Using Choquet integral in Machine Learning: what can MCDA bring?

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Abstract. In this paper we discuss the Choquet integral model in the realm of Preference Learning, and point out advantages of learning simultaneously partial utility functions and capacities rather than sequentially, i.e., first utility functions and then capacities or vice-versa. Moreover, we present possible interpretations of the Choquet integral model in Preference Learning based on Shapley values and interaction indices.

1 Introduction

The first application of the Choquet integral in computer science appeared in the late 80's in the field of decision under uncertainty [21], and early 90's in the fields of multi-criteria decision making (MCDM) [7] and data mining [6, 25]. Recently, it has also been used in machine learning (ML) [24]) and preference learning (PL) [4]. The use of the Choquet integral in MCDM and data mining for almost 20 years has lead to a wide literature dealing with both theoretical (axiomatizations) and practical (methodologies, algorithms) aspects [10]. The new fields of ML and PL can benefit from this huge literature. We focus on two aspects in this paper.

The first aspect concerns partial utility functions. As an aggregation function of n input variables, the Choquet integral requires that these variables are *commensurate*. By commensurate, we mean that a same value (say 0.5) taken by two different input variables must have the same meaning. In MCDA, this meaning refers to the degree of satisfaction to criteria. For instance, value 0.5 corresponds to half-satisfaction. The reason why the Choquet requires commensability is that it compares the values taken by the n variables. This commensurability property is obtained by introducing partial utility functions over the attributes. The use of partial utility functions is well-established in MCDA. They are much less used in ML and PL. Sometimes, the attributes are aggregated without having being normalized. When attributes are normalized, the partial utility functions are fixed a priori and not learnt. The main point of the paper is to show that, fixing utility functions a priori significantly reduces the expressivity of the model. For instance, if we only consider three criteria, when the utility functions are fixed, it is not difficult to find two comparisons that cannot be represented by a Choquet integral model: in fact 2 comparisons are sufficient (see Section 3.1). Now, when utility functions are not fixed, the simplest example we came up with that is not representable by a Choquet integral and partial utility functions is composed of 6 comparisons (see Section 4.1) with only two attributes. Using conditional relative importance, we also give a non-representable example composed of 11 comparisons with three attributes (see Section 4.2). Back to the case of two attributes, we show in Section 4.3, some sufficient conditions under which the preference relation can be represented by a Choquet integral and partial utility functions.

The second aspect is on the interpretation of the model. Murofushi proposed to use the Shapley value as an importance index [18], and later introduced an interaction index [19]. These two concepts are often used to interpret a capacity. The use of these indices might be debatable as one may argue that the user is interested in the interpretation of the Choquet integral and not the capacity. We recall some results – apparently not known from the community in ML and PL – showning that the Choquet integral can be interpreted in terms of Shapley and interaction indices. These results show that the Shapley value is actually equal to the mean value of the discrete derivative of the Choquet integral over all possible vectors in $[0,1]^n$. This assumes that the set of possible alternatives is uniformly distributed in $[0,1]^n$. We show in Section 5 how to extend these results to non uniform distributions which arises often in ML or even in MCDA. Some connections with the definition of the Shapley value and interaction indices on non-Boolean lattices are given.

2 Preliminaries

2.1 The Choquet integral

Let us denote by $N = \{1, \dots, n\}$ a finite set of n criteria and $X = X_1 \times \cdots \times X_n$ the set of actions (also called alternatives or options), where for each $i \in N$, X_i represents the set of possible levels on criterion i. We refer to function $u_i : X_i \to \mathbb{R}$, $i = 1, \dots, n$, as utility function.

The Choquet integral [9, 10, 17, 16] is based on a *capacity* μ defined as a set function from the powerset of criteria 2^N to $[0,1]$ such that:

1. $\mu(\emptyset) = 0$
2. $\mu(N) = 1$
3. $\forall A, B \in 2^N,\ [A \subseteq B \Rightarrow \mu(A) \leq \mu(B)]$ (monotonicity).

For an alternative $x := (x_1, ..., x_n) \in X$, the expression of the Choquet integral w.r.t. a capacity μ is given by:

$$C_\mu((u_1(x_1), \dots, u_n(x_n))) := \sum_{i=1}^{n} (u_{\tau(i)}(x_{\tau(i)}) - u_{\tau(i-1)}(x_{\tau(i-1)}))\ \mu(\{\tau(i), \dots, \tau(n)\})$$

where τ is a permutation on N such that $u_{\tau(1)}(x_{\tau(1)}) \leq u_{\tau(2)}(x_{\tau(2)}) \leq \cdots \leq u_{\tau(n-1)}(x_{\tau(n-1)}) \leq u_{\tau(n)}(x_{\tau(n)})$, and $u_{\tau(0)}(x_{\tau(0)}) := 0$.

The preferential information of the decision maker is represented by a binary relation $\succsim$ over X where $\succ$ is the asymmetric part of $\succsim$.

