Today

- Simple games: a class of TU games for modeling voting.
- Measuring the power of a voter: Shapley Shubik, Banzhaff and Co.

Simple Games

Definition (Simple games)
A game \((N,v)\) is a \textbf{Simple game} when
the valuation function takes two values
- 1 for a winning coalition
- 0 for the losing coalitions
\(v\) satisfies \textbf{unanimity}: \(v(\emptyset) = 1\)
\(v\) satisfies \textbf{monotonicity}: \(S \subseteq T \Rightarrow v(S) \leq v(T)\)

One can represent the game by stating all the winning coalitions.
Thanks to monotonicity, it is sufficient to only write down the minimal winning coalitions defined as follows:

Definition (Minimal winning coalition)
Let \((N,v)\) be a TU game. A coalition \(C\) is a \textbf{minimal winning coalition} iff \(v(C) = 1\) and \(\forall i \in C, v(C \setminus \{i\}) = 0\).

Example

\(N = \{1,2,3,4\}\).
We use majority voting, and in case of a tie, the decision of player 1 wins.
The set of winning coalitions is
\([1,2], [1,3], [1,4], [1,2,3], [1,2,4], [1,3,4], [2,3,4], [1,2,3,4]\).
The set of minimal winning coalitions is
\([1,2], [1,3], [1,4], [2,3,4]\).

Formal definition of common terms in voting

Definition (Dictator)
Let \((N,v)\) be a simple game. A player \(i \in N\) is a \textbf{dictator} iff \(\{i\}\) is a winning coalition.

Note that with the requirements of simple games, it is possible to have more than one dictator!

Definition (Veto Player)
Let \((N,v)\) be a simple game. A player \(i \in N\) is a \textbf{veto player} if \(N \setminus \{i\}\) is a losing coalition. Alternatively, \(i\) is a \textbf{veto player} iff for all winning coalition \(C\), \(i \in C\).

It also follows that a veto player is member of every minimal winning coalitions.

Definition (Blocking coalition)
A coalition \(C \subseteq N\) is a \textbf{blocking coalition} iff \(C\) is a losing coalition and \(\forall S \subseteq N \setminus C, S \cup C\) is a losing coalition.

A class of simple games

Definition (weighted voting games)
A game \((N,w)\) is a \textbf{weighted voting game} when \(v\) satisfies unanimity, monotonicity and the valuation function is defined as
\[
v(S) = \begin{cases} 
1 & \text{when } \sum_{i \in S} w_i \geq q \\
0 & \text{otherwise}
\end{cases}
\]

Unanimity requires that \(\sum_{i \in N} w_i \geq q\).
If we assume that \(\forall i \in N, w_i > 0\), monotonicity is guaranteed.
For the rest of the lecture, we will assume \(w_i > 0\).

We will note a weighted voting game \((N,w)\) as \(\langle q; w_1, w_2, w_3, w_4 \rangle\).
A weighted voting game is a \textbf{succinct} representation, as we only need to define a weight for each agent and a threshold.

Weighted voting game is a strict subclass of voting games.
i.e., all voting games are not weighted voting games.

Example: Let \((\{1,2,3,4\},v)\) a voting game such that the set of minimal winning coalitions is \(\{1,2\}, \{3,4\}\). Let us assume we can represent \((N,v)\) with a weighted voting game \(\langle q; w_1, w_2, w_3, w_4 \rangle\).

\[
\begin{align*}
v(\{1,2\}) & = 1 \text{ then } w_1 + w_2 \geq q \\
v(\{3,4\}) & = 1 \text{ then } w_3 + w_4 \geq q \\
v(\{3\}) & = 0 \text{ then } w_3 + w_4 < q \\
v(\{4\}) & = 0 \text{ then } w_3 + w_4 < q \\
\end{align*}
\]

But then, \(w_1 + w_2 + w_3 + w_4 < 2q\) and \(w_1 + w_2 + w_3 + w_4 \geq 2q\), which is impossible. Hence, \((N,v)\) cannot be represented by a weighted voting game.

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Stability for simple games

A simple game \((N,\nu)\) is convex iff it is a unanimity game \((N,v_N)\) where \(V\) is the set of veto players.

**Proof**

A game is convex iff \(\forall S,T \subseteq N \ (v(S) + v(T)) \leq v(S \cup T) + v(S \cap T)\).

Let us assume \((N,v)\) is convex.

If \(S\) and \(T\) are winning coalitions, \(S \cup T\) is a winning coalition by monotonicity. Then, we have \(2 \leq 1 + v(S \cap T)\) and it follows that \(v(S \cap T) = 1\). The intersection of two winning coalitions is a winning coalition. Moreover, from the definition of veto players, the intersection of all winning coalitions is the set \(V\) of veto players. Hence, \(v(V) = 1\).

