Fine-grained complexity of coloring geometric intersection graphs

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Many problems are solvable in $2^{O(\sqrt{n})}$ in planar graphs, and unlikely solvable in $2^{o(n)}$ in general graphs.
Square root phenomenon on planar graphs

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Max Independent Set, 3-Coloring, Hamiltonian Path...
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Dynamic programming would spare a $\log n$ in the exponent.
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Coloring Unit Disks

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Balanced separators

Theorem (Smith, Wormald ’98)
For every $d \geq 1$ and $B \geq 0$, there exists a constant $c = c(d, B)$, such that for every $B$-fat collection $S$ of $n$ $d$-dimensional convex sets with ply at most $\ell$, there exists a $d$-dimensional sphere $Q$, such that:

- at most $\frac{d+1}{d+2} n$ elements of $S$ are entirely inside $Q$,
- at most $\frac{d+1}{d+2} n$ elements of $S$ are entirely outside $Q$,
- at most $cn^{1-1/d} \ell^{1/d}$ elements of $S$ intersect $Q$. 
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- at most \( \frac{d+1}{d+2} n \) elements of \( S \) are entirely inside \( Q \),
- at most \( \frac{d+1}{d+2} n \) elements of \( S \) are entirely outside \( Q \),
- at most \( cn^{1-1/d} \ell^{1/d} \) elements of \( S \) intersect \( Q \).

**ply:** maximum number of objects covering a point.

**\( B \)-fat** objects: aspect ratios diameter/width are bounded by \( B \).
Balanced separators for unit disks

**Theorem (Smith, Wormald ’98, special case)**

Given a collection $S$ of $n$ disks with ply at most $\ell$, there exists a circle $Q$, such that:

- at most $\frac{3n}{4}$ disks of $S$ are entirely inside $Q$,
- at most $\frac{3n}{4}$ disks of $S$ are entirely outside $Q$,
- at most $O(\sqrt{n\ell})$ disks of $S$ intersect $Q$. 
Standard algorithm for $\ell$-coloring (for unit disks)

If the ply is greater than $\ell$, then more than $\ell$ colors are needed.

Otherwise, there is a balanced separator of size $O(\sqrt{n\ell})$ which can be exhaustively found in time $O(2^{\sqrt{n\ell} \log n})$.

Trying all the $\ell$-colorings on $S$ takes time $O(2^{\sqrt{n\ell} \log \ell})$. 
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Overall running time: $O(2^{\sqrt{n\ell} \log n})$. 
We will see that this running time is optimal up to logarithmic factors in the exponent.
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**Theorem**

For any $\alpha \in [0, 1]$, coloring $n$ unit disks with $\ell = \Theta(n^\alpha)$ colors cannot be solved in time $2^{o(n^{1+\alpha}/2)} = 2^{o(\sqrt{n\ell})}$, under the ETH.
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Constant number of colors $\leadsto$ square root phenomenon.
Linear number of colors $\leadsto$ no subexponential-time algorithm.
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Linear number of colors $\leadsto$ no subexponential-time algorithm.

And everything in between (hard part).
For instance, $\sqrt{n}$-coloring cannot be done in $2^{o(n^{3/4})}$. 
Roadmap

3-SAT $\rightarrow$ 2-grid 3-SAT $\rightarrow$ Partial 2-grid Coloring $\rightarrow$ coloring unit disks
Partial 2-grid Coloring $\rightarrow$ coloring unit disks
Partial 2-Grid Coloring

**Input:** An induced subgraph $G$ of the $g \times g$-grid, a positive integer $\ell$. Each cell of this grid is mapped to a set of $\ell$ points (in a smaller grid $[\ell]^2$).

**Question:** Is there an $\ell$-coloring of all the points such that:

- two points in the same cell get different colors;
- if $v$ and $w$ are adjacent in $G$, say, $w = v + (1, 0)$, $p$, resp. $q$, are points in the smaller grid of $v$ resp. $w$, receiving the same color, then $q$ has at a second coordinate which is at least the second coordinate of $p$?
2-Grid 3-SAT

**Input:** A $g \times g$ grid, a positive integer $k$, each vertex (or cell) of the grid is associated to $k$ variables, and a set $C$ of constraints of two kinds:

- **clause constraints:** for each cell of the grid, a set of pairwise variable-disjoint 3-clauses on its variables;
- **equality constraints:** for two adjacent cells of the grid, a set of pairwise variable-disjoint equality constraints.