Let $\Pi(2^N)$ be the set of permutations on N, and $Z_\tau = \{z \in [0,1]^n : z_{\tau(1)} \geq \cdots \geq z_{\tau(n)}\}$, for $\tau \in \Pi(2^N)$. The Choquet integral $C_\mu(x)$ is clearly a weighted sum in each domain Z_τ. The

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weights of criteria change from a domain Z_τ to another one $Z_{\tau'}$, for $\tau, \tau' \in \Pi(2^N)$. Two alternatives are called *comonotone* if they belong to a same set Z_τ. The Choquet integral is additive for all comonotone alternatives [20].

2.2 Interpretation of a capacity

A capacity is a complex object (it contains 2^n parameters), hence it is useful to provide an interpretation of μ.

The Shapley value [22] is often used in MCDA as a tool to interpret a capacity [19, 7, 8]. Actually, the concept of Shapley value comes from cooperative game theory and has been axiomatized in this framework [26]. The Shapley value describes how the worth obtained by all players shall be fairly redistributed among themselves [27].

Let us give a construction of the Shapley value in the spirit of cost allocation (cooperative game theory). N is interpreted here as the set of players and $\mu(S)$ is the cost of the cheapest way to serve all agents in S, ignoring the players in $N \setminus S$ altogether. All players of N agree to participate in the collective use of the common technology or public goods. Consider an ordering $\tau \in \Pi(2^N)$ of the players. Assume that the players are served in the order given by this permutation. Once the k first players have been served, the marginal cost of serving the next player according to the permutation is $\mu(\{\tau(1), \ldots, \tau(k+1)\}) - \mu(\{\tau(1), \ldots, \tau(k)\}) =: h^\tau_{\tau(k+1)}(\mu)$. The Shapley value allocates to agent i her expected marginal cost over all possible orderings of agents [22]:

$$\begin{aligned}\phi_i(\mu) &:= \frac{1}{n!} \sum_{\tau \in \Pi(2^N)} h^\tau_i(\mu) \\ &= \sum_{S \subseteq N \setminus \{i\}} \frac{|S|!(n-|S|-1)!}{n!} \Delta_i \mu(S)\end{aligned}$$

where $\Delta_i \mu(S) := \mu(S \cup \{i\}) - \mu(S)$. Coefficient $\frac{|S|!(n-|S|-1)!}{n!}$ is the probability that coalition S corresponds precisely to the set of players preceding player i in a giving ordering.

The *interaction index* [19] between criteria i and j is defined by

$$I_{ij}(\mu) := \sum_{A \subset N \setminus \{i,j\}} \frac{|A|!(n-|A|-2)!}{(n-1)!} \Delta_{i,j} \mu(A)$$

where $\Delta_{i,j}\mu(A) := \mu(A \cup \{i,j\}) - \mu(A \cup \{i\}) - \mu(A \cup \{j\}) + \mu(A)$. A positive (resp. negative) interaction depicts a positive (resp. negative) synergy between criteria – both criteria need to be satisfied (resp. it is sufficient that only one criterion is met).

This interaction index was extended to any coalition A of criteria [5]:

$$I_A(\mu) = \sum_{B \subseteq N \setminus A} \frac{(n-|B|-|A|)!|B|!}{(n-|A|+1)!} \Delta_A \mu(B),$$

where $\Delta_A \mu(B) = \sum_{K \subseteq A} (-1)^{|A \setminus K|} \mu(B \cup K)$. In particular we have

$$I_{\{i\}}(\mu) = \phi_i(\mu) \quad \text{and} \quad I_{\{i,j\}}(\mu) = I_{i,j}(\mu).$$

3 Choquet integral: the importance of learning utility functions and capacities simultaneously

3.1 The limitation of Choquet integral: a classical example

A classical example that shows the limitation of the Choquet integral model is [9]:

The students of a faculty are evaluated on three subjects Mathematics (M), Statistics (S) and Language skills (L). All marks are taken from the same scale from 0 to 20. The evaluations of eight students are given by the table below:

	1 : Mathematics (M)	2 : Statistics (S)	3 : Language (L)
A	16	13	7
B	16	11	9
C	6	13	7
D	6	11	9
E	14	16	7
F	14	15	8
G	9	16	7
H	9	15	8

To select the best students, the dean of the faculty expresses his preferences:

- for a student good in Mathematics, Language is more important than Statistics

$$\Longrightarrow A \prec B \quad \text{and} \quad E \prec F,$$

- for a student bad in Mathematics, Statistics is more important than Language

$$\Longrightarrow D \prec C \quad \text{and} \quad H \prec G.$$

The two preferences $A \prec B$ and $D \prec C$ lead to a contradiction with the arithmetic mean model because

$$\begin{cases} A \prec B \Rightarrow 16\ w_M + 13\ w_S + 7\ w_L < 16\ w_M + 11\ w_S + 9\ w_L \\ D \prec C \Rightarrow 6\ w_M + 11\ w_S + 9\ w_L < 6\ w_M + 13\ w_S + 7\ w_L. \end{cases}$$

