Lecture 7

A Special Class of TU games: Voting Games

The formation of coalitions is usual in parliaments or assemblies. It is therefore interesting to consider a particular class of coalitional games that models voting in an assembly. For example, we can represent an election between two candidates as a voting game where the winning coalitions are the coalitions of size at least equal to the half the number of voters.

7.1 Definitions

We start by providing the definition of a voting game, which can be viewed as a special class of TU games. Then, we will formalize some known concepts used in voting. We will see how we can define what a dictator is,

7.1.1. DEFINITION. [voting game] A game (N, v) is a *voting game* when

- the valuation function takes only two values: 1 for the winning coalitions, 0 otherwise.
- v satisfies *unanimity*: $v(N) = 1$
- v satisfies *monotonicity*: $S \subseteq T \subseteq N \Rightarrow v(S) \leq v(T)$.

Unanimity and monotonicity are natural assumptions in most cases. Unanimity reflects the fact that all agents agree; hence, the coalition should be winning. Monotonicity tells that the addition of agents in the coalition cannot turn a winning coalition into a losing one, which is reasonable for voting: more supporters should not harm the coalition. A first way to represent a voting game is by listing all winning coalitions. Using the monotonicity property, a more succinct representation is to list only the *minimal winning coalitions*.

7.1.2. DEFINITION. [Minimal winning coalition] A coalition $\mathcal{C} \subseteq N$ is a minimal winning coalition iff $v(\mathcal{C}) = 1$ and $\forall i \in \mathcal{C}\ v(\mathcal{C} \setminus \{i\}) = 0$.

For example, we consider the game $(\{1,2,3,4\}, v)$ such that $v(\mathcal{C}) = 1$ when $|\mathcal{C}| \geq 3$ or ($|\mathcal{C}| = 2$ and $1 \in \mathcal{C}$) and $v(\mathcal{C}) = 0$ otherwise. The set of winning coalitions is $\{\{1,2\}, \{1,3\}, \{1,4\}, \{1,2,3\}, \{1,2,4\}, \{1,3,4\}, \{2,3,4\}, \{1,2,3,4\}\}$. We can represent the game more succinctly by just writing the set of minimal winning coalitions, which is $\{\{1,2\}, \{1,3\}, \{1,4\}, \{2,3,4\}\}$.

We can now see how we formalize some common terms in voting. We can first express what a dictator is.

7.1.3. DEFINITION. [Dictator] Let (N, v) be a simple game. A player $i \in N$ is a *dictator* iff $\{i\}$ is a winning coalition.

Note that with the requirements of simple games, it is possible to have more than one dictator! The next notion is the notion of veto player, in which a player can block a decision on its own by opposing to it (e.g. in the United Nations Security Council, China, France, Russia, the United Kingdom, and the United States are veto players).

7.1.4. DEFINITION. [Veto Player] Let (N, v) be a simple game. A player $i \in N$ is a *veto* player if $N \setminus \{i\}$ is a losing coalition. Alternatively, i is a *veto* player iff for all winning coalition $\mathcal{C}$, $i \in \mathcal{C}$.

It also follows that a veto player is member of every minimal winning coalitions. Another concept is the concept of a blocking coalition: it is a coalition that, on its own, cannot win, but the support of all its members is required to win. Put another way, the members of a blocking coalition do not have the power to win, but they have the power to lose.

7.1.5. DEFINITION. [blocking coalition] A coalition $\mathcal{C} \subseteq N$ is a *blocking coalition* iff $\mathcal{C}$ is a losing coalition and $\forall S \subseteq N \setminus \mathcal{C}$, $S \setminus \mathcal{C}$ is a losing coalition.

7.2 Stability

We can start by studying what it means to have a stable payoff distribution in these games. The following theorem characterizes the core of simple games.

7.2.1. THEOREM. *Let (N, v) be a simple game. Then*

$$Core(N, v) = \left\{ x \in \mathbb{R}^n \;\middle|\; x \textit{ is an imputation } x_i = 0 \textit{ for each non-veto player } i \right\}$$

Proof.

- $\subseteq$ Let $x \in Core(N, v)$. By definition $x(N) = 1$. Let i be a non-veto player. $x(N \setminus \{i\}) \geq v(N \setminus \{i\}) = 1$. Hence $x(N \setminus \{i\}) = 1$ and $x_i = 0$.

