Complexity of non-preemptive single machine scheduling problems in the presence of uncertainty

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1 Introduction

This paper deals with single machine scheduling problems where some job characteristics are uncertain. This uncertainty is described through a finite set S of well-defined scenarios. We denote by p_j^s, d_j^s and w_j^s, respectively, the processing time, the due date and the weight of job j under scenario $s \in S$. Consider a scheduling problem, denoted by $\alpha|\beta|\gamma$ according to Graham *et al.* notation [4]. Let Π be the set of feasible schedules with respect to the problem constraints. For each scenario $s \in S$, we denote by $OPT(\alpha|\beta|\gamma, s)$ the problem of finding an optimal schedule π_s^* minimizing a cost function $F(\pi, s)$: π_s^* satisfies :

$$F(\pi_s^*, s) = \min_{\pi \in \Pi} F(\pi, s). \tag{1}$$

When problem parameters are uncertain, it is appropriate to search for a solution that is acceptable for any considered scenario. For this purpose, several criteria can be applied to select among solutions. In [6], Kouvelis and Yu proposed three different robustness criteria: the absolute robustness or maximal cost, the maximal regret or robust deviation and the relative robustness. In this paper, we focus on the *absolute robustness* criterion.

To the best of our knowledge, the absolute robustness in single machine scheduling problems has only been considered in [2] and [10] where, for the $1||\sum C_j$ problem with uncertain processing times, two distinct proofs of the NP-hardness even for $|S| = 2$ were provided (notice that the corresponding deterministic version is well known to be polynomially solvable [9]). The maximal regret criterion was instead much more studied (see for instance, again [10] but also [5]).

The absolute robustness of schedule π over all scenarios $s \in S$ is denoted by $\bar{F}(\pi)$. We have

$$\bar{F}(\pi) = \max_{s \in S} F(\pi, s). \tag{2}$$

We denote by $MinMax(\alpha|\beta|\gamma, \theta)$ the problem of finding a schedule π^A minimizing the absolute robustness $\bar{F}(\pi)$ among all schedules $\pi \in \Pi$. Field θ indicates the set of uncertain problem

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parameters. For the problems considered here, $\theta \subseteq \{p_j, d_j, w_j\}$. Sequence π^A is called in [6] *absolute robust sequence.* Its cost $\bar{F}(\pi^A)$ satisfies

$$\bar{F}(\pi^A) = \min_{\pi \in \Pi} \bar{F}(\pi) = \min_{\pi \in \Pi} \max_{s \in S} F(\pi, s). \tag{3}$$

Notice that if problem $\alpha|\beta|\gamma$ is NP-hard, then, in the presence of uncertainty, the corresponding problems $MinMax(\alpha|\beta|\gamma, \theta)$ are also NP-hard. However, if problem $\alpha|\beta|\gamma$ is polynomially solvable, then, the corresponding problems $MinMax(\alpha|\beta|\gamma, \theta)$ are not necessarily polynomially solvable. In this work we establish the complexity status for the absolute robustness versions of the most well known non-preemptive polynomial-time single machine scheduling problems, namely problems $1|prec|f_{\max}$ (with $f_{\max} \in \{C_{\max}, L_{\max}, T_{\max}\}$) , $1||\sum w_j C_j$ and $1||\sum U_j$. Notice that all these problems present regular cost functions non-decreasing in the job completion times. In this context any schedule $\pi \in \Pi$ is completely characterized by the corresponding job sequence. Given a schedule $\pi \in \Pi$, the completion time of job j under scenario s, denoted by $C_j(\pi, s)$, $j = 1, \ldots, n, s \in S$, can easily be determined and the quality of the schedule $\pi \in \Pi$ under scenario s is then evaluated using the regular cost function $F(\pi, s)$.