Furthermore it is not difficult to see that the other two preferences, $E \prec F$ and $H \prec G$, are not representable by a Choquet integral C_μ since

$$\begin{cases} E \prec F \Rightarrow 7 + 7\mu(\{M,S\}) + 2\mu(\{S\}) < 8 + 6\mu(\{M,S\}) + \mu(\{S\}) \\ H \prec G \Rightarrow 8 + \mu(\{M,S\}) + 6\mu(\{S\}) < 7 + 2\mu(\{M,S\}) + 7\mu(\{S\}) \end{cases}$$

$$i.e. \begin{cases} E \prec F \Rightarrow \mu(\{M,S\}) + \mu(\{S\}) < 1 \\ H \prec G \Rightarrow \mu(\{M,S\}) + \mu(\{S\}) > 1 \end{cases}$$

An important remark in this example is that we try to find a capacity by assuming that the utility functions are fixed. If the latter are not fixed, then $E \prec F$ and $H \prec G$ can be modeled by C_μ, for instance, using these following utility functions:

	1 : Mathematics (M)	2 : Statistics (S)	3 : Language (L)
E	$u_M(14) = 16$	$u_S(16) = 16$	$u_L(7) = 7$
F	$u_M(14) = 16$	$u_S(15) = 15$	$u_L(8) = 8$
G	$u_M(9) = 9$	$u_S(16) = 16$	$u_L(7) = 7$
H	$u_M(9) = 9$	$u_S(15) = 15$	$u_L(8) = 8$

Indeed these utility functions lead to the system

$$\begin{cases} E \prec F \Rightarrow 2\mu(\{M,S\}) - \mu(\{M\}) < 1 \\ \\ H \prec G \Rightarrow \mu(\{M,S\}) + \mu(\{S\}) > 1 \end{cases}$$

Hence a capacity μ such that $\mu(\{M,S\}) = \mu(\{M\}) = \mu(\{S\}) = 0.6$ can be found. The utility function given above show that for the DM, the interpretation of "a good mark" in mathematics and "a good mark" in statistics is different. Such an interpretation is not in contradiction with the definition of commensurate scales: for $x_i \in X_i$ and $x_j \in X_j$,

$u_i(x_i) \geq u_j(x_j)$ iff the DM considers x_i at least as good as x_j.

Of course, if we assume that $u_M(a) = u_S(a) = u_L(a)$, for all $a \in [0,20]$, then $E \prec F$ and $H \prec G$ remain not representable by C_μ. This is not surprising because such situations can be viewed as the representation of preferences in decision under uncertainty where the Choquet integral model is well characterized [20, 21]. The four alternatives E, F, G, H are comonotone and thus the preferences $E \prec F$ and $H \prec G$ violate comonotone additivity.

To show the limitation of the Choquet integral in MCDA, we look for an example where the utility functions are not fixed a priori. This is the purpose of the next section.

4 Example non representable by a Choquet integral

We wish to known under which condition $\succsim$ is representable by a Choquet integral, i.e. there exists n utility functions $u_i : X_i \to \mathbb{R}$ and a capacity μ such that for all $x, y \in X$

$$x \succsim y \Longrightarrow C_\mu(u_1(x_1), \dots, u_n(x_n)) \geq C_\mu(u_1(y_1), \dots, u_n(y_n)). \quad (1)$$

4.1 A counter-example with 2 criteria

Let $\succsim$ be a weak order on the set $X = X_1 \times X_2$. We are interested in conditions that would guarantee that $\succsim$ can be represented using a Choquet integral model, i.e., that there is a real valued function u_1 on X_1 a real valued function u_2 on X_2 and positive real numbers λ_1, ω_1, such that:

$$x \succsim y \iff V(x) \geq V(y),$$

where V is a real valued function on X such that:

$$V(x) = \begin{cases} \lambda_1 u_1(x_1) + (1-\lambda_1) u_2(x_2) & \text{if } u_1(x_1) \geq u_2(x_2), \\ \omega_1 u_1(x_1) + (1-\omega_1) u_2(x_2) & \text{otherwise.} \end{cases}$$

Such a model is clearly a particular case of the model studied in [2] in which

$$V(x) = F(u_1(x_1), u_2(x_2)), \quad (2)$$

F being nondecreasing in its two arguments. We suppose that the conditions underlying the latter model hold. They are given in [2]. We now give an example of a weak order on X that cannot be represented using a Choquet integral model whatever the capacity and the functions u_1 and u_2.

Example 1 *Let* $X_1 = \{a_1, b_1, c_1, d_1, e_1, f_1\}$ *and* $X_2 = \{a_2, b_2, c_2, d_2, e_2, f_2\}$.