$\supseteq$ Let x be an imputation and $x_i = 0$ for every non-veto player i. Since $x(N) = 1$, the set V of veto players is non-empty and $x(V) = 1$.

Let $\mathcal{C} \subseteq N$. If $\mathcal{C}$ is a winning coalition then $V \subseteq \mathcal{C}$, hence $x(\mathcal{C}) \geq v(\mathcal{C})$. Otherwise, $v(\mathcal{C})$ is a losing coalition (which may contain veto players), and $x(\mathcal{C}) \geq v(\mathcal{C})$. Hence, x is group rational.

□

We can also study the class of simple convex games. The following theorem shows that they are the games with a single minimal winning coalition.

7.2.2. THEOREM. *A simple game (N, v) is convex iff it is a unanimity game (N, v_V) where V is the set of veto players.*

Proof. A game is convex iff $\forall S, T \subseteq N \; v(S) + v(T) \leq v(S \cap T) + v(S \cup T)$.

$\Rightarrow$ Let us assume (N, v) is convex.

If S and T are winning coalitions, $S \cup T$ is a winning coalition by monotonicity. Then, we have $2 \leq 1 + v(S \cap T)$ and it follows that $v(S \cap T) = 1$. The intersection of two winning coalitions is a winning coalition. Moreover, from the definition of veto players, the intersection of all winning coalitions is the set V of veto players. Hence, $v(V) = 1$. By monotonicity, if $V \subseteq \mathcal{C}$, $v(\mathcal{C}) = 1$. Otherwise, $V \nsubseteq \mathcal{C}$. Then there must be a veto player $i \notin \mathcal{C}$, and it must be the case that $v(\mathcal{C}) = 0$. Hence, for all coalition $\mathcal{C} \subseteq N$, $v(\mathcal{C}) = 1$ iff $V \subseteq \mathcal{C}$.

$\Leftarrow$ Let (N, v_V) a unanimity game. Let us prove it is a convex game. Let $S \subseteq N$ and $T \subseteq N$, and we want to prove that $v(S) + v(T) \leq v(S \cup T) + v(S \cap T)$.

- case $V \subseteq S \cap T$: Then $V \subseteq S$ and $V \subseteq T$, and we have $2 \leq 2$
- case $V \nsubseteq S \cap T \;\wedge\; V \subseteq S \cup T$:
  * if $V \subseteq S$ then $V \nsubseteq T$ and $1 \leq 1$
  * if $V \subseteq T$ then $V \nsubseteq S$ and $1 \leq 1$
  * otherwise $V \nsubseteq S$ and $V \nsubseteq T$, and then $0 \leq 1$
- case $V \nsubseteq S \cup T$: then $0 \leq 0$

For all cases, $v(S) + v(T) \leq v(S \cup T) + v(S \cap T)$, hence a unanimity game is convex. In addition, all members of V are veto players.

□

7.3 Weighted voting games

We now define a class of voting games that has a more succinct representation: each agent has a weight and a coalition needs to achieve a threshold (i.e. a quota) to be winning. This is a much more compact representation as we only use to define a vector of weights and a threshold. The formal definition follows.

7.3.1. DEFINITION. [weighted voting game] A game (N, v, q, w) is a *weighted voting game* when

- $w = (w_1, w_2 \ldots, w_n) \in \mathbb{R}^n_+$ is a vector of weights, one for each voter
- A coalition $\mathcal{C}$ is winning (i.e., $(v(\mathcal{C}) = 1)$ iff $\sum_{i\in\mathcal{C}} w_i \geq q$, it is losing otherwise (i.e., $(v(\mathcal{C}) = 0)$
- v satisfies monotonicity: $\sum_{i\in N} w_i \geq q$

The fact that each agent has a positive (or zero) weight ensures that the game is monotone. We will note a weighted voting game $(N, w_{i\in N}, q)$ as $[q;\ w_1, \ldots, w_n]$. In its early days, the European Union was using a weighted voted games. Now a combination of weighted voting games are used (a decision is accepted when it is supported by 55% of Member States, including at least fifteen of them, representing at the same time at least 65% of the Union's population).

