpha_i \right) + \alpha = \frac{5}{5} + \sum_{i=2}^{6} \alpha_i + \alpha = 1 + \sum_{i=2}^{6} \alpha_i + \alpha \leq 1$.

Then $\sum_{i=2}^{6} \alpha_i \leq -\alpha < \frac{1}{5}$. 

\[ P=\{1,2,3,4,5,6\}. \] We need to find the payoff vector \( y \in \mathbb{R}^6 \) so that \((P,y)\) is an objection. \( y = \langle \alpha, \frac{1}{5} + \alpha_2, \frac{1}{5} + \alpha_3, \ldots, \frac{1}{5} + \alpha_6 \rangle. \)

The conditions for \((P,y)\) to be an objection are the following:

- each agent is as well off as in \( x \): \( \alpha > -\frac{1}{5}, \alpha_i \geq 0 \)
- \( y \) is feasible for coalition \( P \): \( \sum_{i=2}^{6} (\alpha_i + \frac{1}{5}) + \alpha \leq 1. \)

w.l.o.g \( 0 \leq \alpha_2 \leq \alpha_3 \leq \alpha_4 \leq \alpha_5 \leq \alpha_6. \)

Then \( \sum_{i=2}^{6} \left( \frac{1}{5} + \alpha_i \right) + \alpha = \frac{5}{5} + \sum_{i=2}^{6} \alpha_i + \alpha = 1 + \sum_{i=2}^{6} \alpha_i + \alpha \leq 1. \)

Then \( \sum_{i=2}^{6} \alpha_i \leq -\alpha < \frac{1}{5}. \)

\( \Rightarrow \) We need to find a counter-objection for \((P,y)\).
\[ P = \{1,2,3,4,5,6\} \]. We need to find the payoff vector \( y \in \mathbb{R}^6 \) so that \((P, y)\) is an objection. \( y = \langle \alpha, \frac{1}{5} + \alpha_2, \frac{1}{5} + \alpha_3, \ldots, \frac{1}{5} + \alpha_6 \rangle \).

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w.l.o.g \( 0 \leq \alpha_2 \leq \alpha_3 \leq \alpha_4 \leq \alpha_5 \leq \alpha_6 \).

Then \[ \sum_{i=2}^{6} \left( \frac{1}{5} + \alpha_i \right) + \alpha = \frac{5}{5} + \sum_{i=2}^{6} \alpha_i + \alpha = 1 + \sum_{i=2}^{6} \alpha_i + \alpha \leq 1. \]

Then \[ \sum_{i=2}^{6} \alpha_i \leq -\alpha < \frac{1}{5}. \]

We need to find a counter-objection for \((P, y)\).

**claim:** we can choose \( Q = \{2,3,4,7\} \) and \( z = \langle \frac{1}{5} + \alpha_2, \frac{1}{5} + \alpha_3, \frac{1}{5} + \alpha_4, \frac{1}{5} + \alpha_5 \rangle \).
\( P = \{1, 2, 3, 4, 5, 6\} \). We need to find the payoff vector \( y \in \mathbb{R}^6 \) so that \((P, y)\) is an objection. \( y = \langle \alpha, \frac{1}{5} + \alpha_2, \frac{1}{5} + \alpha_3, \ldots, \frac{1}{5} + \alpha_6 \rangle \).

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- each agent is as well off as in \( x \): \( \alpha > -\frac{1}{5}, \alpha_i \geq 0 \)
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w.l.o.g \( 0 \leq \alpha_2 \leq \alpha_3 \leq \alpha_4 \leq \alpha_5 \leq \alpha_6 \).

Then \( \sum_{i=2}^{6} \left( \frac{1}{5} + \alpha_i \right) + \alpha = \frac{5}{5} + \sum_{i=2}^{6} \alpha_i + \alpha = 1 + \sum_{i=2}^{6} \alpha_i + \alpha \leq 1 \).