Suppose that the relation $\succsim$ *is such that:*

$$\begin{array}{c} (a_1, e_2) \sim (b_1, d_2) \\ (c_1, d_2) \sim (a_1, f_2) \\ (c_1, e_2) \not\sim (b_1, f_2) \end{array} \quad (3)$$

and

$$\begin{array}{c} (d_1, b_2) \sim (e_1, a_2) \\ (f_1, a_2) \sim (d_1, c_2) \\ (f_1, b_2) \not\sim (e_1, c_2) \end{array} \quad (4)$$

It is easy to find a weak order on X *that satisfies the conditions in [2] and that includes the relations* (3) *and* (4)*. Moreover, it is not difficult to choose this weak order in such a way as to satisfy* (2) *together with:*

$$\begin{array}{c} u_1(a_1) \leq u_1(b_1) \leq u_1(c_1) \leq u_1(d_1) \leq u_1(e_1) \leq u_1(f_1) \\ u_2(a_2) \leq u_2(b_2) \leq u_2(c_2) \leq u_2(d_2) \leq u_2(e_2) \leq u_2(f_2) \end{array} \quad (5)$$

Since each of triple of relations (3) *and* (4) *violates the Thomsen condition [13], they cannot be represented using an additive model.*

Considering the first triple, this implies that it is impossible that we have

$$u_1(a_1) \geq u_2(f_2), \quad \textit{or} \quad u_2(d_2) \geq u_1(c_1).$$

Indeed, if it were the case the representation for the elements in the triple would be additive, so that the Thomsen condition would be satisfied.

Similarly, considering the second triple, it is impossible that we have

$$u_1(d_1) \geq u_2(c_2) \quad \textit{or} \quad u_2(a_2) \geq u_1(f_1).$$

Hence, we must have:

$$u_1(a_1) < u_2(f_2), \quad \textit{and} \quad u_2(d_2) < u_1(c_1), \quad \textit{and} \quad u_1(d_1) < u_2(c_2) \quad \textit{and} \quad u_2(a_2) < u_1(f_1).$$

It is not difficult to see that, together with (5) *this leads to contradiction.*

Hence any weak order on X *that has a representation in model* (2) *satisfying* (5) *and that contains the relations in* (3) *and* (4) *cannot be represented using the Choquet integral model.*

4.2 A counter-example with 3 criteria

Let $n = 3$. We assume that there is an order on attribute 1. For instance, X_1 is a real interval and the utility function is increasing (e.g. X_1 represents the elements of a revenue). Another example: $X_1 = \{$ "very bad", "bad", "medium", "good", "very good", $\}$, where "very bad" is worse than " bad", etc. The ordering on X_1 is denoted by $\leq$ and the strict ordering by $<$.

We choose two elements on attributes 2 and 3:

$$y_2, z_2 \in X_2 \text{ and } y_3, z_3 \in X_3.$$

We choose now eleven elements on attribute 1:

$$x_1^1, x_1^2, \ldots, x_1^{11} \in X_1 \text{ with } x_1^1 < x_1^2 < \cdots < x_1^{11}.$$

We assume that the decision maker provides the following preferential information:

$$\begin{aligned}
&(x_1^1, y_2, y_3) \succ (x_1^1, z_2, z_3)\\
&(x_1^2, y_2, y_3) \prec (x_1^2, z_2, z_3)\\
&(x_1^3, y_2, y_3) \succ (x_1^3, z_2, z_3)\\
&(x_1^4, y_2, y_3) \prec (x_1^4, z_2, z_3)\\
&(x_1^5, y_2, y_3) \succ (x_1^5, z_2, z_3)\\
&(x_1^6, y_2, y_3) \prec (x_1^6, z_2, z_3)\\
&(x_1^7, y_2, y_3) \succ (x_1^7, z_2, z_3)\\
&(x_1^8, y_2, y_3) \prec (x_1^8, z_2, z_3)\\
&(x_1^9, y_2, y_3) \succ (x_1^9, z_2, z_3)\\
&(x_1^{10}, y_2, y_3) \prec (x_1^{10}, z_2, z_3)\\
&(x_1^{11}, y_2, y_3) \succ (x_1^{11}, z_2, z_3)
\end{aligned}$$

This idea of this example is to introduce sufficiently many comparisons such that there necessarily exist three comparisons of comonotonic alternatives leading to a contradiction.

Lemma 1 *The previous example is not representable by (1).*

Proof: Assume for a contradiction that there exist utility functions u_1, u_2, u_3 and a capacity μ representing the previous example. Note that $u_1(x_1^1) < u_1(x_1^2) < \cdots < u_1(x_1^{11})$. Let $V = \{u_2(y_2), u_2(z_2), u_3(y_3), u_3(z_3)\}$. These four elements split the real line $\mathbb{R}$ into at most five intervals $(-\infty, v_1]$, $[v_1, v_2]$, $[v_2, v_3]$, $[v_3, v_4]$ and $[v_4, +\infty)$, where $v_1 \leq v_2 \leq v_3 \leq v_4$ and $V = \{v_1, v_2, v_2, v_4\}$.