Using the set S of scenarios, we construct a scenario s^w in which parameters k_j take their worst case value, denoted by k_j^w. In our case, we have $p_j^w = \max_{s \in S} p_j^s$, $d_j^w = \min_{s \in S} d_j^s$ and $w_j^w = \max_{s \in S} w_j^s$. Notice that in the context of a discrete set of scenarios, the constructed scenario is not necessarily feasible, i.e. we can have $s^w \notin S$: s^w is called *worst-case artificial scenario.*

Remark 1. When parameters are interval-uncertain, s^w is a feasible scenario. In this case, an absolute robust solution π^A of problem $MinMax(1|\beta|\gamma, \theta)$ is such that

$$\bar{F}(\pi^A) = \min_{\pi \in \Pi} \bar{F}(\pi) = \min_{\pi \in \Pi} \max_{s \in S} F(\pi, s) = \min_{\pi \in \Pi} F(\pi, s^w). \tag{4}$$

Hence π^A is also optimal for problem $OPT(1|\beta|\gamma, s^w)$. This means that the problem of finding an absolute robust sequence can be solved straightforwardly by any algorithm solving the problem without uncertainty applied to the worst-case artificial scenario. ▮

When uncertainty is scenario-based, we cannot apply the same reasoning because scenario s^w is not necessarily feasible. Nevertheless, we show that the above result holds for some problems.

2 Main results

2.1 Problem $MinMax(1|prec|f_{\max}, d_j)$

Theorem 1. *Problem $MinMax(1|prec|f_{\max}, d_j)$ can be optimally solved in $O(n^2 + n|S|)$ time by means of Lawler's algorithm applied to the worst-case artificial scenario s^w.*

Proof. In the following, for the sake of clarity we consider that $f_{\max} = L_{\max}$ but the same analysis holds for $f_{\max} = T_{\max}$. An absolute robust solution π^A of problem $MinMax(1|prec|L_{\max}, d_j)$ is such that

$$\begin{aligned} \bar{L}(\pi^A) &= \min_{\pi \in \Pi} \max_{s \in S} L_{\max}(\pi, s) = \min_{\pi \in \Pi} \max_{s \in S} \max_{j \in N} (C_j(\pi, s) - d_j^s) & (5) \\ &= \min_{\pi \in \Pi} \max_{s \in S} \max_{j \in N} (C_j(\pi) - d_j^s) = \min_{\pi \in \Pi} \max_{j \in N} \max_{s \in S} (C_j(\pi) - d_j^s) & (6) \\ &= \min_{\pi \in \Pi} \max_{j \in N} (C_j(\pi) - d_j^w) = \min_{\pi \in \Pi} L_{\max}(\pi, s^w) & (7) \end{aligned}$$

Hence, π^A is also an optimal solution for problem OPT($1|prec|L_{\max}, s^w$). For the complexity, the construction of the worst-case scenario requires $O(n|S|)$ time and the application of Lawler's algorithm requires $O(n^2)$ time, hence the overall complexity is $O(n^2 + n|S|)$. ∎

We observe that the proof of Theorem 1 can be applied, as it is, to any scheduling problem $\alpha|\beta|f_{\max}$. Hence, we have the following result.

Corollary 1 *Any algorithm solving problem $\alpha|\beta|f_{\max}$ provides an absolute robust solution for problem $MinMax(\alpha|\beta|f_{\max}, d_j)$, when applied to the worst-case artificial scenario s^w.*

2.2 Problem $MinMax(1|prec|f_{\max}, p_j, d_j)$

A robust solution π^A is such that

$$\bar{F}(\pi^A) = \min_{\pi \in \Pi} \max_{s \in S} F_{\max}(\pi, s) = \min_{\pi \in \Pi} \max_{s \in S} \max_{j \in N} f_j(C_j(\pi, s)) \tag{8}$$

We propose an algorithm, called *MinMax-Lawler*, which is an extension of Lawler's algorithm. This algorithm constructs a sequence π in reverse order. Let U be the set of unscheduled jobs. Define $p^s(U) = \sum_{j \in U} p_j^s$ for all $s \in S$. The rule is the following : Schedule last the job $j \in U$, which has no successor in U and such that $\max_{s \in S} f_j^s(p^s(U))$ is minimal. It is immediate to see that the complexity of *MinMax-Lawler* is $O(n^2|S|)$. We have the following results (proof omitted).