By monotonicity, if \(V \subseteq C\), \(v(C) = 1\).

Otherwise, \(V \not\subseteq C\). Then there must be a veto player \(i \not\in C\), and it must be the case that \(v(C) = 0\).

Hence, for all coalition \(C \subseteq N\), \(v(C) = 1\) iff \(V \subseteq C\).

\(\square\)

**Example**

Let us consider the game \([4,2,1]\).

- \(q = 1\): minimal winning coalitions: \([1],[2],[3]\)
- \(q = 2\): minimal winning coalitions: \([1,2]\)
- \(q = 3\): minimal winning coalitions: \([1,2],[3]\)
- \(q = 4\): minimal winning coalition: \([1,2]\)
- \(q = 5\): minimal winning coalitions: \([1,2],[3,1]\)
- \(q = 6\): minimal winning coalition: \([1,2]\)
- \(q = 7\): minimal winning coalition: \([1,2,3]\)

For \(q = 4\) ("majority" weight), 1 is a dictator, 2 and 3 are dummies.

**Weights may be deceptive**

Let us consider the game \([10,7,4,3,1]\).

The set of minimal winning coalitions is \([1],[2],[3],[4],[1,2,3,4]\).

Player 5, although it has some weight, is a dummy.

Player 2 has a higher weight than player 3 and 4, but it is clear that player 2, 3 and 4 have the same influence.

Let us consider the game \([33,49,49,2]\).

The set of winning coalition is \([1,2],[1,3],[2,3]\).

It seems that the players have symmetric roles, but it is not reflected in their weights.

\(\square\)
Consider the game all coalitions are equally likely.

Paradox of new players

intuition: Adding a voter should decrease the power of the original voters. not necessarily true!

Example: [7, 4, 3, 2, 1]

Paradoxes (cont)

Paradox of size

intuition: If a voter splits her identities and share her weights between the new identities, she should not gain or lose power — no necessarily true!

- increase of power
  - n-player game \([n = 1, 2, 1, \ldots, 1]\): all voters have a Shapley value of \(\frac{1}{n}\)
  - Voter 1 splits into two voters with weight of 1. In the new game, each agent has a Shapley value of \(\frac{1}{n}\) — voter 1 gets more power.

- decrease of power
  - n-player game \([n = 2, 1, 2, \ldots, 2]\): all voters have the same Shapley value of \(\frac{1}{n}\)
  - Voter 1 splits into two voters with a weight of 1. These new voters have a Shapley value of \(\frac{1}{2n}\) in the new game — voter 1 loses power by a factor of \(\frac{1}{2n}\).

Other indices

- Coleman indices: all winning coalitions are equally likely. Let \(W(N, v)\) be the set of all winning coalitions.
- The power of collectivity to act \(P_{\text{act}}\) is the probability that a winning vote arise:
  \[
  P_{\text{act}} = \frac{|W(N, v)|}{2^n}
  \]
- The power to prevent an action: \(P_{\text{prevent}}\) captures the power of \(i\) to prevent a coalition to win by withholding its vote:
  \[
  P_{\text{prevent}} = \sum_{\emptyset \neq C \subseteq N, (i,j) \notin C} |\{C \cup (i,j)\}| - |\{C\}|
  \]
- The power to initiate an action: \(P_{\text{initiate}}\) captures the power of \(i\) to join a losing coalition so that it becomes a winning one:
  \[
  P_{\text{initiate}} = \sum_{\emptyset \neq C \subseteq N, (i,j) \in C} |\{C \cup (i,j)\}| - |\{C\}|
  \]

Maybe only minimal winning coalitions are important to measure the power of an agent (non-minimal winning coalitions may form, but only the minimal ones are important to measure power).

Let \((N, v)\) be a simple game, \(i \in N\) be an agent.
- \(M(N, v)\) denotes the set of minimal winning coalitions,
- \(M_i(N, v)\) denotes the set of minimal winning coalitions containing \(i\).
- The Deegan-Packel power index of player \(i\) is:
  \[
  I_{DP}(N, v, i) = \frac{1}{|M(N, v)|} \sum_{C \in M_i(N, v)} \left[ \frac{|M(N, v)|}{n} \right]
  \]
- The public good index of player \(i\) is defined as
  \[
  I_{PG}(N, v, i) = \frac{|M_i(N, v)|}{\sum_{j \in N} |M_j(N, v)|}
  \]
Summary

- We introduced the simple games
- We considered few examples
- We studied some power indices

Coming next

- Representation and Complexity issues
- Are there some succinct representations for some classes of games.
- How hard is it to compute a solution concept?