It is not difficult to see that among $u_1(x_1^1), u_1(x_1^2), \ldots, u_1(x_1^{11})$, at least three of them necessarily belong to the same interval. These three values necessarily correspond to three successive elements, denoted by x_1^k, x_1^{k+1} and x_1^{k+2}. Hence, in the preferential information, the comparison obtained from x_1^k and x_1^{k+2} are the same and are opposite to the comparison obtained with x_1^{k+1}. More precisely, we have two cases:

- In the first case, we have

$$\begin{aligned}
&(x_1^k, y_2, y_3) \succ (x_1^k, z_2, z_3)\ ,\\
&(x_1^{k+1}, y_2, y_3) \prec (x_1^{k+1}, z_2, z_3)\\
&\text{and}\\
&(x_1^{k+2}, y_2, y_3) \succ (x_1^{k+2}, z_2, z_3).
\end{aligned} \tag{6}$$

As $u_1(x_1^k)$, $u_1(x_1^{k+1})$ and $u_1(x_1^{k+2})$ belong to the same interval, the vectors $(u_1(x_1^k), u_2(y_2), u_3(y_3))$, $(u_1(x_1^{k+1}), u_2(y_2), u_3(y_3))$ and $(u_1(x_1^{k+2}), u_2(y_2), u_3(y_3))$ are comonotone, and $(u_1(x_1^k), u_2(z_2), u_3(z_3))$, $(u_1(x_1^{k+1}), u_2(z_2), u_3(z_3))$ and $(u_1(x_1^{k+2}), u_2(z_2), u_3(z_3))$ are comonotone. The Choquet integral is a weighted sum for all comonotone vectors. We denote by (w_1^y, w_2^y, w_3^y) the weights of criteria (obtained from the capacity μ) for the comonotone vectors $(u_1(x_1^k), u_2(y_2), u_3(y_3))$, $(u_1(x_1^{k+1}), u_2(y_2), u_3(y_3))$ and $(u_1(x_1^{k+2}), u_2(y_2), u_3(y_3))$. We denote by (w_1^z, w_2^z, w_3^z) the weights of criteria (obtained from the capacity μ) for the comonotone vectors $(u_1(x_1^k), u_2(z_2), u_3(z_3))$, $(u_1(x_1^{k+1}), u_2(z_2), u_3(z_3))$ and $(u_1(x_1^{k+2}), u_2(z_2), u_3(z_3))$. Hence (6) gives

$$\begin{aligned}
&u_1(x_1^k)\, w_1^y + u_2(y_2)\, w_2^y + u_3(y_3)\, w_3^y\\
&> u_1(x_1^k)\, w_1^z + u_2(z_2)\, w_2^z + u_3(z_3)\, w_3^z
\end{aligned} \tag{7}$$

$$\begin{aligned}
&u_1(x_1^{k+1})\, w_1^y + u_2(y_2)\, w_2^y + u_3(y_3)\, w_3^y\\
&< u_1(x_1^{k+1})\, w_1^z + u_2(z_2)\, w_2^z + u_3(z_3)\, w_3^z
\end{aligned} \tag{8}$$

$$\begin{aligned}
&u_1(x_1^{k+2})\, w_1^y + u_2(y_2)\, w_2^y + u_3(y_3)\, w_3^y\\
&> u_1(x_1^{k+2})\, w_1^z + u_2(z_2)\, w_2^z + u_3(z_3)\, w_3^z
\end{aligned} \tag{9}$$

Combining (7) with (8) gives

$$(u_1(x_1^k) - u_1(x_1^{k+1}))\, (w_1^y - w_1^z) > 0$$

As $u_1(x_1^k) < u_1(x_1^{k+1})$, we obtain $w_1^y < w_1^z$. Combining (9) with (8) gives

$$(u_1(x_1^{k+2}) - u_1(x_1^{k+1}))\, (w_1^y - w_1^z) > 0$$

As $u_1(x_1^{k+2}) > u_1(x_1^{k+1})$, we obtain the opposite inequality $w_1^y > w_1^z$. Hence a contradiction is attained.

- In the second case, we have

$$\begin{aligned}
&(x_1^k, y_2, y_3) \prec (x_1^k, z_2, z_3)\ ,\\
&(x_1^{k+1}, y_2, y_3) \succ (x_1^{k+1}, z_2, z_3) \text{ and}\\
&(x_1^{k+2}, y_2, y_3) \prec (x_1^{k+2}, z_2, z_3).
\end{aligned} \tag{10}$$

We proceed similarly and a contradiction is also raised.

■

From the two counter-examples presented above, we can deduce some necessary conditions to represent a preference by the Choquet integral model when utility functions and capacity are unknown a priori. Therefore we hope to entirely characterized this model in the future works. The search of this characterization led us to obtain a first sufficient condition in the case of two criteria.

4.3 Sufficient conditions for representability by Choquet integrals with 2 criteria

Let X_1, X_2 be two arbitrary chains (linearly ordered sets), and let $\succsim$ be a partial relation on $X_1 \times X_2$, extendable to a (total) preference relation on $X_1 \times X_2$ (i.e., which does not violate reflexivity, transitivity and the Pareto condition), and let $\succ$ be its nonsymmetric part ... Denote by $D(\succsim)$ the universe of R, i.e., the set of elements $x \in X_1 \times X_2$ that appear in some couple in $\succsim$.