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\( z(Q) = \frac{1}{5} + \alpha_2 + \frac{1}{5} + \alpha_3 + \frac{1}{5} + \alpha_4 + \frac{1}{5} + \alpha_5 = \frac{4}{5} + \sum_{i=2}^{5} \alpha_i \leq 1 \) since 
\( \sum_{i=2}^{5} \alpha_i \leq \sum_{i=2}^{6} \alpha_i < \frac{1}{5} \) so \( z \) is feasible.
$P = \{1, 2, 3, 4, 5, 6\}$. We need to find the payoff vector $y \in \mathbb{R}^6$ so that $(P, y)$ is an objection. $y = \langle \alpha, \frac{1}{5} + \alpha_2, \frac{1}{5} + \alpha_3, \ldots, \frac{1}{5} + \alpha_6 \rangle$.

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It is clear that $\forall i \in Q$, $z_i \geq x_i$ ✓ and that $\forall i \in Q \cap P$, $z_i \geq y_i$ ✓
\( P = \{1,2,3,4,5,6\} \). We need to find the payoff vector \( y \in \mathbb{R}^6 \) so that \((P,y)\) is an objection. \( y = \langle \alpha, \frac{1}{5} + \alpha_2, \frac{1}{5} + \alpha_3, \ldots, \frac{1}{5} + \alpha_6 \rangle \).

The conditions for \((P,y)\) to be an objection are the following:

- each agent is as well off as in \( x \): \( \alpha > -\frac{1}{5}, \alpha_i \geq 0 \)
- \( y \) is feasible for coalition \( P \): \( \sum_{i=2}^{6} \left( \alpha_i + \frac{1}{5} \right) + \alpha \leq 1 \).

w.l.o.g \( 0 \leq \alpha_2 \leq \alpha_3 \leq \alpha_4 \leq \alpha_5 \leq \alpha_6 \).

Then \( \sum_{i=2}^{6} \left( \frac{1}{5} + \alpha_i \right) + \alpha = \frac{5}{5} + \sum_{i=2}^{6} \alpha_i + \alpha = 1 + \sum_{i=2}^{6} \alpha_i + \alpha \leq 1 \).

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\( z(Q) = \frac{1}{5} + \alpha_2 + \frac{1}{5} + \alpha_3 + \frac{1}{5} + \alpha_4 + \frac{1}{5} + \alpha_5 = \frac{4}{5} + \sum_{i=2}^{6} \alpha_i \leq 1 \) since \( \sum_{i=2}^{5} \alpha_i \leq \sum_{i=2}^{6} \alpha_i < \frac{1}{5} \) so \( z \) is feasible.

It is clear that \( \forall i \in Q, z_i \geq x_i \) ✓ and that \( \forall i \in Q \cap P, z_i \geq y_i \) ✓

Hence, \((Q,z)\) is a counter-objection. ✓
\[ P = \{1, 2, 3, 4, 5\}. \]
• $P = \{1, 2, 3, 4, 5\}$. The vector 
  
  \[ y = \langle \alpha, \frac{1}{5} + \alpha_2, \frac{1}{5} + \alpha_3, \frac{1}{5} + \alpha_4, \frac{1}{5} + \alpha_5 \rangle \]

  is an objection when
  \[ \alpha > -\frac{1}{5}, \alpha_i \geq 0, \sum_{i=2}^{5} \left( \frac{1}{5} + \alpha_i \right) + \alpha \leq 1 \]
- \( P = \{1, 2, 3, 4, 5\} \). The vector 
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y = \langle \alpha, \frac{1}{5} + \alpha_2, \frac{1}{5} + \alpha_3, \frac{1}{5} + \alpha_4, \frac{1}{5} + \alpha_5 \rangle
\] is an objection when 
\[
\alpha > -\frac{1}{5}, \quad \alpha_i \geq 0, \quad \sum_{i=2}^{5} \left( \frac{1}{5} + \alpha_i \right) + \alpha \leq 1
\]

