Lecture 2

The Core

Let us assume that we have a TU game (N, v) and that we want to form the grand coalition. We model cooperation between all the agents in N and we focus on the sharing problem: how to distribute the payoff $v(N)$ to all agents. The idea for defining one solution is to consider a payoff distribution in which no agent has an incentive to change coalition to gain additional payoff. This is what is called *stability*.

The Core, which was first introduced by Gillies [2], is the most attractive and natural way to define stability. A payoff distribution is in the *Core* when no group of agents has any incentive to form a different coalition. This is a strong condition for stability, so strong that some games may have an empty core. In this lecture, we will first introduce the definition of the core and consider some graphical representations for games with up to three players. Then, we will present some games that are guaranteed to have a non-empty core. Finally, we will present a theorem that characterizes games with non-empty core: the Bondareva-Shapley theorem. We will give some intuition about the proof, relying on results from linear programming, and we will use this theorem to show that market games have a non-empty core.

2.1 Definition and graphical representation for games with up to three players

We consider a TU game (N, v). We assume that all the agents cooperate by forming the *grand coalition* and that they receive a payoff distribution x. We want the grand coalition to be stable, i.e., no agent should have an incentive to leave the grand coalition. We will say that x is in the core of the game (N, v) when no group of agents has an incentive to leave the grand coalition and form a separate coalition.

2.1.1. DEFINITION. [Core] A payoff distribution $x \in \mathbb{R}^n$ is in the *Core* of a TU game (N, v) iff x is an imputation that is group rational, i.e.,

$$Core(N, v) = \{x \in \mathbb{R}^n \mid \sum_{i \in N} x_i = v(N) \ \wedge \ \forall \mathcal{C} \subseteq N \ x(\mathcal{C}) \geq v(\mathcal{C})\}.$$

A payoff distribution is in the Core when no group of agents has any interest in rejecting it, i.e., no group of agents can gain by forming a different coalition. Note that this condition has to be true for all subsets of N (group rationality). As a special case, this ensures individual rationality. Another way to define the Core is in terms of excess:

2.1.2. DEFINITION. [Core] The Core of a TU game (N, v) is the set of payoff distributions $x \in \mathbb{R}^n$, such that $\forall \mathcal{C} \subseteq N$, $e(\mathcal{C}, x) \leq 0$.

In other words, a PC is in the Core when there exists no coalition that has a positive excess. This definition is attractive as it shows that no coalition has any complaint: each coalition's demand can be granted.

To be in the core, a payoff distribution must satisfy a set of 2^n weak linear inequalities: for each coalition $\mathcal{C} \subseteq N$, we have $v(\mathcal{C}) \leq x(\mathcal{C})$. The Core is therefore closed and convex, and we can try to represent it geometrically.

Let us consider the following two-player game $(\{1, 2\}, v)$ where $v(\{1\}) = 5$, $v(\{2\}) = 5$, and $v(\{1, 2\}) = 20$. The core of the game is a segment defined as follows: $core(N, v) = \{(x_1, x_2) \in \mathbb{R}^2 \mid x_1 \geq 5,\ x_2 \geq 5,\ x_1 + x_2 = 20\}$ and is represented in Figure 2.1. This example shows that, although the game is symmetric, most of the payoffs in the core are not fair. Core allocations focus on stability only and they may not be fair.

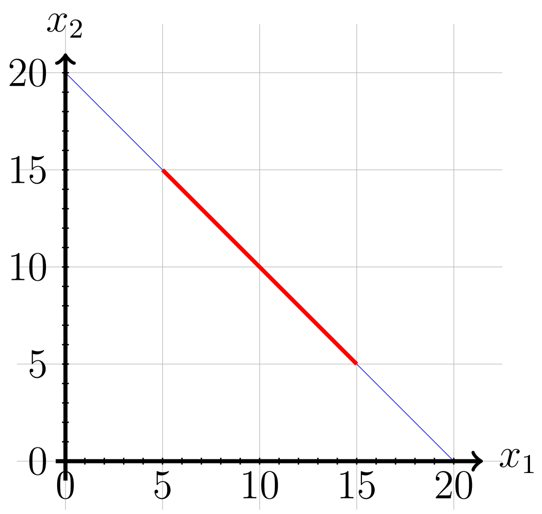
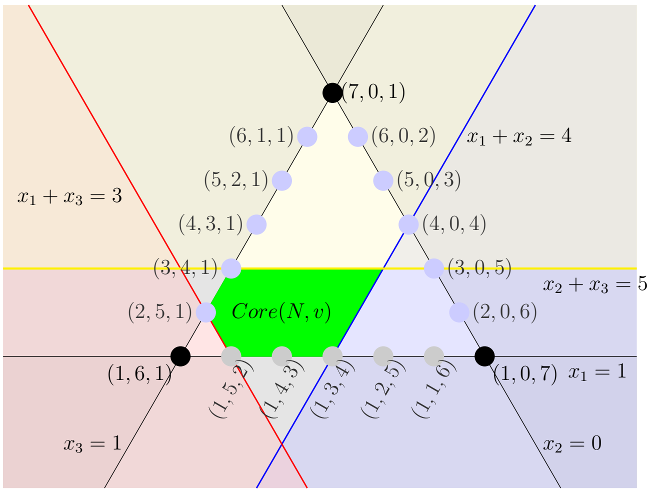


Figure 2.1: Example of a core allocation

It is possible to represent the core for game with three agents. For a game $(\{1, 2, 3\}, v)$, the efficiency condition is $v(\{1, 2, 3\}) = x_1 + x_2 + x_3$, which is a plane in a 3-dimensional space. On this plane, we can draw the conditions for individual rationality and for group rationality. Each of these conditions partitions the space into two regions separated by a line: one region is compatible with a core allocation, the other region is not. The core is the intersection of all the compatible regions. Figure 2.2 represents the core of a three-player game.



$$\begin{array}{cc} v(\{1\}) = 1 & v(\{1,2\}) = 4 \\ v(\{2\}) = 0 & v(\{1,3\}) = 3 \\ v(\{3\}) = 1 & v(\{2,3\}) = 5 \\ v(\emptyset) = 0 & v(\{1,2,3\}) = 8 \end{array}$$

Figure 2.2: Example of a three-player game: The core is the area in green

There are, however, multiple concerns associated with using the notion of the Core. First and foremost, the Core can be empty: the conflicts captured by the characteristic function cannot satisfy all the players simultaneously. When the Core is empty, at least one player is dissatisfied by the utility allocation and therefore blocks the coalition. Let us consider the following example from [3]: $v(\{A, B\}) = 90$, $v(\{A, C\}) = 80$, $v(\{B, C\}) = 70$, and $v(N) = 120$. In this case, the Core is the PC where the grand coalition forms and the associated payoff distribution is $(50, 40, 30)$. If $v(N)$ is increased, the size of the Core also increases. But if $v(N)$ decreases, the Core becomes empty.

