Cooperative Games

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Why study coalitional games ?

Cooperative games are a branch of game theory that models **cooperation** or collaboration between agents.

Coalitional games can also be studied from a computational point of view (e.g., the problem of succint representation).

A coalition may represent a set of :

- persons or group of persons (labor unions, towns)
- objectives of an economic project
- artificial agents

We have a population N of n agents.

Definition (Coalition)

A **coalition** $\mathcal{C}$ is a set of agents : $\mathcal{C} \in 2^N$.

The classic problem

- N is the set of all agents (or players)
- $v : 2^N \to \mathbb{R}$ is the **valuation function**. For $\mathcal{C} \subseteq N$, $v(\mathcal{C})$ is the value obtained by the coalition $\mathcal{C}$

problem : a game (N, v), and we assume all agents in N want to cooperate.

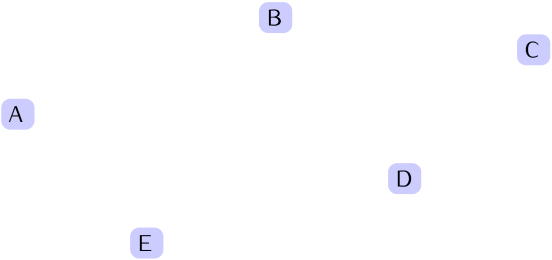
solution : a **payoff distribution** $x \in \mathbb{R}^n$ that provides a value to individual agents.

What are the interesting **properties** that x should satisfy ?

How to **determine** the payoff vector x ?

Example

Sharing a water supply system : n towns consider building a common water treatment facility. The cost of a coalition is the minimum cost of supplying the coalition members by the most efficient means.



no coalition, no facility

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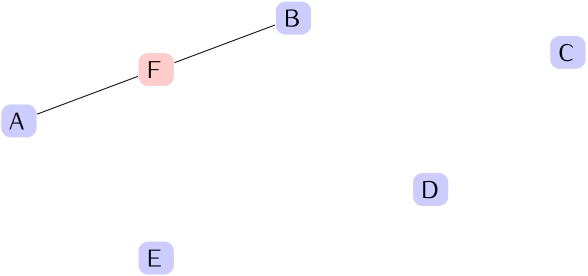

$\{A, B\}$ forms

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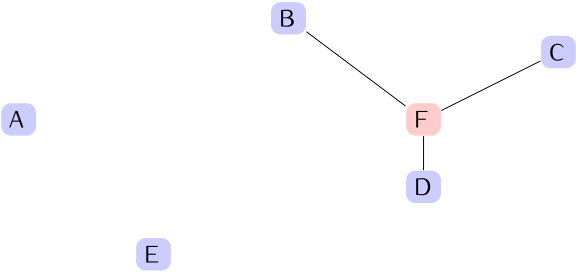

$\{B, C, D\}$ forms

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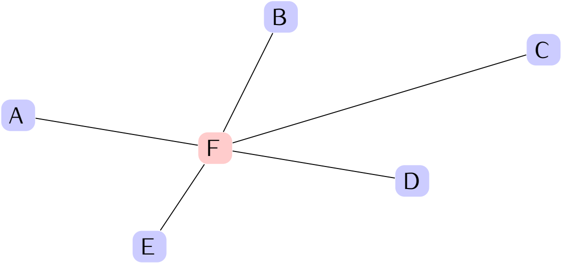

$\{A, B, C, D, E\}$ forms

Today

- Two out of many solutions :
  - One solution that focuses on stability
  - One solution that focuses on fairness
- Application to voting power
- Some issues with representation

The main problem

In the game (N, v) we want to form the **grand coalition**.

Each agent i will get a **personal payoff** x_i.

What are the interesting **properties** that x should satisfy ?

How to **determine** the payoff vector x ?

problem : a game (N, v) in which v is a worth of a coalition
solution : a payoff vector $x \in \mathbb{R}^n$

Some properties

Let x, y be two solutions of a TU-game (N, v).

Efficiency : $x(N) = v(N)$

- ➪ the payoff distribution is an allocation of the entire worth of the grand coalition to all agents.

