The Complexity of Grundy Coloring and its Variants

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Grundy coloring

The worst way of reasonably coloring a graph.
Grundy coloring

The **worst way** of reasonably coloring a graph.

- Order the vertices $v_1, v_2, \ldots, v_n$ to **maximize** #colors used by the greedy coloring.
- Greedy coloring: $v_i$ gets the first color $c(v_i)$ that does not appear in its neighborhood.
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The worst way of \textbf{reasonably} coloring a graph.

- Order the vertices $v_1, v_2, \ldots, v_n$ to \textit{maximize} \#colors used by the \textbf{greedy} coloring.
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- Greedy coloring: $v_i$ gets the first color $c(v_i)$ that does not appear in its neighborhood.
- Connected version: $\forall i, G[v_1 \cup \ldots \cup v_i]$ is connected.
- Weak version: $v_i$ can be colored with any color $\leq c(v_i)$. 
Grundy coloring

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- Greedy coloring: $v_i$ gets the first color $c(v_i)$ that does not appear in its neighborhood.
- Connected version: $\forall i, G[v_1 \cup \ldots \cup v_i]$ is connected.
- Weak version: $v_i$ can be colored with any color $\leq c(v_i)$.

Grundy number $\Gamma(G)$, connected/weak Grundy number
Do you see a Grundy coloring reaching color 6?
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Do you see a Grundy coloring reaching color 6?
Was it a weak Grundy coloring?
Was it a connected Grundy coloring?
(Minimal) witness = (minimal) induced subgraph having the same $X$ Grundy number, where $X \in \{-1, \text{weak, connected}\}$. 
A brief History of Grundy colorings

- 1939: Studied in directed acyclic graphs by Grundy.
- 1979: Formally defined by Kristen and Selkow.
- 1983: Ochromatic number defined by Simmons.
- 1987: Erdős et al. proved that ochromatic number = Grundy number.
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- 1983: Ochromatic number defined by Simmons.
- 1987: Erdös et al. proved that ochromatic number = Grundy number.
- 2011: Weak Grundy defined by Kierstead and Saoub.
- 2014: Connected Grundy defined by Benevides et al.
Algorithmic motivations

- $\Gamma(G)$ upper bounds the number of colors used by any greedy heuristic for \textsc{Min Coloring}.
- $\Gamma(G) \leq C\chi(G)$ on some classes of graphs gives a $C$-approximation for \textsc{Min Coloring}.
- Online coloring.
- see Sampaio’s PhD thesis for further motivations.
More questionable motivations

▶ (Weak) Grundy Coloring is to (Independent) Dominating Set what Coloring is to Independent Set.
▶ Is Sudoku more interesting than Grundy Coloring?
▶ Idea for commercialization: rename color $i$ to $2^i$ and set the goal to 2048.
Complexity of computing the Grundy number

\( k = \Gamma(G) \) and \( w \) denotes the treewidth of the graph

XP algorithm: \( n^{f(\kappa)} \); FPT algorithm: \( f(\kappa)n^{O(1)} \).

- NP-hard on (co-)bipartite, chordal, line, claw-free graphs.
- Solvable in \( n^{2^{k-1}} \) [Z ’06].
- Solvable in \( 2^{O(kw)}n \) and\(^1\) in \( n^{O(w^2)} \) [TP ’97].

\(^1\)one can show that \( k \leq 1 + w \log n \) [TP ’97]
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\[ \begin{array}{ccc}
\text{parameter} & \text{XP} & \text{FPT} \\
\hline
k & n^{2^{k-1}} & ? \\
w & n^{O(w^2)} & ? \\
k + w & - & 2^{O(kw)}n \\
\end{array} \]

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How many vertices (at most) do we need to achieve color $k$?
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A minimal witness is of size at most $2^{k-1}$.
(and its vertices are at distance $\leq k$ of the vertex colored by $k$)

**Theorem (Zaker ’06)**

*The Grundy number can be computed in $O(f(k)n^{2^{k-1}})$.*
Complexity of computing the Grundy number

Outline [BFKS '15]:

- Solvable in time $O^*(2.443^n)$.
- FPT in various classes such as $H$-minor free graphs, chordal graphs, claw-free graphs.
- An $O^*(2^{o(w \log w)})$ algorithm would contradict the ETH.
Complexity of the variants

- **Weak Grundy Coloring** is NP-complete [GV '97].
- **Connected Grundy Coloring** is NP-complete on chordal graphs, co-bipartite graphs [B+ '14].
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Outline [BKFS '15]:

- **Weak Grundy Coloring** is solvable in $O^*(2^{O(k)})$ but not in $2^{o(k)}2^{o(n+m)}$ under the ETH.
- **Connected Grundy Coloring** is NP-complete for $k = 7$. 
Grundy Coloring
Exact exponential algorithm

Try all possible orderings and run the corresponding greedy coloring: $O^*(n!)$. 
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Try all possible orderings and run the corresponding greedy coloring: $O^*(n!)$. 