Exercise: How can you modify the game in Figure 2.2 so that the core becomes empty?

2.2 Games with non-empty core

In the previous section, we saw that some games have an empty core. In this section, we provide examples of some classes of games that are guaranteed to have a non-empty core. In the following we will show that convex games and minimum cost spanning tree games have a non empty core.

We start introducing an example that models bankruptcy: individuals have claims in a resource, but the value of the resource is not sufficient to meet all of the claims (e.g., a man leaves behind an estate worth less than the value of its debts). The problem is then to share the value of the estate among all the claimants. The value of a coalition

$\mathcal{C}$ is defined as the amount of the estate which is not claimed by the complement of $\mathcal{C}$, in other words $v(\mathcal{C})$ is the amount of the estate that the coalition $\mathcal{C}$ is guaranteed to obtain.

2.2.1. DEFINITION. Bankruptcy game A *Bankruptcy game* (N, E, v) where N is the set of claimants, $E \in \mathbb{R}_+$ is the estate and $c \in \mathbb{R}^n_+$ is the claim vector (i.e., c_i is the claim of the i^{th} claimant. The valuation function $v : 2^N \to \mathbb{R}$ is defined as follows. For a coalition of claimants $\mathcal{C}$, $v(\mathcal{C}) = \max \left\{0, E - \sum_{i \in N \setminus \mathcal{C}} c_i \right\}$.

First, we show that a bankruptcy game is convex.

2.2.2. THEOREM. *Every bankruptcy game is convex.*

Proof. Let (N, E, c) be a bankruptcy game. Let $S \subseteq T \subseteq N$, and $i \notin T$. We want to show that

$$v(S \cup \{i\}) - v(S) \leq v(T \cup \{i\}) - v(T),$$

or equivalently that

$$v(S \cup \{i\}) + v(T) \leq v(T \cup \{i\}) + v(S).$$

For all $C \subseteq N$, we note $c(C) = \sum_{j \in C} c_j$, then we can write:

$$E - \sum_{j \in N \setminus C} c_j = E - \sum_{j \in N} c_j + \sum_{j \in C} c_i = E - c(N) + c(C).$$

Let $\Delta = E - \sum_{j \in N} c_j = E - c(N)$. We have $E - \sum_{j \in N \setminus C} c_j = \Delta + c(C)$.

First, observe that $\forall (x, y) \in \mathbb{R}^2$, $\max\{0, x\} + \max\{0, y\} = \max\{0, x, y, x + y\}$.

$$\begin{aligned} v(S \cup \{i\}) + v(T) &= \max \left\{ 0, E - \sum_{j \in N \setminus (S \cup \{i\})} c_j \right\} + \max \left\{ 0, E - \sum_{j \in N \setminus T} c_j \right\} \\ &= \max \{0,\ \Delta + c(S) + c_i\} + \max \{0,\ \Delta + c(T)\} \\ &= \max \{0,\ \Delta + c(S) + c_i,\ \Delta + c(T),\ 2\Delta + c(S) + c_i + c(T)\} \end{aligned}$$

$$\begin{aligned} v(T \cup \{i\}) + v(S) &= \max \left\{ 0, E - \sum_{j \in N \setminus (T \cup \{i\})} c_j \right\} + \max \left\{ 0, E - \sum_{j \in N \setminus S} c_j \right\} \\ &= \max \{0,\ \Delta + c(T) + c_i\} + \max \{0,\ \Delta + c(S)\} \\ &= \max \{0,\ \Delta + c(T) + c_i,\ \Delta + c(S),\ 2\Delta + c(T) + c_i + c(S)\} \end{aligned}$$

Then, note that since $S \subseteq T$, $c(S) \leq c(T)$. Then

$$\max \{0,\ \Delta + c(T) + c_i,\ \Delta + c(S),\ 2\Delta + c(T) + c_i + c(S)\} = \max \{0,\ \Delta + c(T) + c_i,\ 2\Delta + c(T) + c_i + c(S)\}.$$

We also have:
$\Delta + c(S) + c_i \leq \Delta + c(T) + c_i$.
$\Delta + c(T) \leq \Delta + c(T) + c_i$.

It follows that $\max\{0,\ \Delta + c(S) + c_i,\ \Delta + c(T),\ 2\Delta + c(S) + c_i + c(T)\}$
$\leq \max\{0,\ \Delta + c(T) + c_i,\ 2\Delta + c(T) + c_i + c(S)\}$
which proves that $v(S \cup \{i\}) + v(T) \leq v(T \cup \{i\}) + v(S)$. ✔ □

Now, we show an important property of convex games: they are guaranteed to have a non-empty core. We define a payoff distribution where each agent gets its marginal contribution, given that the agents enter the grand coalition one at a time in a given order, and we show that this payoff distribution is an imputation that is group rational.

2.2.3. THEOREM. *A convex game has a non-empty core.*

Proof. Let us assume a convex game (N, v). Let us define a payoff vector x in the following way: $x_1 = v(\{1\})$ and for all $i \in \{2, \ldots, n\}$, $x_i = v(\{1, 2, \ldots, i\}) - v(\{1, 2, \ldots, i-1\})$. In other words, the payoff of the i^{th} agent is its marginal contribution to the coalition consisting of all previous agents in the order $\{1, 2, \ldots, i-1\}$.

Let us prove that the payoff vector is *efficient* by writing up and summing the payoff of all agents:

$$\begin{array}{rcl} x_1 & = & v(\{1\}) \\ x_2 & = & v(\{1,2\} - v(\{1\}) \\ & \ldots & \\ x_i & = & v(\{1,2,\ldots,i\}) - v(\{1,2,\ldots,i-1\}) \\ & \ldots & \\ x_n & = & v(\{1,2,\ldots,n\}) - v(\{1,2,\ldots,n-1\}) \\ \sum_{i \in N} x_n & = & v(\{1,2,\ldots,n\}) = v(N) \end{array}$$

By summing these n equalities, we obtain the efficiency condition:
$\sum_{i \in N} x_n = v(\{1, 2, \ldots, n\}) = v(N)$.✔

Let us prove that the payoff vector is *individually rational*. By convexity, we have $v(\{i\}) - v(\emptyset) \leq v(\{1, 2, \ldots, i\}) - v(\{1, 2, \ldots, i-1\})$, hence $v(\{i\}) \leq x_i$. ✔