Weighted voting games is a succinct representation of a simple game. However, not all the simple games can be represented by a weighted voting game. We say that the representation is not *complete*. For example, consider the voting game $(\{1, 2, 3, 4\}, v)$ such that the set of minimal winning coalitions is $\{\{1, 2\}, \{3, 4\}\}$. Let us assume we can represent (N, v) with a weighted voting game $[q;\ w_1, w_2, w_3, w_4]$. We can form the following inequalities:

$v(\{1, 2\}) = 1$ then $w_1 + w_2 \geq q$
$v(\{3, 4\}) = 1$ then $w_3 + w_4 \geq q$
$v(\{1, 3\}) = 0$ then $w_1 + w_3 < q$
$v(\{2, 4\}) = 0$ then $w_2 + w_4 < q$

But then, $w_1 + w_2 + w_3 + w_4 < 2q$ and $w_1 + w_2 + w_3 + w_4 \geq 2q$, which is impossible. Hence, (N, v) cannot be represented by a weighted voting game.

Not all simple games can be represented by a weighted voting game. However, many weighted voting games represent the same simple game: two weigthed voting games may have different quotas and weights, but they may have exactly the same winning coalitions. Two weighted voting games $G = [q, w_1, ..., w_n]$ and $G' = [q', w'_1, ..., w'_n]$ are said to be *equivalent* when $\forall\mathcal{C} \subseteq N$, $w(C) \geq q$ iff $w'(C) \geq q'$. The definition of weighted voting games allows to choose the weights and the quota as a real number. From a computational point of view, storing and manipulating real number is challenging. However, one do not need to use real numbers. The following result shows that any weighted voting game is equivalent to a weighted voting game with small integer weights and quota.

7.3.2. THEOREM. *For any weighted voting game G, there exists an equivalent weighted voting game* $[q, w_1, ..., w_n]$ *with* $q \in \mathbb{N}$ *and* $\forall i \in N$ $w_i \in \mathbb{N}$ *and* $w_i = O(2^{nlogn})$.

Without loss of generality, we can now study weighted voting games with only integer weights and integer quota, which allows us to represent a weighted voting game with a polynomial number of bits.

We now turn to the question about the meaning of the weight. One intuition may be that the weight represents the importance or the strength of a player. Let us consider some examples to check this intuition.

- $[10;\ 7, 4, 3, 3, 1]$: The set of minimal winning coalitions is $\{\{1, 2\}\{1, 3\}\{1, 4\}\{2, 3, 4\}\}$. Player 5, although it has some weight, is a dummy. Player 2 has a higher weight than player 3 and 4, but it is clear that player 2, 3 and 4 have the same influence.
- $[51;\ 49, 49, 2]$: The set of winning coalition is $\{\{1, 2\}, \{1, 3\}, \{2, 3\}\}$. It seems that the players have symmetric roles, but it is not reflected in their weights.

These examples shows that the weights can be deceptive and may not represent the voting power of a player. Hence, we need different tools to measure the voting power of the voters, which is the goal of the following section.

7.4 Power Indices

The examples raise the subject of measuring the voting power of the agents in a voting game. Multiple indices have been proposed to answer these questions. In the following, we introduce few of them, and we will discuss some weaknesses (some paradoxical situations may occur). Finally, we briefly describe some applications.

7.4.1 Definitions

One central notion to define the power of a voter is the notion of being a *Swing or Pivotal Voter*. Informally, when a coalition $\mathcal{C}$ is losing, a pivotal voter for that coalition is a voter that makes the coalition $\mathcal{C} \cup \{i\}$ win. The presence of the members of $\mathcal{C}$ is not sufficient to win the election, but with the presence of i, $\mathcal{C} \cup \{i\}$ wins and i can be seen as an important voter.

7.4.1. DEFINITION. [Swing or Pivotal Voter] A voter i is *pivotal* or *swing* for a coalition $\mathcal{C}$ when i turns the coalition from a losing to a wining one, i.e., $v(\mathcal{C}) = 0$ and $v(\mathcal{C} \cup \{i\}) = 1$.

In the following, w is the number of winning coalitions and for a voter i, η_i is the number of coalitions for which i is pivotal, i.e., $\eta_i = \sum_{S \subseteq N \setminus \{i\}} v(S \cup \{i\}) - v(S)$. We are now ready to define some power indices.