Proposition 1 *Every partial relation $\succsim$ on $X_1 \times X_2$ for which $D(\succsim)$ is a finite antichain (w.r.t. the componentwise ordering of $X_1 \times X_2$) can be extended to a (total) preference relation on $X_1 \times X_2$ that is representable by a Choquet integral.*

Proof: Suppose that $D(\succsim) = \{x_1, \ldots, x_n\}$, $x_i = (x_{i1}, x_{i2})$, is an antichain. Without loss of generality, we assume that $x_{11} < \ldots < x_{n1}$ and $x_{12} > \ldots > x_{n2}$; the other possible case can be dealt with similarly. We shall construct utility functions $u_t \colon X_t \to \mathbb{R}$, $t = 1, 2$, and a capacity $\mu \colon 2^N \to [0, 1]$ such that $x_i \succsim x_j$ implies $C_\mu(u_1(x_{i1}), u_2(x_{i2})) \leq C_\mu(u_1(x_{j1}), u_2(x_{j2}))$.

Since $\succsim$ is extendable to a (total) preference relation on $X_1 \times X_2$, we can partition $D(\succsim)$ into (indifference) classes $C_0, C_1, \ldots$ that are defined recursively as follows:

1. C_0 contains all maximal elements for $\succsim$, i.e., elements $y \in D(\succsim)$ such that there is no $z \in D(\succsim)$ for which $y \succ z$;
2. if $C_0, C_1, \ldots, C_K$ have been defined, then C_{K+1} contains all $y \in D(\succsim)$ such that $y \succ z$ for some $z \in C_K$, and there is no $z' \in D(\succsim) \setminus \bigcup_{0 \leq t \leq K} C_t$ such that $y \succ z'$.

Let $C_0, C_1, \ldots, C_T$ be the thus defined classes.

Consider the capacity $\mu \colon 2^N \to [0, 1]$ given by $\mu(\{1\}) = \mu(\{2\}) = \frac{1}{3}$ and $\mu(\{1, 2\}) = 1$. Hence, $C_\mu(u_1(a_1), u_2(a_2)) = \frac{2}{3}u_1(a_1) + \frac{1}{3}u_2(a_2)$ if $u_1(a_1) \leq u_2(a_2)$; otherwise, $C_\mu(u_1(a_1), u_2(a_2)) = \frac{2}{3}u_2(a_2) + \frac{1}{3}u_1(a_1)$.

We construct $u_t \colon X_t \to \mathbb{R}$ on $\{x_{1t}, \ldots, x_{nt}\}$, $t = 1, 2$, as follows. Let $s := \min\{k : x_k \in C_0\}$. Set $u_1(x_{s1}) = u_2(x_{s2}) = n$. Hence $C_\mu(u_1(x_{j1}), u_2(x_{j2})) = n$.

Also, note that $u_1(x_{k1}) \leq u_2(x_{k2})$ if $k < s$, and $u_1(x_{k1}) \geq u_2(x_{k2})$ if $k > s$.

Now, take a sufficiently small $\epsilon > 0$, say $\epsilon = \frac{1}{n}$. For each $1 \leq i \leq n$ such that $x_i \in C_K$, define

1. $u_1(x_{i1}) = \left(n - (\frac{|j-i|}{2}) - K\epsilon\right)$ and $u_2(x_{i2}) = \left(n + |j-i| + K\epsilon\right)$ if $k < s$, and
2. $u_1(x_{i1}) = \left(n + |j-i| + K\epsilon\right)$ and $u_2(x_{i2}) = \left(n - (\frac{|j-i|}{4}) - K\epsilon\right)$, otherwise.

It is not difficult to verify that for every $0 \leq S \leq T$ and $y = (y_1, y_2) \in C_S$, we have $C_\mu(u_1(y_1), u_2(y_2)) = n - \frac{S\epsilon}{3}$, and the proof is now complete. ∎

5 Interpretation of the Choquet integral model

5.1 The $[0, 1]^n$ case

In the context of MCDA, the Shapley value can be seen as the mean importance of criteria and is thus a useful tool to interpret a capacity [7, 8]. The interpretation of Section 2.2 of the Shapley value is not satisfactory in MCDA since it completely ignores the use of the Choquet integral.

The interpretation of the Shapley value (and the Shapley interaction indices) for the Choquet integral is basically due to J.L. Marichal who noticed that (see [15, proposition 5.3.3 page 141] and also [11, Definition 10.41 and Proposition 10.43 page 369])

$$I_S(\mu) = \int_{[0,1]^n} \Delta_S C_\mu(z) dz$$

where, for any function f, $\Delta_S f$ is defined recursively by

$$\begin{aligned} \Delta_S f(z) &= \Delta_i(\Delta_{S\setminus\{i\}} f)(x) \quad \text{for any } i \in S \\ \Delta_i f(z) &= f(z|z_i = 1) - f(z|z_i = 0) \end{aligned}$$

The Shapley value appears as the mean of relative amplitude of the range of C_μ w.r.t. criterion i, when the remaining variables take random values. What is true with Shapley value is also true for interaction indices.

The following lemma is not difficult to prove:

Lemma 2 *We have*

$$I_S(\mu) = \int_{[0,1]^n} \frac{\partial^{|S|} C_\mu}{\partial z_S}(z)\, dz$$

where the partial derivative is piecewise continuous.

Here the partial derivative is the local importance of C_μ at point z.