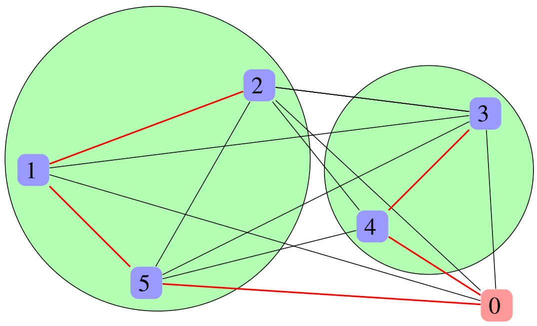
Finally, let us prove that the payoff vector is *group rational*. Let $C \subseteq N$, $C = \{a_1, a_2, \ldots, a_k\}$ and let us consider that $a_1 < a_2 < \ldots < a_k$. It is obvious that $\{a_1, a_2, \ldots, a_k\} \subseteq \{1, 2, \ldots, a_k\}$. Using the convexity assumption, we obtain the following:

$$\begin{array}{rcl} v(\{a_1\}) - v(\emptyset) & \leq & v(\{1,2,\dots,a_1\}) - v(\{1,2,\dots,a_1-1\}) = x_{a_1} \\ v(\{a_1,a_2\}) - v(\{a_1\}) & \leq & v(\{1,2,\dots,a_2\}) - v(\{1,2,\dots,a_2-1\}) = x_{a_2} \\ & \dots & \\ v(\{a_1,a_2,\dots,a_l\}) - v(\{a_1,a_2,\dots,a_{l-1}\}) & \leq & v(\{1,2,\dots,a_l\}) - v(\{1,2,\dots,a_l-1\}) = x_{a_l} \\ & \dots & \\ v(\{a_1,a_2,\dots,a_k\}) - v(\{a_1,a_2,\dots,a_{k-1}\}) & \leq & v(\{1,2,\dots,a_k\}) - v(\{1,2,\dots,a_k-1\}) = x_{a_k} \\ v(C) = v(\{a_1,a_2,\dots,a_k\}) & \leq & \sum_{i=1}^{k} x_{a_k} = x(C) \end{array}$$

By summing these k inequalities, we obtain:
$v(C) = v(\{a_1, a_2, \dots, a_k\}) \leq \sum_{i=1}^{k} x_{a_k} = x(C)$, which is the group rationality condition. ✔ □

Consequently, if a game is convex, we know that we can guarantee a stable payoff distribution. Moreover, we now know one easy way to compute one of these stable payoffs.

Another example of games that have a non-empty core are the class of *minimum cost spanning tree game*. This game features a set of houses that have to be connected to a power plant. The houses can be directly linked to the power plant, or to another house. Let N be the set of houses, and let P be the power plant. Let us define $N_* = N \cup \{0\}$. For $(i,j) \in N_*^2, i \neq j$, the cost of connecting i and j by the edge e_{ij} is $c_{i,j}$. For a coalition of houses $\mathcal{C} \subseteq N$, let $\Gamma(\mathcal{C})$ be a minimum cost spanning tree over the set of edges $\mathcal{C} \cap \{P\}$. In other words, when the houses form a coalition $\mathcal{C}$, they try to minimize the cost of connected them to the power plant. Let (N, c) be the corresponding cost game in which the cost of coalition $\mathcal{C} \subseteq N$ is defined as $c(\mathcal{C}) = \sum_{(i,j) \in \Gamma(\mathcal{C})} c_{ij}$.



2.2.4. THEOREM. *Every minimum cost spanning tree game has a non-empty core.*

Proof. Let us define a cost distribution x and then we will show that x is in the core.

Let $T = (N, E_N)$ a minimum cost spanning tree for the graph $\left(N_*, c_{\{ij\} \subseteq N_*^2}\right)$. Let i be a customer. Since T is a tree, there is a unique path $(0, a_1, \dots, a_k, i)$ from 0 to i. The cost paid by agent i is defined by $x_i = c_{a_k,i}$.

This cost allocation is efficient by construction of x.

We need to show the cost allocation is group rational, i.e., for all coalition S, we have $x(S) \leq v(S)$ (it is a cost, which explains the inequality).

Let $S \subset N$ and $T_S = (S \cup \{0\}, E_s)$ be a minimum cost spanning tree of the graph $\left(S \cup \{0\}, c_{\{ij\} \in S \cup \{0\}}\right)$. Let extand the tree T_S to a graph $T_S^+ = (N_*, E_N^+)$ by adding the remaining customers $N \setminus S$, and for each customer $i \in N \setminus S$, we add the edge of E_N ending in i, i.e., we add the edge (a_k, i). The graph T_S^+ has $|S| + |N \setminus S|$ edges an is connected. Hence, T_S^+ is a spanning tree. Now, we note that $c(S) + x(N \setminus S) = \sum_{e_{ij} \subseteq E_N^+} c_{ij} \geq \sum_{e_{ij} \subseteq E_N} = c(N) = x(N)$. The inequality is due to the fact that T_S^+ is a spanning tree, and T is a minimum spanning tree. It follows that $x(S) \leq v(S)$. ✔ □

2.3 Characterization of games with a non-empty core

We saw that the core may be empty, but that some classes of games have a non-empty core. The next issue is whether we can characterize the games with non-empty core. It turns out that the answer is yes, and the characterization has been found independently by Bondareva (1963) and Shapley (1967), resulting in what is now known as the Bondareva–Shapley theorem. This result connects results from linear programming with the concept of the core. In the following, we will first write the definition of elements in the core as an optimization problem. Then, we will briefly introduce linear programming and we will use a result to charaterize the games with non-empty core, which is the Bondareva-Shapley theorem. Finally, we will apply the Bondareva-Shapley theorem to market games.

2.3.1 Expressing the core as an optimization problem

The main idea is to consider that the core can be expressed as a solution of a constraint linear optimization problem where the condition imposed by group rationality are the constraints of the optimization problem and the objective function is the sum of the payoffs of the agents. Let us consider a TU game (N, v), let x denote a payoff distribution and let us consider the following optimization problem:

$$(LP) \begin{cases} \min x(N) \\ \text{subject to } x(\mathcal{C}) \geq v(\mathcal{C}) \text{ for all } \mathcal{C} \subseteq N,\, S \neq \emptyset \end{cases}$$

The linear constraints are the constraints of group rationality: for each coalition $\mathcal{C} \subseteq N$, we want $x(\mathcal{C}) \geq v(\mathcal{C})$. Satisfying these constraints only is easy: one simply needs to choose large enough values for each x_i. If an element $y \in \mathbb{R}^n$ satisfies all these constraints (this is called a feasible solution), then y is group rational. The group rationality assumption for the grand coalition guarantees that we have $y(N) \geq v(N)$. For y to be in the core, it also needs to be efficient. This forces us to choose values that are not too large for the y_i. The idea is then to search for the elements that minimize $y(N) = \sum_{i \in N} y_i$.