This time, we have 
\[
\sum_{i=2}^{5} \left( \frac{1}{5} + \alpha_i \right) + \alpha = \frac{4}{5} + \sum_{i=2}^{5} \alpha_i + \alpha \leq 1
\]
• $P = \{1, 2, 3, 4, 5\}$. The vector
  
  $y = \langle \alpha, \frac{1}{5} + \alpha_2, \frac{1}{5} + \alpha_3, \frac{1}{5} + \alpha_4, \frac{1}{5} + \alpha_5 \rangle$ is an objection when

  $\alpha > -\frac{1}{5}, \; \alpha_i \geq 0, \; \sum_{i=2}^{5} (\frac{1}{5} + \alpha_i) + \alpha \leq 1$

  This time, we have $\sum_{i=2}^{5} (\frac{1}{5} + \alpha_i) + \alpha = \frac{4}{5} + \sum_{i=2}^{5} \alpha_i + \alpha \leq 1$

  then $\sum_{i=2}^{5} \alpha_i \leq 1 - \frac{4}{5} - \alpha = \frac{1}{5} - \alpha$ and finally $\sum_{i=2}^{5} \alpha_i \leq \frac{1}{5} - \alpha < \frac{2}{5}$. 
\( P = \{1, 2, 3, 4, 5\} \). The vector
\[ y = \langle \alpha, \frac{1}{5} + \alpha_2, \frac{1}{5} + \alpha_3, \frac{1}{5} + \alpha_4, \frac{1}{5} + \alpha_5 \rangle \] is an objection when
\[ \alpha > -\frac{1}{5}, \quad \alpha_i \geq 0, \quad \sum_{i=2}^{5} \left( \frac{1}{5} + \alpha_i \right) + \alpha \leq 1 \]

This time, we have
\[ \sum_{i=2}^{5} \left( \frac{1}{5} + \alpha_i \right) + \alpha = \frac{4}{5} + \sum_{i=2}^{5} \alpha_i + \alpha \leq 1 \]

then
\[ \sum_{i=2}^{5} \alpha_i \leq 1 - \frac{4}{5} - \alpha = \frac{1}{5} - \alpha \] and finally
\[ \sum_{i=2}^{5} \alpha_i \leq \frac{1}{5} - \alpha < \frac{2}{5}. \]

We need to find a counter-objection to \((P, y)\).
- $P = \{1, 2, 3, 4, 5\}$. The vector $y = \langle \alpha, \frac{1}{5} + \alpha_2, \frac{1}{5} + \alpha_3, \frac{1}{5} + \alpha_4, \frac{1}{5} + \alpha_5 \rangle$ is an objection when

$$\alpha > -\frac{1}{5}, \quad \alpha_i \geq 0, \quad \sum_{i=2}^{5} \left( \frac{1}{5} + \alpha_i \right) + \alpha \leq 1$$

This time, we have

$$\sum_{i=2}^{5} \left( \frac{1}{5} + \alpha_i \right) + \alpha = \frac{4}{5} + \sum_{i=2}^{5} \alpha_i + \alpha \leq 1$$