Shapley-Shubik index: it is the Shapley value of the voting game, its interpretation in this context is the percentage of the permutations of all players in which i is pivotal.

$$I_{SS}(N, v, i) = \sum_{\mathcal{C} \subseteq N \setminus \{i\}} \frac{|\mathcal{C}|!(n - |C| - 1)!}{n!} \left(v(\mathcal{C} \cup \{i\}) - v(\mathcal{C}) \right).$$

"For each permutation, the pivotal player gets one more point.". One issue is that the voters do not trade the value of the coalition, though the decision that the voters vote about is likely to affect the entire population.

Banzhaff index: For each coalition, we determine which agent is a swing agent (more than one agent may be pivotal). The *raw Banzhaff index* of a player i is

$$\beta_i = \frac{\sum_{\mathcal{C} \subseteq N \setminus \{i\}} v(\mathcal{C} \cup \{i\}) - v(\mathcal{C})}{2^{n-1}}.$$

The interpretation is that the Banzhaff index is the percentage of coalitions for which a player is pivotal. The raw Banzhaff index does not necessarily sum up to one. However, for a simple game (N, v), $v(N) = 1$ and $v(\emptyset) = 0$, at least one player i has a power index $\beta_i \neq 0$. Hence, $B = \sum_{j \in N} \beta_j > 0$. The *normalized Banzhaff index* of player i for a simple game (N, v) is defined as

$$I_B(N, v, i) = \frac{\beta_i}{B}.$$

Coleman index: Coleman defines three indices [1]: the power of the collectivity to act $A = \frac{w}{2^n}$ (A is the probability of a winning vote occurring); the power to prevent action $P_i = \frac{\eta_i}{w}$ (it is the ability of a voter to change the outcome from winning to losing by changing its vote); the power to initiate action $I_i = \frac{\eta_i}{2^n - w}$(it is the ability of a voter to change the outcome from losing to winning by changing its vote, the numerator is the same as in P, but the denominator is the number of losing coalitions, i.e., the complement of the one of P)

We provide in Table 7.1 an example of computation of the Shapley-Schubik and Banzhaff indices. This example shows that both indices may be different. There is a slight difference in the probability model between the Banzhaf β_i and Coleman's index P_i: in Banzhaf's, all the voters but i vote randomly whereas in Coleman's, the assumption of random voting also applies to the voter i. Hence, the Banzhaf index can be written as $\beta_i = 2P_i \cdot A = 2I_i \cdot (1 - A)$.

$\{1, \underline{2}, 3, 4\}$
$\{1, \underline{2}, 4, 3\}$
$\{1, 3, \underline{2}, 4\}$
$\{1, 3, \underline{4}, 2\}$
$\{1, 4, \underline{2}, 3\}$
$\{1, 4, \underline{3}, 2\}$
$\{2, \underline{1}, 3, 4\}$
$\{2, \underline{1}, 4, 3\}$
$\{2, 3, \underline{1}, 4\}$
$\{2, 3, 4, \underline{1}\}$
$\{2, 4, \underline{1}, 3\}$
$\{2, 4, 3, \underline{1}\}$

$\{3, 1, \underline{2}, 4\}$
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$\{3, 2, \underline{1}, 4\}$
$\{3, 2, 4, \underline{1}\}$
$\{3, 4, \underline{1}, 2\}$
$\{3, 4, 2, \underline{1}\}$
$\{4, 1, \underline{2}, 3\}$
$\{4, 1, \underline{3}, 2\}$
$\{4, 2, \underline{1}, 3\}$
$\{4, 2, 3, \underline{1}\}$
$\{4, 3, \underline{1}, 2\}$
$\{4, 3, 2, \underline{1}\}$

In red and underlined, the pivotal agent

	1	2	3	4
Sh	$\frac{7}{12}$	$\frac{1}{4}$	$\frac{1}{12}$	$\frac{1}{12}$

winning coalitions:
$\{\underline{1}, \underline{2}\}$
$\{\underline{1}, \underline{2}, 3\}$
$\{\underline{1}, \underline{2}, 4\}$
$\{\underline{1}, \underline{3}, \underline{4}\}$
$\{\underline{1}, 2, 3, 4\}$
In red and underlined, the pivotal agents

	1	2	3	4
β	$\frac{5}{8}$	$\frac{3}{8}$	$\frac{1}{8}$	$\frac{1}{8}$
$I_B(N, v, i)$	$\frac{1}{2}$	$\frac{3}{10}$	$\frac{1}{10}$	$\frac{1}{10}$

Table 7.1: Shapley-Schubik and the Banzhaff indices for the weighted voting game $[7;\ 4, 3, 2, 1]$.