When solving this optimization problem, two things may happen. Either the minimum value found is $v(N)$, or it is a value strictly greater.

- In the first case, the solutions x of the optimization problems are elements of the core: they satisfy the constraints – hence, they are group rational – and since the minimum is $v(N)$, x is efficient as well.
- In the second case, it is not possible to satisfy both group rationality and efficiency, and the core of the game is empty.
- there is no other cases as a solution of the optimization problem would satisfy all the constraints, in particular the one for the grand coalition.

The optimization problem we wrote is called a linear program. It minimizes a linear function of a vector x subject to a set of constraints where each constraint is an inequality: a linear combination of x is larger than a constant. This problem is a well established problem in optimization and in the following, we give a brief introduction to such problems.

2.3.2 A very brief introduction to linear programming

The goal of this section is to briefly introduce linear programming, which is a special kind of optimization problems: the problem is about maximizing a linear function subject to linear constraints. More formally, a *linear program* has the following form:

$$\begin{cases} \max c^T x \\ \text{subject to} \begin{cases} Ax \leq b, \\ x \geq 0 \end{cases} \end{cases} \quad \text{where}$$

- $x \in \mathbb{R}^n$ is a vector of n variables
- $c \in \mathbb{R}^n$ is the objective function
- A is a $m \times n$ matrix
- $b \in \mathbb{R}^n$ is a vector of size n

A and b represent the *linear constraints*. Let us look at a simple example:

$$\begin{cases} \text{maximize } 8x_1 + 10x_2 + 5x_3 \\ \text{subject to} \begin{cases} 3x_1 & + & 4x_2 & + & 2x_3 & \leq & 7 & (1) \\ x_1 & + & x_2 & + & x_3 & \leq & 2 & (2) \end{cases} \end{cases}$$

In this example, we can recognize the different components A, B and C to be:

$$A = \begin{pmatrix} 3 & 4 & 2 \\ 1 & 1 & 1 \end{pmatrix} \quad b = \begin{pmatrix} 7 \\ 2 \end{pmatrix} \quad c = \begin{pmatrix} 8 \\ 10 \\ 5 \end{pmatrix}.$$

We say that a solution is *feasible* when it satisfies the constraints. For our example, we have:

- $\langle 0, 1, 1 \rangle$ is feasible, with objective function value 15.
- $\langle 1, 1, 0 \rangle$ is feasible, with objective function value 18, hence it is a better solution.

Next, we introduce the notion of the *dual* of a LP: it is another linear program which goal is to find an upper bound to the objective function of the original LP. Let us first look at our example and let us consider the following two linear transformations:

$$\begin{array}{lcl} (1) \times 1 + (2) \times 6 & \Rightarrow & 9x_1 + 10x_2 + 8x_3 \leq 19 \\ (1) \times 2 + (2) \times 2 & \Rightarrow & 8x_1 + 10x_2 + 6x_3 \leq 18 \end{array}$$

by taking linear combinations over the constraints, we are able to form a new constraint that provides an upper bound for the objective function. The reason is that in the new constraint we formed, the coefficients for x_1, x_2 and x_3 are larger or equal to the ones of the objective function, hence, it must be the case that the bound is an upper bound for the objective function. Using the second new constraint, we observe that the solution cannot be better than 18. But we already found one feasible solution with a value of 18, so we have solved the problem! ✔

Hence, one idea of the dual is to find a new constraint that is a linear combination of all the constraints of the primal: $y^T A \leq y^T b$ (where $y \in \mathbb{R}^m$). This new constraint must generate the lowest value – as $y^T b$ will be the upper bound of a solution –, and the coefficient of $y^T A$ must be larger than the coefficients of the objective function, i.e., $y^T A \geq c^T$. Hence, the dual can be written in the following way:

Primal	Dual
$\begin{cases} \max c^T x \\ \text{subject to} \begin{cases} Ax \leq b, \\ x \geq 0 \end{cases} \end{cases}$	$\begin{cases} \min y^T b \\ \text{subject to} \begin{cases} y^T A \geq c^T, \\ y \geq 0 \end{cases} \end{cases}$

The following theorems link the solution of the primal and the dual problems.

2.3.1. THEOREM (DUALITY THEOREM). *When the primal and the dual are feasible, they have optimal solutions with equal value of their objective function.*

2.3.2. THEOREM (WEAK LP DUALITY). *For a pair x, y of feasible solutions of the primal LP and its dual LP, the objective functions are mutual bounds:*

$$y^T b \leq c^T x$$

If thereby $c^T x = y^T b$ (equality holds), then these two solutions are optimal for both LPs.

Proof. We have $y^T Ax \geq y^T b$ since $Ax \geq b$, $y \geq 0$, and $y^T Ax \leq c^T x$ since $y^T A \leq c$, $x \geq 0$. It is immediate that equality of the objective functions implies optimality. □

2.3.3 Linear programming and the core

Now, let us go back to the core. The *linear programming* problem that corresponds to the core is:

$$(LP) \begin{cases} \min x(N) \\ \text{subject to } x(\mathcal{C}) \geq v(\mathcal{C}) \text{ for all } \mathcal{C} \subseteq N,\, S \neq \emptyset \end{cases}$$

First, this formulation is not exactly the one what have just introduced since it is a minimization and the constraints are of the form: "a linear combination of x is greater than a constant". It should not be difficult to get convinced that these two kinds of optimization problems are symmetrical and have similar properties. In terms of the conventional way to write the primal, we identify the following components:

- the vector $c \in \mathbb{R}^n$ is the vector $\langle 1, 1, \ldots, 1 \rangle$.
- the vector $b \in \mathbb{R}^{2^n}$ contains the value of each coalition, i.e., we can index the elements of b by using a coalition and the elements of b are $b_\mathcal{C} = v(\mathcal{C})$.
- The matrix A has 2^n rows (one for each coalition) and n columns (one for each agent). The entries of A are either 0 or 1. Let us consider one coalition $\mathcal{C}$, the corresponding constraint for the core is $\sum_{k \in \mathcal{C}} x_k \geq v(\mathcal{C})$. Let us say that the value of coalition $\mathcal{C}$ appears in row i of vector b, i.e. the constraint about $\mathcal{C}$ is expressed in the i^{th} row of $Ax \geq b$. Consequently, the i^{th} row of A encodes which agent are present in coalition $\mathcal{C}$: the entry $A(i, j)$ is 1 if $j \in \mathcal{C}$ and 0 otherwise.