**Question:** Is there an assignment of the variables such that all constraints are satisfied?
3-SAT $\rightarrow$ 2-Grid 3-SAT

3-SAT on $N$ variables with bounded number of occurrences (Sparsification Lemma) $\leadsto$

split the variables into $\approx k$ blocks $\leadsto$ split the clauses on one block into a constant number of sub-blocks (clauses vertex-disjoint)

The size of the created instance is $n = g^2 k$.

$N = \Theta(gk) = \Theta(\sqrt{nk})$
2-Grid 3-SAT $\rightarrow$ Partial 2-Grid Coloring

- Clause checking gadget
- Even variable assignment cell
- Odd variable assignment cell
- Local reference cell
- Wires
- Consistency checking gadget
Encoding information and reference coloring
Wires
Permutation
Forget
Independence
Clauses

reference coloring

variable assignment

clause gadget

variable assignment

1
2
3
4
5
6

x1
x2
x3

[6] \ e

Clauses
Consistency gadget (also crossing)
Higher dimension

Theorem

For $\alpha \in [0, 1]$ and dimension $d \geq 2$, coloring $n$ unit $d$-balls with $\ell = \Theta(n^\alpha)$ colors cannot be solved in time $2^n \frac{d-1+\alpha}{d} - \epsilon$ for any $\epsilon > 0$, under the ETH.

The first step in the chain is trickier: the higher dimensional grid should embed the SAT instance in a more compact way.

The second and third steps work similarly.
(Longer and longer) Segments

Theorem
6-coloring 2-Dir is not solvable in $2^{o(n)}$, under the ETH.

Reduction from 3-coloring on degree-4 graphs to list 6-coloring of segment intersection graphs.
The $x_i$’s lists are $[1, 2, 3]$, the $y_j$’s lists are $[4, 5, 6]$.
Circles are equality gadgets ($1 \equiv 4, 2 \equiv 5, 3 \equiv 6$), squares are inequality gadgets.
Equality

<table>
<thead>
<tr>
<th>vertex</th>
<th>list</th>
</tr>
</thead>
<tbody>
<tr>
<td>$x_i$</td>
<td>1,2,3</td>
</tr>
<tr>
<td>$y_i$</td>
<td>4,5,6</td>
</tr>
<tr>
<td>$a_1$</td>
<td>1,4</td>
</tr>
<tr>
<td>$b_1$</td>
<td>4,5</td>
</tr>
<tr>
<td>$c_1$</td>
<td>4,6</td>
</tr>
<tr>
<td>$a_2$</td>
<td>2,5</td>
</tr>
<tr>
<td>$b_2$</td>
<td>4,5</td>
</tr>
<tr>
<td>$c_2$</td>
<td>5,6</td>
</tr>
<tr>
<td>$a_3$</td>
<td>3,6</td>
</tr>
<tr>
<td>$b_3$</td>
<td>4,6</td>
</tr>
<tr>
<td>$c_3$</td>
<td>5,6</td>
</tr>
</tbody>
</table>
Inequality

\[ x \leq \frac{1}{2} (x' + p_1 p_2 q_1 q_2 r_1 r_2) \]

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<tbody>
<tr>
<td>( x_i )</td>
<td>1,2,3</td>
</tr>
<tr>
<td>( y_j )</td>
<td>4,5,6</td>
</tr>
<tr>
<td>( x' )</td>
<td>4,5,6</td>
</tr>
<tr>
<td>( p_1 )</td>
<td>1,5</td>
</tr>
<tr>
<td>( p_2 )</td>
<td>1,6</td>
</tr>
<tr>
<td>( q_1 )</td>
<td>2,4</td>
</tr>
<tr>
<td>( q_2 )</td>
<td>2,6</td>
</tr>
<tr>
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</tbody>
</table>
Inequality

Some extra gadgets permit to remove the lists.
Thanks for your attention!