Individual rationality : $\forall i \in N, x(i) \geqslant v(\{i\})$

- ➪ agent obtains at least its self-value as payoff.

Group rationality : $\forall \mathcal{C} \subseteq N, \sum_{i \in \mathcal{C}} x(i) = v(\mathcal{C})$

- ➪ if $\sum_{i \in \mathcal{C}} x(i) < v(\mathcal{C})$ some utility is lost.
- ➪ if $\sum_{i \in \mathcal{C}} x(i) > v(\mathcal{C})$ is not possible.

The core

D Gillies, **Some theorems in n-person games**. *PhD thesis, Department of Mathematics, Princeton, N.J.*, 1953.

- A condition for a coalition to form :

  all participants prefer to be in it.

  i.e., none of the participants wishes she were in a different coalition or by herself ⇒ **Stability**.
- Stability is a necessary but not sufficient condition, (e.g., there may be multiple stable coalitions).
- The **core** is a stability concepts where no agents prefer to deviate to form a different coalition.
- For simplicity, we will only consider the problem of the stability of the grand coalition :

⇒ Is the grand coalition stable ? ⇔ Is the core non-empty ?

Example of bargaining

$$N = \{1, 2, 3\}$$
$$v(\{1\}) = 0,\ v(\{2\}) = 0,\ v(\{3\}) = 0$$
$$v(\{1, 2\}) = 90$$
$$v(\{1, 3\}) = 80$$
$$v(\{2, 3\}) = 70$$
$$v(\{1, 2, 3\}) = 105$$

What should we do ?

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What should we do ?

- form $\{1,2,3\}$ and share equally $\langle 35,35,35\rangle$?

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What should we do ?

- form $\{1,2,3\}$ and share equally $\langle 35, 35, 35\rangle$?
- 3 can say to 1 "let's form $\{1,3\}$ and share $\langle 40, 0, 40\rangle$".

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- form $\{1,2,3\}$ and share equally $\langle 35,35,35\rangle$?
- 3 can say to 1 “let's form $\{1,3\}$ and share $\langle 40,0,40\rangle$”.
- 2 can say to 1 “let's form $\{1,2\}$ and share $\langle 45,45,0\rangle$”.

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- 2 can say to 1 "let's form $\{1,2\}$ and share $\langle 45,45,0\rangle$".
- 3 can say to 2 "OK, let's form $\{2,3\}$ and share $\langle 0,46,24\rangle$".

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- 1 can say to 2 and 3, "fine ! Let it be $\{1,2,3\}$ with $\langle 33,47,25 \rangle$"

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- 1 can say to 2 and 3, “fine ! Let it be $\{1,2,3\}$ with $\langle 33,47,25\rangle$”
- ... is there a “good” final solution ?

The core relates to the stability of the grand coalition :
No group of agents has any incentive to change coalition.

Definition (*core* of a game (N, v))

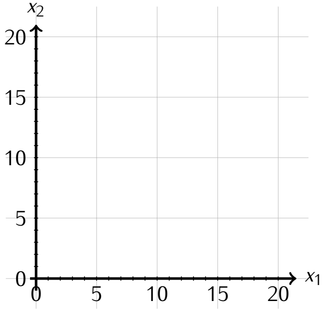
Let (N, v) be a TU game, and assume we form the grand coalition N. The **core** of (N, v) is the set :

$$Core(N, v) = \{x \in \mathbb{R}^n \mid x(N) \leqslant v(N) \wedge x(\mathcal{C}) \geqslant v(\mathcal{C}) \; \forall \mathcal{C} \subseteq N\}$$

Example : a two-player game

$$N = \{1,2\}$$
$$v(\{1\}) = 5,\ v(\{2\}) = 5$$
$$v(\{1,2\}) = 20$$

$$core(N,v) = \{(x_1,x_2) \in \mathbb{R}^2 \mid x_1 \geqslant 5,\ x_2 \geqslant 5,\ x_1 + x_2 = 20\}$$