then

$$\sum_{i=2}^{5} \alpha_i \leq 1 - \frac{4}{5} - \alpha = \frac{1}{5} - \alpha$$

and finally

$$\sum_{i=2}^{5} \alpha_i \leq \frac{1}{5} - \alpha < \frac{2}{5}.$$

- We need to find a counter-objection to $(P,y)$

claim: we can choose $Q = \{2, 3, 6, 7\}$, $z = \langle \frac{1}{5} + \alpha_2, \frac{1}{5} + \alpha_3, \frac{1}{5}, \frac{1}{5} \rangle$
\( P = \{1, 2, 3, 4, 5\} \). The vector
\( y = \langle \alpha, \frac{1}{5} + \alpha_2, \frac{1}{5} + \alpha_3, \frac{1}{5} + \alpha_4, \frac{1}{5} + \alpha_5 \rangle \) is an objection when
\[
\alpha > -\frac{1}{5}, \quad \alpha_i \geq 0, \quad \sum_{i=2}^{5} \left( \frac{1}{5} + \alpha_i \right) + \alpha \leq 1
\]
This time, we have
\[
\sum_{i=2}^{5} \left( \frac{1}{5} + \alpha_i \right) + \alpha = \frac{4}{5} + \sum_{i=2}^{5} \alpha_i + \alpha \leq 1
\]
then
\[
\sum_{i=2}^{5} \alpha_i \leq 1 - \frac{4}{5} - \alpha = \frac{1}{5} - \alpha
\]
and finally
\[
\sum_{i=2}^{5} \alpha_i \leq \frac{1}{5} - \alpha < \frac{2}{5}.
\]
We need to find a counter-objection to \((P, y)\)

**Claim:** we can choose \( Q = \{2, 3, 6, 7\}, \ z = \langle \frac{1}{5} + \alpha_2, \frac{1}{5} + \alpha_3, \frac{1}{5}, \frac{1}{5} \rangle \)

It is clear that \( \forall i \in Q, \ z_i \geq x_i \) and \( \forall i \in P \cap Q, \ z_i \geq y_i \) (for agent 2 and 3).
$P = \{1, 2, 3, 4, 5\}$. The vector

\[ y = \langle \alpha, \frac{1}{5} + \alpha_2, \frac{1}{5} + \alpha_3, \frac{1}{5} + \alpha_4, \frac{1}{5} + \alpha_5 \rangle \]

is an objection when

\[ \alpha > -\frac{1}{5}, \, \alpha_i \geq 0, \, \sum_{i=2}^{5} \left( \frac{1}{5} + \alpha_i \right) + \alpha \leq 1 \]

This time, we have

\[ \sum_{i=2}^{5} \left( \frac{1}{5} + \alpha_i \right) + \alpha = \frac{4}{5} + \sum_{i=2}^{5} \alpha_i + \alpha \leq 1 \]

\[ \sum_{i=2}^{5} \alpha_i \leq 1 - \frac{4}{5} - \alpha = \frac{1}{5} - \alpha \]

and finally

\[ \sum_{i=2}^{5} \alpha_i \leq \frac{1}{5} - \alpha < \frac{2}{5}. \]

We need to find a counter-objection to $(P, y)$

**Claim:** we can choose $Q = \{2, 3, 6, 7\}$, $z = \langle \frac{1}{5} + \alpha_2, \frac{1}{5} + \alpha_3, \frac{1}{5}, \frac{1}{5} \rangle$.

It is clear that $\forall i \in Q$, $z_i \geq x_i$ ✓ and $\forall i \in P \cap Q$ $z_i \geq y_i$ (for agent 2 and 3).

\[ z(Q) = \frac{1}{5} + \alpha_2 + \frac{1}{5} + \alpha_3 + \frac{1}{5} + \frac{1}{5} = \frac{4}{5} + \alpha_2 + \alpha_3. \]

We have $\alpha_2 + \alpha_3 < \frac{1}{5}$, otherwise, we would have $\alpha_2 + \alpha_3 \geq \frac{1}{5}$ and since the $\alpha_i$ are ordered, we would then have $\sum_{i=2}^{5} \alpha_i \geq \frac{2}{5}$, which is not possible. Hence $z(Q) \leq 1$ which proves $z$ is feasible ✓
Using similar arguments, we find a counter-objection for each other objections (you might want to fill in the details at home).

- $P = \{1,2,3,4\}, y = \langle \alpha, \frac{1}{5} + \alpha_1, \frac{1}{5} + \alpha_2, \frac{1}{5} + \alpha_3 \rangle$, $\alpha > -\frac{1}{5}$, $\alpha_i \geq 0$, 
  $\sum_{i=2}^{4} \alpha_i + \alpha \leq \frac{2}{5} \Rightarrow \sum_{i=2}^{4} \alpha_i \leq \frac{2}{5} - \alpha < \frac{3}{5}$.