7.4.2 Paradoxes

The power indices may behave in an unexpected way if we modify the game. For example, we might expect that adding voters to a game would reduce the power of those voters that are present in the original game, but this may not be the case.

Consider the game $[4; 2, 2, 1]$. Player 3 is a dummy in this game, so her Shapley Shubik or Banzhaff indices are zero. Now assume that a voter joins the game with a weight of 1. In the resulting game G', player 3 becomes pivotal for a coalition consisting of one of the two voters of the original game and the new player. Hence, her index must now be positive. This situation is known as the paradox of new player.

Another unexpected behaviour may occur when a voter i splits her identity and weight between two voters. The sum of the new identities' Shapley value may be quite different from the Shapley value of voter i. This situation is known as the paradox of size.

- *increase of power by splitting identities* Consider a game with $|N| = n$ voters $[n+1; 2, 1, \dots, 1]$. In this game, the only winning coalition is the grand coalition, so $I_{SS}(N, v, i) = \frac{1}{n}$. Now suppose that voter 1 splits into two voters of weight one. We have a new game game with n+1 voters $[n + 1; 1, \dots, 1]$. Using a similar argument, the Shapley Shubik index for each voter is $\frac{1}{n+1}$. Hence, the joint power of the new identities is $\frac{2}{n+1}$, almost twice the power of agent 1 by herself!

- *decrease of power by splitting identities* Consider an n-voter voting game in which all voters have a weight of 2 and the quota is $2n-1$, i.e., we have the game $[2n-1; 2, \ldots, 2]$. All the players being symmetric, the Shapley value is $\frac{1}{n}$. If player 1 splits into two voters of weight 1, each of her identities has a Shapley value of $\frac{1}{n(n+1)}$ in the new game. Hence, the sum of the Shapley values of the two identities is smaller than the value in the original game, by a factor of $\frac{n+1}{2}$.

7.4.3 Applications

When designing a weighted voting game, for example to decide on the weights for a vote for the European Union or at the United Nations, one needs to choose which weights are to be attributed to each nation. The problem of choosing the weights so that they corresponds to a given power index has been tackled in [2]. If the number of country changes, you do not want to re-design and negotiate over a new game each time. Each citizen vote for a representative and the representatives for each country vote. It may be desirable that each citizen, irrespective of her/his nationality, has the same voting power. If β_x is the normalized Banzhaf index for a person in a country i in EU with population n_i, and β_i is the normalized Banzhaf index of a representative for country i, then Felsenthal and Machover have shown that $\beta_x \propto \beta_i \sqrt{\frac{2}{\pi n_i}}$. Thus the Banzhaf index of each representative β_i should be proportional to n_i for each person in the EU to have equal power.

7.4.4 Complexity

The computational complexity of voting and weighted voting games have been studied in [3, 4]. For example, the problem of determining whether the core is empty is polynomial. The argument for this result is the following theorem: the core of a weighted voting game is non-empty iff there exists a veto player. When the core is non-empty, the problem of computing the nucleolus is also polynomial, otherwise, it is an $\mathcal{NP}$-hard problem.

Bibliography

[1] James S. Coleman. The benefits of coalition. *Public Choice*, 8:45–61, 1970.

[2] Bart de Keijzer, Tomas Klos, and Yingqian Zhang. Enumeration and exact design of weighted voting games. In *Proc. of the 9th Int. Conf. on Autonomous Agents and Multiagent Systems (AAMAS-2010)*, pages 391–398, 2010.

[3] Xiaotie Deng and C H Papadimitriou. On the complexity of cooperative solution concetps. *Mathematical Operation Research*, 19(2):257–266, 1994.

[4] Edith Elkind, Leslie Ann Goldberg, Paul Goldberg, and Michael Wooldridge. Computational complexity of weighted threshold games. In *Proceedings of the Twenty-Second AAAI Conference on Artificial Intelligence (AAAI-07)*, pages 718–723, 2007.