Now, we write the dual which maximises $y^T b$ over all vectors $y \in \mathbb{R}_+^{2^n}$.

$$\begin{cases} \max y^T b \\ \text{subject to } \begin{cases} y^T A \leq c^T, \\ y \geq 0 \end{cases} \end{cases}$$

Now, let us introduce some notations to help us write the matrix A.

2.3.3. DEFINITION. [Characteristic vector] Let $\mathcal{C} \subseteq N$. The *characteristic vector* $\chi_\mathcal{C} \in \mathbb{R}^N$ of $\mathcal{C}$ is the member of $\mathbb{R}^N$ defined by

$$\chi_\mathcal{C}^i = \begin{cases} 1 \text{ if } i \in \mathcal{C} \\ 0 \text{ if } i \in N \setminus \mathcal{C} \end{cases}$$

The characteristic vector of a coalition simply encodes which agents are present in a coalition. For example, for $n = 4$, $\chi_{\{2,4\}} = \langle 0, 1, 0, 1 \rangle$. This will be helpful to express the rows of A.

2.3.4. DEFINITION. [Map] A *map* is a function $2^N \setminus \emptyset \to \mathbb{R}_+$ that gives a positive weight to each coalition.

A map can be seen as a positive weight that is given to each coalition. Hence, the solution y of the dual can be called a map.

2.3.5. DEFINITION. [Balanced map] A function $\lambda : 2^N \setminus \emptyset \to \mathbb{R}_+$ is a *balanced map* iff $\sum_{\mathcal{C}\subseteq N} \lambda(\mathcal{C})\chi_\mathcal{C} = \chi_N$. For convenience, we will write $\lambda(\mathcal{C}) = \lambda_\mathcal{C}$.

We provide an example in Table 2.1 for a three-player game. $\lambda_\mathcal{C}$ is a scalar and $\chi_\mathcal{C}$ is a vector of $\mathbb{R}^n$, so the condition features the equality between a sum over 2^n vectors of $\mathbb{R}^n$ and $\chi_N \in \mathbb{R}^n$ that is nothing but the vector of $\mathbb{R}^n$ containing the value 1 for each entry. This will be useful to write the constraints of the dual (we will give further explanation in the following).

i	1	2	3
$\lambda_{\{1,2\}}\chi^i_{\{1,2\}}$	$\frac{1}{2}$	$\frac{1}{2}$	0
$\lambda_{\{1,3\}}\chi^i_{\{1,3\}}$	$\frac{1}{2}$	0	$\frac{1}{2}$
$\lambda_{\{2,3\}}\chi^i_{\{2,3\}}$	0	$\frac{1}{2}$	$\frac{1}{2}$

$\lambda_\mathcal{C} = \begin{cases} \frac{1}{2} \text{ if } |\mathcal{C}| = 2 \\ 0 \text{ otherwise} \end{cases}$

Each of the column sums up to 1.

$\frac{1}{2}\chi_{\{1,2\}} + \frac{1}{2}\chi_{\{1,3\}} + \frac{1}{2}\chi_{\{2,3\}} = \chi_{\{1,2,3\}}$

Table 2.1: Example of a balanced map for $n = 3$

One can interpret a balanced map as a percentage of time spent by each agent in each possible coalition: for each agent i, the sum of the map for all coalitions containing agent i must sum up to one.

Given these notational tools, let us re-write the dual.

- for the objective function: $y^T \cdot b$ is the dot product of the variable y with the value of each coalition. If we use a coalition to index the entries of the vector y – or if we say we are using a map y –, the objective function can be written as $\sum_{\mathcal{C}\subseteq N} y_\mathcal{C} v(\mathcal{C})$.

- for the constraints, we have $y^T A \leq c^T$. First c^T is a vector composed of 1. It is also the vector χ_N as all the agents are present in N.

  Then we have the dot product between y^T and A: the result of this product is a vector of size n. Let us consider the i^{th} entry of the product: it is the dot product between y^T and the i^{th} column of A (both vectors are of size 2^n and we can indexed them using coalition). We can write this as $\sum_{\mathcal{C}\subseteq N} y_\mathcal{C} A(\mathcal{C}, i)$ and we note that $A(\mathcal{C}, i) = 1$ if $i \in \mathcal{C}$ and 0 otherwise. That is here that our notation comes handy and we can write $\sum_{\mathcal{C}\subseteq N} y_\mathcal{C} A(\mathcal{C}, i) = \sum_{\mathcal{C}\subseteq N} y_\mathcal{C}\chi^i_\mathcal{C}$. Writing for the entire vector, we finally have $y^T A = \sum_{\mathcal{C}\subseteq N} y_\mathcal{C}\chi_\mathcal{C}$.

  Finally, we have shown that the constraints become $\sum_{\mathcal{C}\subseteq N} y_\mathcal{C}\chi_\mathcal{C} \leq \chi_N$.

With our notation, we can now write the dual of LP as:

$$(DLP)\left\{\begin{array}{l}\max \sum_{\mathcal{C}\subseteq N} y_{\mathcal{C}} v(\mathcal{C})\\ \text{subject to}\left\{\begin{array}{l}\sum_{\mathcal{C}\subseteq N} y_{\mathcal{C}}\chi_{\mathcal{C}} \leq \chi_N \text{ and,}\\ y_{\mathcal{C}} \geq 0 \text{ for all } \mathcal{C}\subseteq N, \mathcal{C}\neq\emptyset.\end{array}\right.\end{array}\right.$$

Let us consider a game (N, v) with a non-empty core. This means that the dual is feasible (there are payoff distributions that satisfy the constraints) and that the optimal payoff has a value of $v(N)$, i.e. it is efficient.

Note that the dual is also feasible. Since one can always define a balanced map, we are guaranteed that there exists some $y \in \mathbb{R}_+^{2^n}$ such that $\sum_{\mathcal{C}\subseteq N} y_{\mathcal{C}}\chi_{\mathcal{C}} \leq \chi_N$).

Since $v(N)$ is the minimum of the primal, by Theorem 2.3.2 it is an upper bound of the dual and it follows that $\max \sum_{\mathcal{C}\subseteq N} y_{\mathcal{C}} v(\mathcal{C}) \leq v(N)$. With this, we conclude that if a game has a non-empty core we have $\max \sum_{\mathcal{C}\subseteq N} y_{\mathcal{C}} v(\mathcal{C}) \leq v(N)$. To characterize games with a non-empty core, we need to prove the converse. First, let us give a name to our condition.