5.2 The case of a subset of $[0, 1]^n$

The set of options that the decision maker finds feasible is often far from covering the whole space X. The following example shows that only a subset denoted by Ω of X may be realistic.

Example 2 (Situation awareness) *Consider a surveillance system that generates alerts from the information provided by several sensors such as cameras and radars. The system provides a situation awareness of the environment, gathering the identification of the intruder and its accurate localization [23].*

We are interested in assessing the quality of information provided by the system. To this end, we access the difference between what the system displays to the user and the real situation. Three criteria are considered.

- Relevance of identity information*: this is the difference between the identity that is obtained by the system and the real identity of the intruder. The determination of a wrong identity has strong consequences on the level of threat associated to the intruder.*
- Rough localization*: There are several particular assets that must be protected in the area that is covered by the surveillance system. Three areas of interest are defined around the assets: the alert zone which is the area at close range of the assets, the warning zone which is the area at medium range of the assets, and the rest of the area. There is a procedure which indicates the action that must be performed by an operator when an intruder is in one of these three zones. The system identifies the area to which intruders belong. Clearly, the identification of a wrong area has a critical consequence on the safety (if an intruder at close range is not seen as being in that area) or the relevance (false alarms) of the system.*

- Fine localization*: This is the accuracy of the intruders localization made by the system, i.e., the distance between the localization given by the system and the real one. The decision maker needs to know the accurate location of intruders, especially, in the alert area in order to perform a dissuasive action on the intruders.*

The last two criteria quantify the consequence with respect to two different points of view attached to localization. These two criteria are statistically correlated. Indeed, when the second criteria is not met, which entails a crude error, then it is not possible that the last criteria is well-satisfied. This implies that the satisfaction of last criterion cannot be better than that of the second criterion. Hence $x \in X$ such that $x_3 > x_2$ is not feasible.

In MCDA, our starting point is the subset Ω of realistic options in $[0,1]^n$. One can assume that Ω is convex and has a non-zero measure, as it is the case in Example 2. For $\tau \in \Pi(2^N)$ and $z \in Z_\tau$, those coalitions used in the computation of the Choquet integral w.r.t. z are $\emptyset, \{\tau(1)\}, \{\tau(1), \tau(2)\}, \ldots, \{\tau(1), \ldots, \tau(n)\}$. The following set

$$\mathcal{T} := \{\tau \in \Pi(2^N)\ ,\ \mathrm{Int}(X_\tau) \cap \Omega \neq \emptyset\}$$

contains those permutations that are reached when computing the Choquet integral of the elements of Ω. The set of coalitions that are used in the previous computations is then

$$\mathcal{F} = D(\mathcal{T}) := \bigcup_{\tau \in \mathcal{T}} \{\emptyset, \{\tau(1)\}, \{\tau(1), \tau(2)\}, \\ \ldots, \{\tau(1), \ldots, \tau(n)\}\}.$$

A *convex geometry* on N is a family $\mathcal{F}$ of subsets of N satisfying the following properties [3, 12]

(*i*) $\emptyset \in \mathcal{F}$, $N \in \mathcal{F}$,
(*ii*) $S \in \mathcal{F}$ and $T \in \mathcal{F}$ implies that $S \cap T \in \mathcal{F}$,
(*iii*) $\forall S \in \mathcal{F} \setminus \{N\}$, $\exists i \in N \setminus S$ such that $S \cup \{i\} \in \mathcal{F}$.

Note that properties (*i*) and (*ii*) allow to define a closure operator $\overline{S} = \bigcap\{T \in \mathcal{F}\ :\ S \subseteq T\}$ for every $S \subseteq N$. The set of *extreme points* of a subset $S \in \mathcal{F}$ is defined as $\mathrm{ext}(S) = \{i \in S\ :\ S \setminus \{i\} \in \mathcal{F}\}$. From this set, one can define a shelling process. Starting with the whole set N, one may successively eliminate extreme points until the empty set is obtained. This process defines maximal chains of $\mathcal{F}$. A *chain in* $\mathcal{F}$ *from* $S \in \mathcal{F}$ *to* $T \in \mathcal{F}$, with $S \subset T$, is a set of nested elements of $\mathcal{F}$ of the form $S = K_{i_1} \subset K_{i_2} \subset \cdots \subset K_{i_{m-1}} \subset K_{i_m} = T$ with $|K_{i_l}| = i_l$ for $l = 1, \ldots, m$, and $i_1 < i_2 < \cdots < i_m$. A *maximal chain of* $\mathcal{F}$ *from* $S \in \mathcal{F}$ *to* $T \in \mathcal{F}$, with $S \subset T$, is a chain of $\mathcal{F}$ from S to T for which $m = t-s+1$ (i.e. $i_l = s+l-1$ for all l) with the previous notation. A *maximal chain of* $\mathcal{F}$ is a maximal chain of $\mathcal{F}$ from $\emptyset$ to N.