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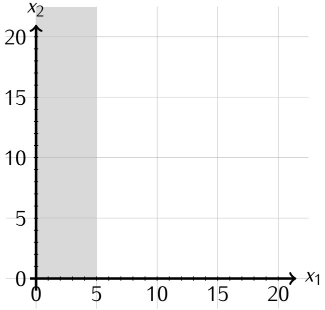

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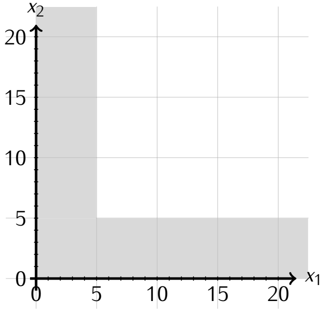

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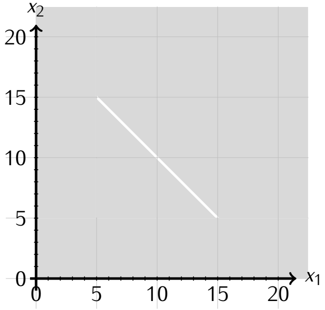

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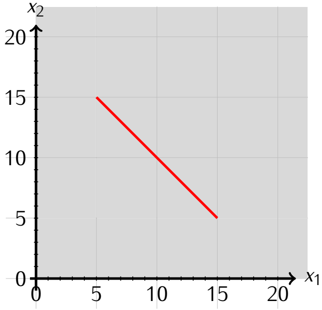

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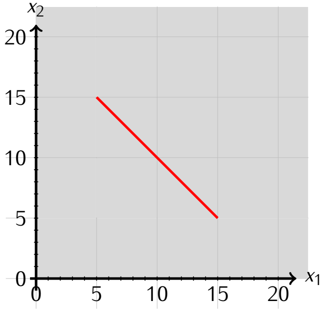

The core may not be fair : the core only considers stability.

Example : a two-player game

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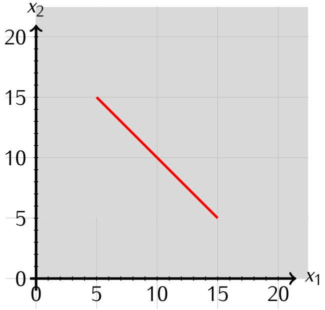

if $v(\{1\}) = 11$, $v(\{2\}) = 11$, and $v(\{1,2\}) = 20$, the core becomes **empty** !

Definition (Convex games)

A game (N,v) is **convex** iff
$\forall \mathcal{C} \subseteq \mathcal{T}$ and $i \notin \mathcal{T}$, $v(\mathcal{C} \cup \{i\}) - v(\mathcal{C}) \leqslant v(\mathcal{T} \cup \{i\}) - v(\mathcal{T})$.

TU-game is convex if the marginal contribution of each player increases with the size of the coalition he joins.

Theorem

A TU game (N,v) is convex iff for all coalition S and T $v(S) + v(T) \leqslant v(S \cup T) + v(S \cap T)$.

Theorem

A convex game has a non-empty core.

We consider the following **linear programming** problem :

$$(LP) \begin{cases} \min x(N) \\ \text{subject to } x(\mathcal{C}) \geqslant v(\mathcal{C}) \text{ for all } \mathcal{C} \subseteq N,\ S \neq \emptyset \end{cases}$$

(N, v) has a non-empty core iff the value of (LP) is $v(N)$.

Theorem (Bondareva Shapley)

A TU game has a non-empty core iff it is balanced.

Issues with the core

- The core may not always be non-empty.
- When the core is not empty, it may not be 'fair'.
- It may not be easy to compute.
- There are classes of games that have a non-empty core.
- It is possible to characterize the games with non-empty core.

The Shapley value

Lloyd S. Shapley. **A Value for *n*-person Games**. In *Contributions to the Theory of Games, volume II (Annals of Mathematical Studies)*, 1953.

Definition (marginal contribution)

The **marginal contribution** of agent i for a coalition $\mathcal{C} \subseteq N \setminus \{i\}$ is $mc_i(\mathcal{C}) = v(\mathcal{C} \cup \{i\}) - v(\mathcal{C})$.