2.3.6. DEFINITION. [Balanced game] A game is *balanced* iff for each balanced map λ we have

$$\sum_{\mathcal{C}\subseteq N, \mathcal{C}\neq\emptyset} \lambda(\mathcal{C}) v(\mathcal{C}) \leq v(N).$$

Let us consider that a game (N, v) is balanced, i.e., for each balanced map λ, we have $\sum_{\mathcal{C}\subseteq N} \lambda_{\mathcal{C}} v(\mathcal{C}) \leq v(N)$. We know that the dual is feasible (using any balanced map). With the use of a balanced map, we reach the equality for the constraints (i.e. each constraint is an inequality, but with the balanced map we reach an equality). Since the coefficients are positive, we will not be able to improve the optimal value of the dual. Hence, $v(N)$ is the optimal value.

Now, let us go back to the primal. The vector $\langle 0, \ldots, 0\rangle$ is feasible, so the primal is feasible. Using theorem 2.3.2, we know that $v(N)$ is a lower bound for the primal, i.e $v(N) \leq x(N)$. Applying group rationality to the grand coalition, we also know that a solution must satisfy $x(N) \geq v(N)$. Consequently, $v(N)$ is also the solution to the primal. Hence, the core is non-empty.

We have thus proved a characterization of games with non-empty core. This results was established independently by Bondareva (1963) and Shapley (1967).

2.3.7. THEOREM (BONDAREVA-SHAPLEY THEOREM). *A TU game has a non-empty core iff it is balanced.*

This theorem completely characterizes the set of games with a non-empty core. However, it is not always easy or computationally feasible to check that it is a balanced game.

2.3.4 Application to market games

One example of coalitional games coming from the field of economics is a market game. This game models an environment where there is a given, fixed quantity of a set of continuous good. Initially, these goods are distributed among the players in an arbitrary way. The quantity of each good is called the endowment of the good. Each agent i has a valuation function that takes as input a vector describing its endowment for each good and that output a utility for possessing these goods (the agents do not perform any transformation, i.e., the goods are conserved as they are). To increase their utility, the agents are free to trade goods. When the agents are forming a coalition, they are trying to allocate the goods such that the social welfare of the coalition (i.e. the sum of the utility of each member of the coalition) is maximized. We now provide the formal definition.

A *market* is a quadruple (N, M, A, F) where

- N is a set of traders
- M is a set of m continuous good
- $A = (a_i)_{i \in N}$ is the initial endowment vector
- $F = (f_i)_{i \in N}$ is the valuation function vector, each f_i is continuous and concave.
- $v(S) = \max \left\{ \sum_{i \in S} f_i(x_i) \;\middle|\; x_i \in \mathbb{R}^m_+, \sum_{i \in S} x_i = \sum_{i \in S} a_i \right\}$
- we further assume that the f_i are continuous and concave.

Let us assume that the players form the grand coalition: all the players are in the market and try to maximize the sum of utility of the market. How should this utility be shared amond the players? One way to answer this question is by using an allocation that is in the core. One interesting property is that the core of such game is guaranteed to be non-empty, and one way to prove it is to use the Bondareva-Shapley theorem.

2.3.8. THEOREM. *Every Market Game is balanced.*

Proof.

$f : \mathbb{R}^n \to \mathbb{R}$ is concave iff $\forall \alpha \in [0, 1]$, $\forall (x, y) \in \mathbb{R}^n$, $f(\alpha x + (1 - \alpha)y) \geq \alpha f(x) + (1 - \alpha)f(y)$. It follows from this definition that for $f : \mathbb{R} \to \mathbb{R}$, $\forall x \in \mathbb{R}^n$, $\forall \lambda \in \mathbb{R}^n_+$ such that $\sum_{i=1}^n \lambda_i = 1$, we have $f(\sum_{i=1}^n \lambda_i x_i) \geq \sum_{i=1}^n \lambda_i f(x_i)$.

Since the f_is are continuous, $\sum_{i \in S} f_i(x_i)$ is a continuous mapping from T=$\left\{ (x_i)_{i \in S} \mid \forall i \in \mathbb{R}^k_+, \forall x_i \in \mathbb{R}^k_+, \sum_{i \in S} x_i = \sum_{i \in S} a_i \right\}$ to $\mathbb{R}$. Moreover, T is compact (it is closed and bounded). Thanks to the extreme value theorem from calculus, we conclude that $\sum_{i \in S} f_i(x_i)$ attains a maximum.

For a coalition $S \subseteq N$, let $x^S = \langle x_1^S, \ldots, x_n^S \rangle$ be the endowment that achieves the maximum value for the coalition S, i.e., $v(S) = \sum_{i \in S} f_i(x_i^S)$. In other words, the members of S have made some trades that have improved the value of the coalition S up to its maximal value.

Let λ be a balanced map. Let $y \in \mathbb{R}_+^n$ defined as follows: $y_i = \sum_{S \in \mathcal{C}_i} \lambda_S x_i^S$ where $\mathcal{C}_i$ is the set of coalitions that contains agent i.

First, note that y is a feasible payoff function.

$$\begin{aligned}\sum_{i \in N} y_i &= \sum_{i \in N} \sum_{S \in \mathcal{C}_i} \lambda_S x_i^S = \sum_{S \subseteq N} \sum_{i \in S} \lambda_S x_i^S = \sum_{S \subseteq N} \lambda_S \sum_{i \in S} x_i^S \\ &= \sum_{S \subseteq N} \lambda_S \sum_{i \in S} a_i \text{ since } x_i^S \text{ was achieved by a sequence of trades within the members of } S \\ &= \sum_{i \in N} a_i \sum_{S \in \mathcal{C}_i} \lambda_S \\ &= \sum_{i \in N} a_i \text{ as } \lambda \text{ is balanced, } \left(\begin{array}{l} \text{i.e., the sum of the weights over all coalitions} \\ \text{of one agent sums up to 1} \end{array} \right)\end{aligned}$$

Then, by definition of v, we have $v(N) \geq \sum_{i \in N} f_i(y_i)$. ✔

The f_i are concave and since $\sum_{S \in \mathcal{C}_i} \lambda_S = 1$, we have

$$f_i(\sum_{S \in \mathcal{C}_i} \lambda_S x_i^S) \geq \sum_{S \in C_i} \lambda_S f_i(x_i^S).$$

It follows:

$$v(N) \geq \sum_{i \in N} f_i(y_i) \geq \sum_{i \in N} f_i(\sum_{S \in \mathcal{C}_i} \lambda_S x_i^S) \geq \sum_{i \in N} \sum_{S \in \mathcal{C}_i} \lambda_S f_i(x_i^S) \geq \sum_{S \subseteq N} \lambda_S \sum_{i \in S} f_i(x_i^S) \geq \sum_{S \subseteq N} \lambda_S v(S).$$