Can we improve on this trivial algorithm?
Solving Grundy Coloring

In a minimal witness:

- $C_i$ is an independent dominating set in $G[\bigcup_{i \leq j \leq k} C_j]$. 
Solving Grundy Coloring

In a minimal witness:

- $C_i$ is an independent dominating set in $G[\bigcup_{i \leq j \leq k} C_j]$.
- $\Gamma(S) = \max\{\Gamma(S \setminus X), X \text{ ind. dom. set in } G[S]\} + 1.$
Solving Grundy Coloring

- Enumerating the independent dominating sets takes time $O^*(3^{n^3}) = O(1.443^n)$.
- So, filling a cell of the table takes $1.443^i$ for a subset of size $i$.
- Hence, the total running time is
  $$\sum_{i=0}^{n} \binom{n}{i} 1.443^i = (1 + 1.443)^n = 2.443^n.$$
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$O^*(2^n)$ algorithms? Polynomial space? Connected Grundy?
On $H$-minor free graphs

Induced Subgraph Isomorphism: $B$ induced subgraph of $A$?

Theorem (FG ’01)

ISI is FPT in $|V(B)|$ on $H$-minor-free graphs.
On $H$-minor free graphs

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Why does it imply an FPT algorithm for Grundy Coloring?
On $H$-minor free graphs

**Induced Subgraph Isomorphism**: $B$ induced subgraph of $A$?

**Theorem (FG ’01)**

ISI is FPT in $|V(B)|$ on $H$-minor-free graphs.

Why does it imply an FPT algorithm for Grundy Coloring?

Minimal witnesses have size at most $2^{k-1}$.
So, there are less than $k2^{2k}$ graphs $B$ to try.
On chordal graphs

Fact: for any chordal graph $G$, $tw(G) = \omega(G) - 1$. 
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Besides, $\omega(G) \leq \Gamma(G)$.

Therefore, $tw(G) \leq \Gamma(G) - 1$

$\leadsto$ run FPT algorithm in $2^{O(tw(G)\Gamma(G))} = 2^{O(\Gamma(G)^2)}$. 
On claw-free graphs

Observation

Grundy Coloring is solvable in time $O^*(\Delta^{\Delta^O(\Delta)})$. 
On claw-free graphs

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Guess a vertex $v$ colored by $k := \Gamma(G)$ in a minimal witness.
On claw-free graphs

Observation
Grundy Coloring is solvable in time $O^*(\Delta^{\mathcal{O}(\Delta)})$.

Guess a vertex $v$ colored by $k := \Gamma(G)$ in a minimal witness. This minimal witness is in $\mathcal{N}^k[v]$ of size roughly $\Delta^k$. 

Observation
For any class such that $\Delta(G) \leq f(\Gamma(G))$, Grundy Coloring is FPT.
On claw-free graphs

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For any class such that $\Delta(G) \leq f(\Gamma(G))$, Grundy Coloring is FPT.

In a claw-free graph, if $d(v) = \Delta(G)$, then $\chi(G[N(v)]) \geq \frac{\Delta(G)}{2}$. Besides, $\Gamma(G) \geq \chi(G) \geq \chi(G[N(v)])$ holds in any graph.
Weak Grundy Coloring
Color Coding

**Theorem**

Weak Grundy Coloring *is solvable in* $O^*(k^{2k-1})$.

- Color each vertex uniformly at random between 1 and $k$. 