Theorem 2. *Problem $MinMax(1|prec|f_{\max}, p_j, d_j)$ is optimally solved by algorithm MinMax-Lawler.*

Corollary 2 *Problem $MinMax(1|prec|f_{\max}, p_j)$ is optimally solved by algorithm MinMax-Lawler.*

2.3 Problem $MinMax(1||\sum U_j, d_j)$

The following result holds (proof omitted).

Lemma 1. *There exist an optimal solution of problem $MinMax(1||\sum U_j, d_j)$ in which on-time jobs are scheduled in a non-decreasing order of their worst-case artificial scenario due dates d_j^w.*

Following the proof of Moore's algorithm optimality for problem $1||\sum U_j$, the following result also holds (proof omitted):

Theorem 3. *Problem $MinMax(1||\sum U_j, d_j)$ can be optimally solved in $O(n \log n + n|S|)$ time by means of Moore's algorithm applied to the worst-case artificial scenario s^w.*

2.4 Problem $MinMax(1||\sum w_j C_j, w_j)$

Lemma 2. *The $1||\sum C_j$ problem and the $1|p_j = 1|\sum w_j C_j$ problem are equivalent.*

Proof. Given any instance of the $1||\sum C_j$ problem where each job j has processing time p'_j, generate an instance of the $1|p_j = 1|\sum w_j C_j$ problem where each job j has weight $w''_j = p'_{n-j+1}$. Consider a generic sequence $(1, 2, ..., n-1, n)$. For the $1||\sum C_j$ problem the corresponding cost function value is $Z_1 = \sum_{j=1}^n (n-j+1)p'_j$. For the $1|p_j = 1|\sum w_j C_j$ problem the corresponding cost function value is $Z_2 = \sum_{j=1}^n j w''_j$. We show that $Z_2 = Z_1$. Indeed, $Z_2 = \sum_{j=1}^n j w''_j = \sum_{j=1}^n j p'_{n-j+1} = \sum_{j=1}^n (n-j+1)p'_j = Z_1$. ∎

Due to Lemma 2 and the NP-hardness of problem $MinMax(1||\sum C_j, p_j)$ from [2, 10], we have :

Theorem 4. *Problem $MinMax(1||\sum w_j C_j, w_j)$ is NP-hard even when $|S| = 2$ and $p_j = 1$ $\forall j$.*

2.5 Problem $MinMax(1||\sum U_j, p_j)$

By reduction from the NP-complete even-odd partition problem [3], we have (proof omitted) the following result.

Theorem 5. *Problem* $MinMax(1||\sum U_j, p_j)$ *is NP-hard even when* $|S| = 2$.

Correspondingly, the following corollary also holds.

Corollary 3 *Problem MinMax(*$1||\sum U_j, p_j, d_j$*) is NP-hard even when* $|S| = 2$.

2.6 Summary

Table 1 summarizes the above results presenting the complexity status for the absolute robustness versions of the most well known non-preemptive polynomial-time single machine scheduling problems, where an entry ”-” indicates that the considered case does not apply (for instance problem $1||\sum w_j C_j$ cannot have uncertainty on due dates as due dates are not present in the problem).

Table 1: Summary of the obtained results

Uncertain	corresponding deterministic problem		
parameter	$1\|\|\sum w_j C_j$	$1\|prec\|f_{\max}$	$1\|\|\sum U_j$
d_j	–	$O(n^2 + n\|S\|)$ (Theor. 1)	$O(n \log n + n\|S\|)$ (Theor. 3)
p_j	NP-hard (ref. [2, 10])	$O(n^2\|S\|)$ (Corol. 2)	NP-hard (Theor. 5)
w_j	NP-hard (Theor. 4)	–	–
p_j and d_j	–	$O(n^2\|S\|)$ (Theor. 2)	NP-hard (Corol. 3)
p_j and w_j	NP-hard (ref. [2, 10])	–	–

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