This inequality proves that the game is balanced. ✔ □

2.4 Extension of the core

There are few extensions to the concept of the Core. As discussed above, one main issue of the Core is that it can be empty. In particular, a member of a coalition may block the formation so as to gain a very small payoff. When the cost of building a coalition is considered, it can be argued that it is not worth blocking a coalition for a small utility gain. The strong and weak ϵ-Core concepts model this possibility. The constraints defining the strong (respectively the weak) ϵ-Core become $\forall T \subseteq N, x(T) \geq v(T) - \epsilon$, (respectively $\forall T \subseteq N, x(T) \geq v(T) - |T| \cdot \epsilon$). In the weak Core, the minimum amount of utility required to block a coalition is per player, whereas for the strong Core, it is a fixed amount. If one picks ϵ large enough, the strong or weak ϵ-core will exist. When decreasing the value of ϵ, there will be a threshold $\epsilon^\star$ such that for $\epsilon < \epsilon^\star$ the ϵ core ceases to be non-empty. This special ϵ-core is then called the the *least core*.

2.5 Games with Coalition Structure

Thus far, we stated that the grand coalition is formed. With this definition, checking whether the core is empty amounts to checking whether the grand coalition is stable. In many studies in economics, the superadditivity of the valuation function is not explicitly stated, but it is implicitly assumed and hence, it makes sense to consider only the grand coalition. But when the valuation function is not superadditive, agents may have an incentive to form a different partition.

We recall that a coalition structure (CS) is a partition of the grand coalitions. If $\mathcal{S}$ is a CS, then $\mathcal{S} = \{\mathcal{C}_1, \ldots, \mathcal{C}_m\}$ where each $\mathcal{C}_i$ is a coalition such that $\cup_{i=1}^m \mathcal{C}_i = N$ and $i \neq j \Rightarrow \mathcal{C}_i \cap \mathcal{C}_j = \emptyset$.

Aumann and Drèze discuss why the coalition formation process may generate a CS that is not the grand coalition [1]. One reason they mention is that the valuation may not be superadditive (and they provide some discussion about why it may be the case). Another reason is that a CS may "reflect considerations that are excluded from the formal description of the game by necessity (impossibility to measure or communicate) or by choice" [1]. For example, the affinities can be based on location, or trust relations, etc.

2.5.1. DEFINITION. [Game with coalition structure] A *game with coalition structure* is a triplet (N, v, S), where (N, v) is a TU game, and S is a particular CS. In addition, transfer of utility is only permitted within (not between) the coalitions of S, i.e., $\forall \mathcal{C} \in S, x(\mathcal{C}) \leq v(\mathcal{C})$.

Another way to understand this definition is to consider that the problems of deciding which coalition forms and how to share the coalition's payoff are decoupled: the choice of the coalition is made first and results in the CS. Only the payoff distribution choice is left open. The agents are allowed to refer to the value of coalition with agents oustide of their coalition (i.e., opportunities they would get outside of their coalition) to negotiate a better payoff. Aumann and Drèze use an example of researchers in game theory that want to work in their own country, i.e., they want to belong to the coalition of game theorists of their country. They can refer to offers from foreign countries in order to negotiate their salaries. Note that the agents' goal is not to change the CS, but only to negotiate a better payoff for themselves.

First, we need to define the set of possible payoffs: the payoff distributions such that the sum of the payoff of the members of a coalition in the CS does not exceed the value of that coalition. More formally:

2.5.2. DEFINITION. [Feasible payoff] Let $(N, v, \mathcal{S})$ be a TU game with CS. The set of *feasible payoff distributions* is $X_{(N,v,\mathcal{S})} = \{x \in \mathbb{R}^n \mid \forall \mathcal{C} \in \mathcal{S} x(\mathcal{C}) \leq v(\mathcal{C})\}$.

A payoff distribution x is *efficient* with respect to a CS $\mathcal{S}$ when $\forall \mathcal{C} \in \mathcal{S}$, $\sum_{i\in\mathcal{C}} x_j = v(\mathcal{C})$. A payoff distribution is an *imputation* when it is efficient (with respect to the current CS) and individually rational (i.e., $\forall i \in N$, $x_i \geq v(\{i\})$). The set of all imputations for a CS $\mathcal{S}$ is denoted by $\mathcal{I}mp(\mathcal{S})$. We can now state the definition of the core:

2.5.3. DEFINITION. [Core] The core of a game $(N, v, \mathcal{S})$ is the set of all PCs $(\mathcal{S}, x)$ such that $x \in \mathcal{I}mp(\mathcal{S})$ and $\forall \mathcal{C} \subseteq N$, $\sum_{i\in\mathcal{C}} x_j \geq v(\mathcal{C})$, i.e.,

$$core(N, v, \mathcal{S}) = \{x \in \mathbb{R}^n \mid (\forall \mathcal{C} \in \mathcal{S}, x(\mathcal{C}) \leq v(\mathcal{C})) \wedge (\forall \mathcal{C} \subseteq N, x(\mathcal{C}) \geq v(\mathcal{C}))\}.$$

We now provide a theorem by Aumann and Drèze which shows that the core satisfies a desirable properties: if two agents can be substituted, then a core allocation must provide them identical payoffs.

2.5.4. DEFINITION. [Substitutes] Let (N, v) be a game and $(i, j) \in N^2$. Agents i and j are *substitutes* iff $\forall \mathcal{C} \subseteq N \setminus \{i, j\}$, $v(\mathcal{C} \cup \{i\}) = v(\mathcal{C} \cup \{j\})$.

Since the agents have the same impact on all coalitions that do not include them, it would be fair if they obtained the same payoff. For the core of a game in CS, this is indeed the case.