- $Q = \{2,5,6,7\}, z = \langle \frac{1}{5} + \alpha_2, \frac{1}{5}, \frac{1}{5}, \frac{1}{5} \rangle$ since $\alpha_2 \leq \frac{1}{5}$

- $|P| \leq 3$ $P = \{1,2,3\}, v(P) = 0$, $y = \langle \alpha, \alpha_1, \alpha_2 \rangle$, $\alpha > -\frac{1}{5}$, $\alpha_i \geq 0$, $\alpha_1 + \alpha_2 \leq -\alpha < \frac{1}{5}$

- $Q = \{4,5,6,7\}, z = \langle \frac{1}{5}, \frac{1}{5}, \frac{1}{5}, \frac{1}{5} \rangle$ will be a counter argument (1 cannot provide more than $\frac{1}{5}$ to any other agent).

- For each possible objection of 1, we found a counter-objection. Using similar arguments, we can find a counter-objection to any objection of player 7 against player 1.

- $x \in \text{preBS}(N,v,S)$.✅
Using similar arguments, we find a counter-objection for each other objections (you might want to fill in the details at home).

- \( P = \{1,2,3,4\}, \ y = \langle \alpha, \frac{1}{5} + \alpha_1, \frac{1}{5} + \alpha_2, \frac{1}{5} + \alpha_3 \rangle, \ \alpha > -\frac{1}{5}, \ \alpha_i \geq 0, \ \sum_{i=2}^{4} \alpha_i + \alpha \leq \frac{2}{5} \Rightarrow \sum_{i=2}^{4} \alpha_i \leq \frac{2}{5} - \alpha < \frac{3}{5} \).

- \( Q = \{2,5,6,7\}, \ z = \langle \frac{1}{5} + \alpha_2, \frac{1}{5}, \frac{1}{5}, \frac{1}{5} \rangle \) since \( \alpha_2 \leq \frac{1}{5} \).

- \( |P| \leq 3 \ P = \{1,2,3\}, \ v(P) = 0, \ y = \langle \alpha, \alpha_1, \alpha_2 \rangle, \ \alpha > -\frac{1}{5}, \ \alpha_i \geq 0, \ \alpha_1 + \alpha_2 \leq -\alpha < \frac{1}{5} \).

- \( Q = \{4,5,6,7\}, \ z = \langle \frac{1}{5}, \frac{1}{5}, \frac{1}{5}, \frac{1}{5} \rangle \) will be a counter argument (1 cannot provide more than \( \frac{1}{5} \) to any other agent).

- For each possible objection of 1, we found a counter-objection. Using similar arguments, we can find a counter-objection to any objection of player 7 against player 1.

- \( x \in \text{preBS}(N,v,S) \). ✓
Using similar arguments, we find a counter-objection for each other objections (you might want to fill in the details at home).

- \( P = \{1, 2, 3, 4\}, y = \langle \alpha, \frac{1}{5} + \alpha_1, \frac{1}{5} + \alpha_2, \frac{1}{5} + \alpha_3 \rangle, \alpha > -\frac{1}{5}, \alpha_i \geq 0, \sum_{i=2}^{4} \alpha_i + \alpha \leq \frac{2}{5} \Rightarrow \sum_{i=2}^{4} \alpha_i \leq \frac{2}{5} - \alpha < \frac{3}{5}. \)

- \( Q = \{2, 5, 6, 7\}, z = \langle \frac{1}{5} + \alpha_2, \frac{1}{5}, \frac{1}{5} \rangle \) since \( \alpha_2 \leq \frac{1}{5} \)

- \(|P| \leq 3 \ P = \{1, 2, 3\}, v(P) = 0, y = \langle \alpha, \alpha_1, \alpha_2 \rangle, \alpha > -\frac{1}{5}, \alpha_i \geq 0, \alpha_1 + \alpha_2 \leq -\alpha < \frac{1}{5} \)

- \( Q = \{4, 5, 6, 7\}, z = \langle \frac{1}{5}, \frac{1}{5}, \frac{1}{5}, \frac{1}{5} \rangle \) will be a counter argument (1 cannot provide more than \( \frac{1}{5} \) to any other agent).

