Fine-grained complexity of coloring geometric intersection graphs

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NP-hardness vs ETH-hardness

NP-hardness:
- your problem is not solvable in polynomial, unless 3-SAT is
- very widely believed but do not give evidence against algorithms running in say, $2^{n^{1/100}}$. 
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ETH-hardness:

- stronger assumption than $P \neq NP$ is ETH asserting that no $2^{o(n)}$ algorithm exists for 3-SAT
- Allows to prove stronger conditional lower bounds
- linear reduction from 3-SAT: no $2^{o(n)}$ algorithm for your problem, quadratic reduction: no $2^{o(\sqrt{n})}$ algorithm, etc.
Square root phenomenon on planar graphs

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Dynamic programming would spare a log $n$ in the exponent.
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Coloring (Unit) Disks

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Coloring Unit Disks

- Frequency assignment in broadcast networks
- $k$-COLORING is NP-hard for any integer $k \geq 3$
- the problem can be 3-approximated
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Is 3-COLORING as hard as 100-COLORING? $\lceil \sqrt{n} \rceil$-COLORING?
Coloring Unit Disks

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Is \textsc{3-Coloring} as hard as \textsc{100-Coloring}? \textsc{\lceil \sqrt{n} \rceil-Coloring}?

For general graphs, the answer is yes: for any integer \( k \),
- there is an \( O^*(2^n) \) algorithm for \textsc{k-Coloring}
- and no \( 2^{o(n)} \) algorithm under the ETH.

For planar graphs,
Coloring Unit Disks

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Is \texttt{3-Coloring} as \textit{hard} as \texttt{100-Coloring?} \( \lceil \sqrt{n} \rceil \)-\texttt{Coloring}? 

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- there is an \( O^*(2^n) \) algorithm for \texttt{k-Coloring}
- and no \( 2^{o(n)} \) algorithm under the ETH.

For planar graphs, only \texttt{3-Coloring} is hard!
Balanced separators for unit disks

**Theorem (Smith, Wormald ’98, special case)**

Given a collection $S$ of $n$ disks with ply at most $\ell$, there exists a circle $Q$, such that:

- at most $3n/4$ disks of $S$ are entirely inside $Q$,
- at most $3n/4$ disks of $S$ are entirely outside $Q$,
- at most $O(\sqrt{n\ell})$ disks of $S$ intersect $Q$. 
Standard algorithm for $\ell$-coloring (for unit disks)

If the ply is greater than $\ell$, then more than $\ell$ colors are needed. Otherwise, there is a balanced separator of size $O(\sqrt{n\ell})$ which can be exhaustively found in time $O(2^{\sqrt{n\ell} \log n})$.

Trying all the $\ell$-colorings on $S$ takes time $O(2^{\sqrt{n\ell} \log \ell})$. 
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Overall running time: $O(2^{\sqrt{n\ell} \log n})$. 
We will see that this running time is optimal up to logarithmic factors in the exponent.
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**Theorem**

For any $\alpha \in [0, 1]$, coloring $n$ unit disks with $\ell = \Theta(n^\alpha)$ colors cannot be solved in time $2^{o(n^{1+\frac{\alpha}{2}})} = 2^{o(\sqrt{n\ell})}$, under the ETH.
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**Theorem**

For any $\alpha \in [0, 1]$, coloring $n$ unit disks with $\ell = \Theta(n^\alpha)$ colors cannot be solved in time $2^{o(n^{\frac{1}{2+\alpha}})} = 2^{o(\sqrt{n\ell})}$, under the ETH.

Constant number of colors $\leadsto$ square root phenomenon.
Linear number of colors $\leadsto$ no subexponential-time algorithm.
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Constant number of colors $\leadsto$ square root phenomenon.
Linear number of colors $\leadsto$ no subexponential-time algorithm.

And everything in between (hard part).
For instance, $\sqrt{n}$-coloring cannot be done in $2^{o(n^{3/4})}$. 
Roadmap

3-SAT $\rightarrow$ 2-grid 3-SAT $\rightarrow$ Partial 2-grid Coloring $\rightarrow$ coloring unit disks
Partial 2-grid Coloring $\rightarrow$ coloring unit disks
Partial 2-Grid Coloring

**Input:** An induced subgraph $G$ of the $g \times g$-grid, a positive integer $\ell$. Each cell of this grid is mapped to a set of $\ell$ points (in a smaller grid $[\ell]^2$).