2.5.5. THEOREM. *Let $(N, v, \mathcal{S})$ be a game with coalition structure, let i and j be substitutes, and let $x \in core(N, v, \mathcal{S})$. If i and j belong to different members of $\mathcal{S}$, then $x_i = x_j$.*

Proof. Let $(i, j) \in N^2$ be substitutes, $\mathcal{C} \in \mathcal{S}$ such that $i \in \mathcal{C}$ and $j \notin \mathcal{C}$. Let $x \in Core(N, v, \mathcal{S})$. Since i and j are substitutes, we have

$$v((\mathcal{C} \setminus \{i\}) \cup \{j\}) = v((\mathcal{C} \setminus \{i\}) \cup \{i\}) = v(\mathcal{C}).$$

Since $x \in Core(N, v, \mathcal{S})$, we have $\forall \mathcal{C} \subseteq N$, $x(\mathcal{C}) \geq v(\mathcal{C})$, we apply this to the coalition $(\mathcal{C} \setminus \{i\}) \cup \{j\}$:
$0 \geq v((\mathcal{C} \setminus \{i\}) \cup \{j\}) - x((\mathcal{C} \setminus \{i\}) \cup \{j\}) = v(\mathcal{C}) - x(\mathcal{C}) + x_i - x_j$. Since $\mathcal{C} \in \mathcal{S}$ and $x \in Core(N, v, \mathcal{S})$, we have $x(\mathcal{C}) = v(\mathcal{C})$. We can then simplified the previous expression and we obtain $x_j \geq x_i$.

Since i and j play symmetric roles, we have also $x_i \geq x_j$ and finally, we obtain $x_i = x_j$. ✔ □

Aumann and Drèze made a link from a game with CS to a special superadditive game $(N, \hat{v})$ called the superadditive cover [1].

2.5.6. DEFINITION. [Superadditive cover] The *superadditive cover* of (N, v) is the game $(N, \hat{v})$ defined by

$$\begin{cases} \hat{v}(\mathcal{C}) = \max_{\mathcal{P} \in \mathscr{S}_\mathcal{C}} \left\{ \sum_{T \in \mathcal{P}} v(T) \right\} \forall \mathcal{C} \subseteq N \setminus \emptyset \\ \hat{v}(\emptyset) = 0 \end{cases}$$

In other words, $\hat{v}(\mathcal{C})$ is the maximal value that can be generated by any partition of $\mathcal{C}$[1]. The superadditive cover is a superadditive game. The following theorem, from [1] shows that a necessary condition for $(N, v, \mathcal{S})$ to have a non empty core is that $\mathcal{S}$ is an optimal CS.

2.5.7. THEOREM. *Let $(N, v, \mathcal{S})$ be a game with coalition structure. Then*

a) $Core(N, v, \mathcal{S}) \neq \emptyset$ *iff* $Core(N, \hat{v}) \neq \emptyset \wedge \hat{v}(N) = \sum_{\mathcal{C} \in \mathcal{S}} v(\mathcal{C})$

b) if $Core(N, v, \mathcal{S}) \neq \emptyset$*, then* $Core(N, v, \mathcal{S}) = Core(N, \hat{v})$

Proof. Proof of part **a)**

⇒ Let $x \in Core(N, v, \mathcal{S})$. We show that $x \in Core(N, \hat{v})$ as well. Let $\mathcal{C} \subseteq N \setminus \emptyset$ and $P_\mathcal{C} \in \mathscr{S}_\mathcal{C}$ be a partition of $\mathcal{C}$. By definition of the core, for every $S \subseteq N$ we have $x(S) \geq v(S)$. The payoff of coalition $\mathcal{C}$ is

$$x(\mathcal{C}) = \sum_{i \in \mathcal{C}} x_i = \sum_{S \in P_\mathcal{C}} x(S) \geq \sum_{S \in P_\mathcal{C}} v(S),$$

which is valid for all partitions of $\mathcal{C}$. Hence, $x(\mathcal{C}) \geq \max_{\mathcal{P}_\mathcal{C} \in \mathscr{S}_\mathcal{C}} \sum_{S \in \mathcal{P}_\mathcal{C}} v(S) = \hat{v}(\mathcal{C})$.

We have just proved $\forall \mathcal{C} \subseteq N \setminus \emptyset$, $x(\mathcal{C}) \geq \hat{v}(\mathcal{C})$, and so x is *group rational*. ✔

We now need to prove that $\hat{v}(N) = \sum_{\mathcal{C} \in \mathcal{S}} v(\mathcal{C})$.

$x(N) = \sum_{\mathcal{C} \in \mathcal{S}} v(\mathcal{C})$ since x is in the core of $(N, v, \mathcal{S})$ (efficient). Applying the inequality above, we have $x(N) = \sum_{\mathcal{C} \in \mathcal{S}} v(\mathcal{C}) \geq \hat{v}(N)$.
Applying the definition of the valuation function $\hat{v}$, we have $\hat{v}(N) \geq \sum_{\mathcal{C} \in \mathcal{S}} v(\mathcal{C})$.
Consequently, $\hat{v}(N) = \sum_{\mathcal{C} \in \mathcal{S}} v(\mathcal{C})$ and it follows that x is *efficient* for the game $(N, \hat{v})$✔

Hence $x \in Core(N, \hat{v})$.

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[1] Note that for the grand coalition, we have $\hat{v}(N) = \max_{\mathcal{P} \in \mathscr{S}_N} \left\{ \sum_{T \in \mathcal{P}} v(T) \right\}$, i.e., $\hat{v}(N)$ is the maximum value that can be produced by N. We call it the *value of the optimal coalition structure*. For some application, on issue (that will be studied later) is to find this value.

$\Leftarrow$ Let's assume $x \in Core(N, \hat{v})$ and $\hat{v}(N) = \sum_{\mathcal{C} \in \mathcal{S}} v(\mathcal{C})$. We need to prove that $x \in Core(N, v, \mathcal{S})$.

For every $\mathcal{C} \subseteq N$, $x(\mathcal{C}) \geq \hat{v}(\mathcal{C})$ since x is in the core of $Core(N, \hat{v})$. Then $x(\mathcal{C}) \geq \max_{\mathcal{P}_\mathcal{C} \in \mathscr{S}_\mathcal{C}} \sum_{S \in \mathcal{P}_\mathcal{C}} v(S) \geq v(\mathcal{C})$ using $\{\mathcal{C}\}$ as a partition of $\mathcal{C}$, which proves x is group rational. ✔

$x(N) = \hat{v}(N) = \sum_{\mathcal{C} \in \mathcal{S}} v(\mathcal{C})$ since x is efficient. It follows that $\forall \mathcal{C} \in \mathcal{S}$, we must have $x(\mathcal{C}) = v(\mathcal{C})$, which proves x is feasible for the CS $\mathcal{S}$, and that x is efficient.✔

Hence, $x \in Core(N, v, \mathcal{S})$. ✔

proof of part **b)**:

We have just proved that $x \in Core(N, \hat{v})$ implies that $x \in Core(N, v, \mathcal{S})$ and $x \in Core(N, v, \mathcal{S})$ implies that $x \in Core(N, \hat{v})$. This proves that if $Core(N, v, \mathcal{S}) \neq \emptyset$, $Core(N, \hat{v}) = Core(N, v, \mathcal{S})$.

□

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