- For each possible objection of 1, we found a counter-objection. Using similar arguments, we can find a counter-objection to any objection of player 7 against player 1.

- \( x \in preBS(N, v, S). \)
Bargaining set

In the example, agent 1 gets $-\frac{1}{3}$ when $v(C) \geq 0$ for all coalition $C \subseteq N$! This shows that the pre-bargaining set may not be individually rational.

Let $I(N,v,S) = \{ x \in X(N,v,S) | x_i \geq v(\{i\}) \forall i \in N \}$ be the set of individually rational payoff vector in $X(N,v,S)$.

**Lemma**

If a game is weakly superadditive, $I(N,v,S) \neq \emptyset$.

**Definition** (Bargaining set)

Let $(N,v,S)$ a game in coalition structure.

The **bargaining set** ($BS$) is defined by

$$BS(N,v,S) = I(N,v,S) \cap preBS(N,v,S).$$

**Lemma**

We have $Core(N,v,S) \subseteq BS(N,v,S)$. 
Theorem

Let \((N, v, S)\) a game with coalition structure. Assume that \(I(N, v, S) \neq \emptyset\). Then the bargaining set \(BS(N, v, S) \neq \emptyset\).

Proof

It is possible to give a direct proof of this theorem (a bit long, (Section 4.2 in Introduction to the Theory of Cooperative Games)).

We will show this result in a different way in the lecture about the nucleolus next week.

\[\square\]

Definition (weighted voting games)

A game \((N, w_i \in N, q, v)\) is a **weighted voting game** when \(v\) satisfies unanimity, monotonicity and the valuation function is defined as

\[
v(S) = \begin{cases} 
1 & \text{when } \sum_{i \in S} w_i \geq q \\
0 & \text{otherwise}
\end{cases}
\]

We note such a game by \((q : w_1, \ldots, w_n)\)

Let \((N, v)\) be the game associated with the 6-player weighted majority game \((3:1,1,1,1,1,0)\).

Agent 6 is a null/dummy player since its weight is 0.

Nevertheless \(\langle \frac{1}{7}, \ldots, \frac{1}{7}, \frac{2}{7} \rangle \in BS(N,v)\).

**Proof**

This will be part of homework 2

Agent 6 is a dummy, however, it receives a payoff of \(\frac{2}{7}\), which is larger than agents who are not dummy!
Remember: $mc_{i}^{max} = \max_{C \subseteq N \{i\}} v(C \cup \{i\}) - v(C)$

$x$ is **reasonable from above** if $\forall i \in N \ x^i < mc_{i}^{max}$

$mc_{i}^{max}$ is the strongest threat that an agent can use against a coalition.

The bargaining set is not **Reasonable from above**: the dummy agent gets more than $\max_{C \subseteq N \{6\}} (v(C \cup \{6\}) - v(C)) = 0$.

**Lemma**

The core is reasonable for above and from below.

**Proof**

Since the core satisfies IR, it must be reasonable from below. Let $(N, v)$ be a game, $x \in Core(N, v)$ and $i \in N$. Then $x(N) = v(N)$ and $x(N \{i\}) \geq v(N \{i\})$. Then $x_i = v(N) - x(N \{i\}) \leq v(N) - v(N \{i\}) \leq mc_{i}^{max}$.
We introduced the bargaining set, and looked at some examples.

**Pros:** it is guaranteed to be non-empty, when the core is non-empty, it is contained in the bargaining set.

**Cons:** it may not be reasonable from above.
We will consider the Nucleolus. It can also be defined in terms of objections and counter objections, but the nature of the objection is different from the bargaining set.