**Question:** Is there an $\ell$-coloring of all the points such that:

- two points in the same cell get different colors;
- if $v$ and $w$ are adjacent in $G$, say, $w = v + (1, 0)$, $p$, resp. $q$, are points in the smaller grid of $v$ resp. $w$, receiving the same color, then $q$ has at a second coordinate which is at least the second coordinate of $p$?
2-Grid 3-SAT

**Input:** A $g \times g$ grid, a positive integer $k$, each vertex (or cell) of the grid is associated to $k$ variables, and a set $C$ of constraints of two kinds:

- **clause constraints:** for each cell of the grid, a set of pairwise variable-disjoint 3-clauses on its variables;
- **equality constraints:** for two adjacent cells of the grid, a set of pairwise variable-disjoint equality constraints.

**Question:** Is there an assignment of the variables such that all constraints are satisfied?
3-SAT $\rightarrow$ 2-Grid 3-SAT

3-SAT on $N$ variables with bounded number of occurrences (Sparsification Lemma) $\leadsto$

split the variables into $\approx k$ blocks $\leadsto$
split the clauses on one block into a constant number of sub-blocks (clauses vertex-disjoint)

The size of the created instance is $n = g^2 k$.

$N = \Theta(gk) = \Theta(\sqrt{nk})$
2-Grid 3-SAT $\rightarrow$ Partial 2-Grid Coloring

- clause checking gadget
- even variable assignment cell
- odd variable assignment cell
- local reference cell
- wires
- consistency checking gadget
Encoding information and reference coloring

A

bottom of reference coloring

x1
x2

B
top of reference coloring

y1
y2
Wires
Permutation

- **A**
  - a
  - b
  - c
  - d

- **B**
  - c
  - d
  - a
  - b

- **C**
  - c
  - d
  - a
  - b
Forget

\begin{figure}
\centering
\begin{tikzpicture}
\node (A) {$\begin{array}{c}
\bullet a \\
\bullet b \\
\bullet c \\
\bullet d \\
\end{array}$};
\node (B) at (2,0) {$\begin{array}{c}
\bullet a | b \\
\bullet a | b \\
\bullet a | b \\
\bullet d \\
\end{array}$};
\node (C) at (0,-2) {$\begin{array}{c}
\bullet a | b \\
\bullet a | b \\
\bullet a | b \\
\bullet d \\
\end{array}$};
\end{tikzpicture}
\end{figure}
Independence
Clauses

reference coloring

variable assignment

clause gadget

variable assignment
Consistency gadget (also crossing)

- $A(v)$
- $A(w)$
- $S$

**Colors:**
- Green: forget
- Gray: clause
- Blue: wires
- Blue: variable assignments
- Red: bottom of reference coloring
- Orange: top of reference coloring

**Cells:**
- Even variable assignment cell
- Odd variable assignment cell
- Local reference cell
- Combined assignment
Higher dimension

Theorem
For $\alpha \in [0, 1]$ and dimension $d \geq 2$, coloring $n$ unit $d$-balls with $\ell = \Theta(n^\alpha)$ colors cannot be solved in time $2^n \frac{d-1+\alpha}{d} - \epsilon$ for any $\epsilon > 0$, under the ETH.

The first step in the chain is trickier: the higher dimensional grid should embed the SAT instance in a more compact way.

The second and third steps work similarly.
(Longer and longer) Segments

**Theorem**

6-coloring 2-Dir is not solvable in $2^{o(n)}$, under the ETH.

Reduction from 3-coloring on degree-4 graphs to list 6-coloring of segment intersection graphs.

The $x_i$'s lists are $[1, 2, 3]$, the $y_j$'s lists are $[4, 5, 6]$. Circles are equality gadgets ($1 \equiv 4, 2 \equiv 5, 3 \equiv 6$), squares are inequality gadgets.
Equality

<table>
<thead>
<tr>
<th>vertex</th>
<th>list</th>
</tr>
</thead>
<tbody>
<tr>
<td>$x_i$</td>
<td>1,2,3</td>
</tr>
<tr>
<td>$y_i$</td>
<td>4,5,6</td>
</tr>
<tr>
<td>$a_1$</td>
<td>1,4</td>
</tr>
<tr>
<td>$b_1$</td>
<td>4,5</td>
</tr>
<tr>
<td>$c_1$</td>
<td>4,6</td>
</tr>
<tr>
<td>$a_2$</td>
<td>2,5</td>
</tr>
<tr>
<td>$b_2$</td>
<td>4,5</td>
</tr>
<tr>
<td>$c_2$</td>
<td>5,6</td>
</tr>
<tr>
<td>$a_3$</td>
<td>3,6</td>
</tr>
<tr>
<td>$b_3$</td>
<td>4,6</td>
</tr>
<tr>
<td>$c_3$</td>
<td>5,6</td>
</tr>
</tbody>
</table>
Some extra gadgets permit to remove the lists.
Inequality

Some extra gadgets permit to remove the lists.
Same lower bound for 4 colors.
What happens with 3-colors? (whiteboard)
Thanks for your attention!