$\langle mc_1(\emptyset), mc_2(\{1\}), mc_3(\{1,2\})\rangle$ is an efficient payoff distribution for any game $(\{1,2,3\}, v)$. This payoff distribution may model a dynamic process in which 1 starts a coalition, is joined by 2, and finally 3 joins the coalition $\{1,2\}$, and where the incoming agent gets its marginal contribution.

An agent's payoff depends on which agents are already in the coalition. This payoff may not be **fair**. To increase fairness, one could take the average marginal contribution over all possible joining orders.

Let σ represent a joining order of the grand coalition N, i.e., σ is a permutation of $\langle 1, \ldots, n\rangle$.
We write $mc(\sigma) \in \mathbb{R}^n$ the payoff vector where agent i obtains $mc_i(\{\sigma(j) \mid j < i\})$.
The vector mc is called a **marginal vector**.

Shapley value : version based on marginal contributions

Let (N, v) be a TU game. Let $\Pi(N)$ denote the set of all permutations of the sequence $\langle 1, \ldots, n \rangle$.

$$Sh(N, v) = \frac{\sum_{\sigma \in \Pi(N)} mc(\sigma)}{n!}$$

the Shapley value is a **fair** payoff distribution based on marginal contributions of agents averaged over joining orders of the coalition.

An example

$$N = \{1,2,3\},\ v(\{1\}) = 0,\ v(\{2\}) = 0,\ v(\{3\}) = 0,$$
$$v(\{1,2\}) = 90,\ v(\{1,3\}) = 80,\ v(\{2,3\}) = 70,$$
$$v(\{1,2,3\}) = 120.$$

	1	2	3
$1 \leftarrow 2 \leftarrow 3$	0	90	30
$1 \leftarrow 3 \leftarrow 2$	0	40	80
$2 \leftarrow 1 \leftarrow 3$	90	0	30
$2 \leftarrow 3 \leftarrow 1$	50	0	70
$3 \leftarrow 1 \leftarrow 2$	80	40	0
$3 \leftarrow 2 \leftarrow 1$	50	70	0
total	270	240	210
Shapley value	45	40	35

This example shows that the Shapley value may not be in the core (as $v(\{1,2\} - x_1 - x_2 = 5 > 0)$.

The Shapley value $Sh_i(N,v)$ of the TU game (N,v) for player i can also be written

$$Sh_i(N,v) = \sum_{\mathcal{C} \subseteq N \setminus \{i\}} \frac{|\mathcal{C}|!(|N|-|\mathcal{C}|-1)!}{|N|!} \left(v(\mathcal{C} \cup \{i\}) - v(\mathcal{C})\right).$$

Proof

- There are $|\mathcal{C}|!$ permutations in which all members of $\mathcal{C}$ precede i.
- There are $|N \setminus (\mathcal{C} \cup \{i\})|!$ permutations in which the remaining members succede i, i.e. $(|N|-|\mathcal{C}|-1)!$.

□

Using definition, the sum is over $2^{|N|-1}$ instead of $|N|!$.

Some interesting properties

Let (N,v) and (N,u) be TU games and $\phi = (\phi_1, \ldots, \phi_n)$ be function that takes as input a valution function over n players.

- **Symmetry or substitution (SYM) :** If $\forall (i,j) \in N$, $\forall \mathcal{C} \subseteq N \setminus \{i,j\}$, $v(\mathcal{C} \cup \{i\}) = v(\mathcal{C} \cup \{j\})$ then $\phi_i(N,v) = \phi_j(N,v)$
- **Dummy (DUM) :** If $\forall \mathcal{C} \subseteq N \setminus \{i\}$ $v(\mathcal{C}) = v(\mathcal{C} \cup \{i\})$, then $\phi_i(N,v) = 0$.
- **Additivity (ADD) :** Let $(N, u+v)$ be a TU game defined by $\forall \mathcal{C} \subseteq N$, $(u+v)(N) = u(N) + v(N)$. $\phi(u+v) = \phi(u) + \phi(v)$.

Theorem

The Shapley value is the unique value function ϕ that satisfies (SYM), (DUM) and (ADD).