Another interesting case is when the points x are not uniformly spread over $[0,1]^n$. A particular case is when there are some values in $[0,1]^n$ that are infeasible. Then, when computing the Choquet integral, some permutations may never occur, and thus some terms $\mu(S)$ (for some coalitions S) may never be used. The Shapley value has been defined for the situation of "forbidden" coalitions.

Let $\mathcal{F}$ be a convex geometry defined on N. A capacity on $\mathcal{F}$ is a function $\mu : \mathcal{F} \to \mathbb{R}$ satisfying the boundary and monotonicity conditions. Bilbao [1] defined the Shapley value of μ as follows

$$\phi_i^{\mathcal{F}}(\mu) = \sum_{S \subseteq N \setminus \{i\}\ :\ S, S \cup \{i\} \in \mathcal{F}} \frac{C_{\mathcal{F}}(S, S \cup \{i\})}{C_{\mathcal{F}}} \left[\mu(S \cup \{i\}) - \mu(S)\right], \tag{11}$$

where $C_{\mathcal{F}}$ is the total number of maximal chains of $\mathcal{F}$ and $C_{\mathcal{F}}(S, S \cup \{i\})$ is the number of maximal chains of $\mathcal{F}$ going through S and $S \cup \{i\}$.

A *reduced game* describes the situation where the players in a coalition P never play separately. As a consequence, they can be identified to a unique player denoted by $[P]$. Let $N_{[P]} := (N \backslash P) \cup \{[P]\}$. Let $\eta_P : \mathcal{P}_P(N) \to 2^{N_{[P]}}$ be defined by $\eta_P(S) = S$ if $P \not\subseteq S$ and $\eta_P(S) = (S \setminus P) \cup \{[P]\}$ otherwise. Given $\mathcal{F}$, the definition of the set of allowed coalitions $\mathcal{F}^{N_{[P]}}$ on $N_{[P]}$ is as follows [14]

$$\mathcal{F}^{N_{[P]}} = \eta_P(\mathcal{T}_P) = \{\eta_P(S)\ :\ S \in \mathcal{T}_P\},$$

where $\mathcal{T}_P$ is the set of the elements of all chains $\emptyset = S_0 \subset \cdots \subset S_k \subset S_{k+p} \subset \cdots \subset S_n = N$ of elements of $\mathcal{F}$ with $|S_i| = i$, $S_{k+p} = S_k \cup P$ and $S_k \in \mathcal{M}_P$, and where $\mathcal{M}_P := \{S \subseteq N \setminus P\ :\ \forall T \subseteq P\ ,\ S \cup T \in \mathcal{F}\}$. The interaction index $I_P^{\mathcal{F}}(\mu)$ of coalition P w.r.t. a capacity μ has the expression [14]

$$I_P^{\mathcal{F}}(\mu) = \sum_{S \in \mathcal{M}_P} \frac{C_{\mathcal{F}^{N_{[P]}}}(S, S \cup \{[P]\})}{C_{\mathcal{F}^{N_{[P]}}}} \Delta_P \mu(S). \tag{12}$$

Lemma 2 can be extended to the current setting in the following way:

Lemma 3 *We have*

$$I_S^{\mathcal{F}}(\mu) = \int_{\cup_{\tau \in \mathcal{T}} Z_\tau} \frac{\partial^{|S|} C_\mu}{\partial z_S}(z)\, dz$$

where the partial derivative is piecewise continuous.

This formula clearly shows that $I_S^{\mathcal{F}}(\mu)$ is interpreted as the interaction among criteria S for the Choquet integral. Expression (12) provides a combinatorial formulae to compute $I_S^{\mathcal{F}}(\mu)$.

Note that $\cup_{\tau \in \mathcal{T}} Z_\tau$ appears as an approximation of the feasibility domain Ω. When this approximation is not so good, it is possible to compute the interaction index by the following expression

$$\int_{\Omega} \frac{\partial^{|S|} C_\mu}{\partial z_S}(z)\, dz.$$

This computation might be complex when Ω is itself complex.

6 Conclusion

We have shown the gain in terms of expressivity that is obtained when the partial utility functions are constructed at the same time as the Choquet integral. With only two attributes, an example of non representativity is constructed. It is very special in the sense that the alternatives take special values on a grid. Moreover, again with two attributes, when the learning examples use only alternatives that belong to an antichain, then we have shown that any weak order over these alternatives, that does not violates Pareto condition and transitivity, is representable by a Choquet integral and partial utility functions. Clearly this result is wrong when the partial utility functions are a priori fixed. With three criteria, using the idea of conditional relative importance, we need 11 learning examples to contradicts the Choquet integral model. We believe that these examples are important to construct axiomatic characterizations of the Choquet integral and its utility functions.

Next, the Shapley index and interaction indices often used to interpret a capacity can also be used to interpret a Choquet integral. Actually the interaction index among criteria S is the integral over $[0,1]^n$ of the partial derivative of the Choquet integral w.r.t. criteria in S. This can be easily extended to the cases when the set of feasible alternatives is not $[0,1]^n$ but a subset. The corresponding Shapley and interaction indices are then extension of the original indices on convex geometries.

We hope we have convinced the community working on ML and PL on the importance of learning not only the capacity but also partial utility functions.

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