Color Coding

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- The probability that a witness is well colored is at least $\frac{1}{k^{2^{k-1}}}$.
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- The probability that a witness is well colored is at least $\frac{1}{k^{2k-1}}$.
- Solving the instance is easier with this extra information.
- Repeat $100k^{2k-1}$ tries to have a small probability of failure.
Guess #1
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![Graph Diagram]
Guess #2
Guess #2
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Guess #2
\ldots O(k^{2^k}) \text{ unsuccessful guesses later} \ldots
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A *unique* optimal (weak) Grundy Coloring.
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A *unique* optimal (weak) Grundy Coloring.
Dominant subtree: largest among its siblings.
The binomial tree with missing dominant subtrees

$$G - T$$

User guide: show that only $v$ can get color 4, with degree-based considerations.
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How do you want to complete the construction for Weak Grundy?
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Degree-based considerations...
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Number of vertices and edges? $O(n + m)$
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Value of the parameter?
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How many dominant $T_3$ in $T_{\lceil \log m \rceil + 5}^\ast$? $2^{\lceil \log m \rceil} \geq m$

Solving (Weak) Grundy in $2^{2^{o(k)}} 2^{o(n+m)}$ would disprove the ETH.
Back to Grundy Coloring parameterized by treewidth

Binomial tree gadgetry + tricks for fine-grained lower bounds:

**Theorem**

_Solving Grundy Coloring in \(O^*(2^{o(w \log w)})\) would disprove the ETH._
Back to Grundy Coloring parameterized by treewidth

Binomial tree gadgetry + tricks for fine-grained lower bounds:

Theorem

Solving Grundy Coloring in $O^*(2^{o(w \log w)})$ would disprove the ETH.

- **Grouping technique**: partition the variables into $k$ sets of size $n/k \leadsto$ you can add one gadget per group assignment.
Back to Grundy Coloring parameterized by treewidth

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- **Grouping technique:** partition the variables into $k$ sets of size $n/k \leadsto$ you can add one gadget per group assignment.
- **"Compression by permutation" trick:** $(3^{n/k}/\log n/k)! > 2^n/k \leadsto$ encode a group of $n/k$ variables with a clique of size only $3^{n/k}/\log n/k$. 


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- **”Compression by permutation” trick**: $(3 \frac{n}{k}/ \log \frac{n}{k})! > 2^n \implies$ encode a group of $n/k$ variables with a clique of size only $3 \frac{n}{k}/ \log \frac{n}{k}$.

Might become tight if the $k + w$ algorithm is improved to $O^*(k^w) = O^*(w^w (\log n)^w) = O^*(w^w w^{2w}) = O^*(w^{O(w)})$. 
<table>
<thead>
<tr>
<th>Introduction</th>
<th>Grundy Coloring</th>
<th>Weak Grundy Coloring</th>
<th>Connected Grundy Coloring</th>
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</table>

**Connected Grundy Coloring**
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Connected Grundy number $= 3$, unbounded witness
paraNP-hardness of Connected Grundy Coloring

- Reduction from 3Sat-3Occ.
paraNP-hardness of Connected Grundy Coloring

- Reduction from $3\text{Sat-3Occ}$.
- We move along a "path" $P_1$ of literal vertices: coloring such a vertex by $3 \equiv$ setting the literal to true.
paraNP-hardness of Connected Grundy Coloring

▶ Reduction from $3\text{Sat-3Occ}$.  
▶ We move along a "path" $P_1$ of literal vertices: coloring such a vertex by 3 $\equiv$ setting the literal to true.  
▶ We then move along a "path" $P_2$ of clause vertices $c_j$s: coloring such a vertex by 4 $\equiv$ satisfying the clause.
paraNP-hardness of Connected Grundy Coloring

- Reduction from $3\text{Sat}-3\text{Occ}$.
- We move along a "path" $P_1$ of literal vertices: coloring such a vertex by $3 \equiv$ setting the literal to true.
- We then move along a "path" $P_2$ of clause vertices $c_j$'s: coloring such a vertex by $4 \equiv$ satisfying the clause.
- To achieve color $7$, three special neighbors of the $c_j$'s should be colored by $1$, $2$ and $3$ respectively.
$P_1$ and $P_2$ for the instance

\[ \{x_1 \lor \neg x_2 \lor x_3\}, \{x_1 \lor x_2 \lor \neg x_4\}, \{\neg x_1 \lor x_3 \lor x_4\}, \{x_2 \lor \neg x_3 \lor x_4\}. \]
A connected Grundy coloring setting all the $c_j$s to 4.
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Open Questions

- Is Grundy Coloring FPT in the highest color $k$?
- Is Grundy Coloring FPT in the treewidth $w$?
- Is (Weak) Grundy Coloring solvable in $O^*(2^n)$?
- Is Connected Grundy Coloring solvable in $O^*(c^n)$?