Discussion about the axioms

- SYM : it is desirable that two subsitute agents obtain the same value ✔
- DUM : it is desirable that an agent that does not bring anything in the cooperation does not get any value. ✔
- What does the addition of two games mean ?
  - \+ if the payoff is interpreted as an expected payoff, ADD is a desirable property.
  - \+ for cost-sharing games, the interpretation is intuitive : the cost for a joint service is the sum of the costs of the separate services.
  - – there is no interaction between the two games.
  - – the structure of the game $(N, v+w)$ may induce a behavior that has may be unrelated to the behavior induced by either games (N, v) or (N, w).
- The axioms are independent. If one of the axiom is dropped, it is possible to find a different value function satisfying the remaining two axioms.

Some properties

Note that other axiomatisations are possible.

Theorem

For superadditive games, the Shapley value is an efficient and individually rational.

Lemma

For convex game, the Shapley value is in the core.

Simple games

Simple Game

Definition (Simple games)

A game (N,v) is a **Simple game** when

- the valuation function takes two values
  - 1 for a winning coalitions
  - 0 for the losing coalitions
- v satisfies *unanimity* : $v(N) = 1$
- v satisfies *monotonicity* : $S \subseteq T \Rightarrow v(S) \leqslant v(T)$

Formal definition of common terms in voting

Definition (Dictator)

Let (N, v) be a simple game. A player $i \in N$ is a **dictator** iff $\{i\}$ is a winning coalition.

Note that with the requirements of simple games, it is possible to have more than one dictator !

Definition (Veto Player)

Let (N, v) be a simple game. A player $i \in N$ is a **veto** player if $N \setminus \{i\}$ is a losing coalition. Alternatively, i is a **veto** player iff for all winning coalition $\mathcal{C}$, $i \in \mathcal{C}$.

It also follows that a veto player is member of every minimal winning coalitions.

Definition (blocking coalition)

A coalition $\mathcal{C} \subseteq N$ is a **blocking coalition** iff $\mathcal{C}$ is a losing coalition and $\forall S \subseteq N \setminus \mathcal{C}$, $S \setminus \mathcal{C}$ is a losing coalition.

Theorem

Let (N, v) be a simple game. Then

$$Core(N,v) = \left\{ x \in \mathbb{R}^n \;\middle|\; \begin{array}{l} x \text{ is an imputation} \\ x_i = 0 \text{ for each non-veto player } i \end{array} \right\}$$

Definition (Pivotal or swing player)

Let (N, v) be a simple game. A agent i is **pivotal** or **a swing agent** for a coalition $\mathcal{C} \subseteq N \setminus \{i\}$ if agent i turns the coalition $\mathcal{C}$ from a losing to a winning coalition by joining $\mathcal{C}$, i.e., $v(\mathcal{C}) = 0$ and $v(\mathcal{C} \cup \{i\}) = 1$.

Given a **permutation** σ on N, there is a single pivotal agent.

The Shapley-Shubik index of an agent i is the percentage of permutation in which i is pivotal, i.e.

$$I_{SS}(N, v, i) = \sum_{\mathcal{C} \subseteq N \setminus \{i\}} \frac{|\mathcal{C}|!(|N| - |C| - 1)!}{|N|!} \left(v(\mathcal{C} \cup \{i\}) - v(\mathcal{C})\right).$$

“For each permutation, the pivotal player gets a point.”

The Shapley-Shubik power index is the Shapley value.
The index corresponds to the expected marginal utility assuming all join orders to form the grand coalitions are equally likely.

Banzhaff power index

Let (N, v) be a TU game.

- We want to count the **number of coalitions** in which an agent is **a swing agent**.
- For each coalition, we determine which agent is a swing agent (more than one agent may be pivotal).
- The **raw Banzhaff index** of a player i is $\beta_i = \dfrac{\sum_{\mathcal{C}\subseteq N\setminus\{i\}} v(\mathcal{C}\cup\{i\}) - v(\mathcal{C})}{2^{n-1}}$.
- For a simple game (N, v), $v(N) = 1$ and $v(\emptyset) = 0$, at least one player i has a power index $\beta_i \neq 0$. Hence, $B = \sum_{j\in N} \beta_j > 0$.
- The **normalized Banzhaff index** of player i for a simple game (N, v) is defined as $I_B(N, v, i) = \dfrac{\beta_i}{B}$.

The index corresponds to the expected marginal utility assuming all coalitions are equally likely.

Examples : [7; 4, 3, 2, 1]

{1, 2, 3, 4}
{1, 2, 4, 3}
{1, 3, 2, 4}
{1, 3, 4, 2}
{1, 4, 2, 3}
{1, 4, 3, 2}
{2, 1, 3, 4}
{2, 1, 4, 3}
{2, 3, 1, 4}
{2, 3, 4, 1}
{2, 4, 1, 3}
{2, 4, 3, 1}
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{4, 1, 3, 2}
{4, 2, 1, 3}
{4, 2, 3, 1}
{4, 3, 1, 2}
{4, 3, 2, 1}

	1	2	3	4
Sh	$\frac{7}{12}$	$\frac{1}{4}$	$\frac{1}{12}$	$\frac{1}{12}$

winning coalitions :
{1, 2}
{1, 2, 3}
{1, 2, 4}
{1, 3, 4}
{1, 2, 3, 4}

	1	2	3	4
β	$\frac{5}{8}$	$\frac{3}{8}$	$\frac{1}{8}$	$\frac{1}{8}$
$I_B(N, v, i)$	$\frac{1}{2}$	$\frac{3}{10}$	$\frac{1}{10}$	$\frac{1}{10}$

The Shapley-Shubik index and Banzhaff index may be different.

Representation and Complexitity issues

Representation by enumeration

- Let us assume we want to write a program for computing a solution concept.
- How do we represent the input of a TU game ?
- Straighforward representation by enumeration requires **exponential space**.
- Brute force approach may appear good as complexity is measured in term of the **input size**.

⇨ we need **compact** or **succinct** representation of coalitional games.

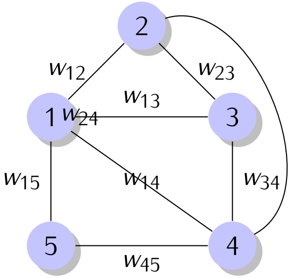
⇨ e.g., a representation so that the input size is a polynomial in the number of agents.

- In general, the more succinct a representation is, the harder it is to compute, hence we look for a balance between succinctness and tractability.

Induced subgraph games

An induced subgraph game is a coalitional game defined by an undirected weighted graph $\mathcal{G} = (V, W)$ where V is the set of vertices and $W : V \to V$ is the set of edges weights. For $(i, j) \in V^2$, w_{ij} is the weight of the edge between i and j.

- $N = V$, i.e., each agent is a node in the graph.
- for all $\mathcal{C} \subseteq N$, $v(\mathcal{C}) = \sum_{(i,j)\in\mathcal{C}} w_{ij}$.



It is a **succinct** representation : using an adjacency matrix, we need to provide n^2 entries.
However, it is **not complete**. Some TU games cannot be represented by a induced subgraph game (e.g., a majority voting game).

Subgraph Game and the Core

Proposition

Let (V, W) be a induced subgraph game. If all the weights are nonnegative then the game is convex.

Proposition

Let (V, W) be a induced subgraph game. If all the weights are nonnegative then membership of a payoff vector in the core can be tested in polynomial time.

Theorem

Let (V, W) be an induced subgraph game. Testing the nonemptyness of the core is NP-complete.

Subgraph Game and the Shapley Value

Theorem

Let (V,W) an induced subgraph game. The Shapley value of an agent $i \in V$ is $Sh_i(N,v) = \frac{1}{2} \sum_{(i,j) \in N^2 \mid i \neq j} w_{ij}$.

The Shapley value can be computed in $O(n^2)$ time.

Conclusion

- Game theory proposes many solution concepts (some of which were not introduced : nucleolus, bargaining sets, kernel, ϵ-core, least-core, Owen value). Each solution concept has pros and cons.
- There are many extensions : NTU games, games with coalition structures.
- Work in AI has dealt with representation issues, finding optimal partition of agents, games with overlaping coalitions, games under different types of uncertainty, and practical coalition formation protocols.