Chapter 2

Binary Relations and Preference Modeling

2.1. Introduction

This volume is dedicated to concepts, results, procedures and software aiming at helping people make a decision. It is then natural to investigate how the various courses of action that are involved in this decision compare in terms of preference. The aim of this chapter is to propose a brief survey of the main tools and results that can be useful to do so.

The literature on preference modeling is vast. This can first be explained by the fact that the question of modeling preferences occurs in several disciplines, e.g.

– in Economics, where one tries to model the preferences of a ‘rational consumer’ [e.g. DEB 59];

– in Psychology in which the study of preference judgments collected in experiments is quite common [KAH 79, KAH 81];

– in Political Sciences in which the question of defining a collective preference on the basis of the opinion of several voters is central [SEN 86];

– in Operational Research in which optimizing an objective function implies the definition of a direction of preference [ROY 85]; and

– in Artificial Intelligence in which the creation of autonomous agents able to take decisions implies the modeling of their vision of what is desirable and what is less so [DOY 92].

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Moreover, the question of preference modeling can be studied from a variety of perspectives [BEL 88], including:

– a *normative* perspective, where one investigates preference models that are likely to lead to a 'rational behavior';

– a *descriptive* perspective, in which adequate models to capture judgements obtained in experiments are sought; or

– a *prescriptive* perspective, in which one tries to build a preference model that is able to lead to an adequate recommendation.

Finally, the preferences that are to be modeled can be expressed on a variety of objects depending on the underlying decision problem. For instance, one may compare:

– vectors in $\mathbb{R}^p$ indicating the consumption of p perfectly divisible goods;

– candidates in an election;

– probability distributions modeling the possible financial results of various investment prospects;

– alternatives evaluated on several criteria expressed in incommensurable units when comparing sites for a new factory;

– projects evaluated on a monetary scale conditionally on the occurrence of various events or on the actions of other players.

It would be impossible within the scope of this chapter to exhaustively summarize the immense literature on the subject. More realistically, we will try here to present in a simple way the main concepts used in building models of preference. This will give the reader the necessary background to tackle the remaining chapters in this book. The reader willing to deepen their understanding of the subject is referred to [ALE 06, FIS 70, FIS 85, KRA 71, PIR 97, ROB 79, ROU 85].

This chapter is organized as follows. Section 2.2 is devoted to the concept of *binary relation* since this is the central tool in most models of preference. Section 2.3 defines a 'preference structure'. Section 2.4 introduces two classical preference structures: complete orders and weak orders. Sections 2.5 and 2.6 introduce several more general preference structures. Section 2.7 concludes with the mention of several important questions that we cannot tackle here.

2.2. Binary relations

2.2.1. *Definitions*

A binary relation T on a set A is a subset of the Cartesian product $A \times A$, i.e. a set of ordered pairs (a, b) of elements of A. If the ordered pair (a, b) belongs to the set T, we will often write $a\ T\ b$ instead of $(a, b) \in T$. In the opposite case, we write

$(a, b) \notin T$ or $a \neg T\ b$. Except when explicitly mentioned otherwise, we will suppose in all what follows that the set A is *finite*.

Remark 2.1. Since binary relations are sets, we can apply the classical operations of set theory to them. For instance, given any two binary relations T_1 and T_2 on A, we will write:

$$T_1 \subset T_2 \Leftrightarrow [a\ T_1\ b \Rightarrow a\ T_2\ b, \forall a, b \in A],$$

$$a\ (T_1 \cup T_2)\ b \Leftrightarrow a\ T_1\ b \text{ or } a\ T_2\ b,$$

$$a\ (T_1 \cap T_2)\ b \Leftrightarrow a\ T_1\ b \text{ and } a\ T_2\ b.$$

Moreover, the *product* $T_1 \cdot T_2$ will be defined by:

$$a\ T_1 \cdot T_2\ b \Leftrightarrow \exists c \in A : a\ T_1\ c \text{ and } c\ T_2\ b.$$

We denote by T^2 the relation $T \cdot T$, i.e. the product of the relation T with itself.

Given a binary relation T on A, we define:

– its inverse relation T^- such that:

$$a\ T^-\ b \Leftrightarrow b\ T\ a;$$

– its complement, i.e. the binary relation T^c such that:

$$a\ T^c\ b \Leftrightarrow a \neg T\ b;$$

– its dual relation T^d such that:

$$a\ T^d\ b \Leftrightarrow b \neg T\ a;$$

– its symmetric part I_T such that:

$$a\ I_T\ b \Leftrightarrow [a\ T\ b \text{ and } b\ T\ a];$$

– its asymmetric part P_T such that:

$$a\ P_T\ b \Leftrightarrow [a\ T\ b \text{ and } b \neg T\ a];$$

– its associated equivalence relation E_T such that:

$$a\ E_T\ b \Leftrightarrow \left\{ \begin{array}{l} a\ T\ c \Leftrightarrow b\ T\ c, \\ c\ T\ a \Leftrightarrow c\ T\ b, \end{array} \right\}, \forall c \in A.$$

Remark 2.2. It is easy to check that we have:

$$T^d = T^{-c} = T^{c-},$$

$$I_T = T \cap T^-,$$

$$P_T = T \cap T^d.$$

2.2.2. *Properties of a binary relation*

A binary relation T on A is said to be:

– *reflexive* if $a\ T\ a$;

– *irreflexive* if $a\ \neg T\ a$;

– *symmetric* if $a\ T\ b \Rightarrow b\ T\ a$;

– *antisymmetric* if $a\ T\ b$ and $b\ T\ a \Rightarrow a = b$;

– *asymmetric* if $a\ T\ b \Rightarrow b\ \neg T\ a$;

– *weakly complete* if $a \neq b \Rightarrow a\ T\ b$ or $b\ T\ a$;

– *complete* if $a\ T\ b$ or $b\ T\ a$;

– *transitive* if $a\ T\ b$ and $b\ T\ c \Rightarrow a\ T\ c$;

– *negatively transitive* if $a\ \neg T\ b$ and $b\ \neg T\ c \Rightarrow a\ \neg T\ c$;

– *Ferrers* if $[a\ T\ b$ and $c\ T\ d] \Rightarrow [a\ T\ d$ or $c\ T\ d]$; and

– *semitransitive* if $[a\ T\ b$ and $b\ T\ c] \Rightarrow [a\ T\ d$ or $d\ T\ c]$

for all $a, b, c, d \in A$.

Remark 2.3. The above properties are not independent. For instance, it is easy to check that

– a relation is asymmetric $\Leftrightarrow$ it is irreflexive and antisymmetric;

– a relation is complete $\Leftrightarrow$ it is weakly complete and reflexive;

– an asymmetric and negatively transitive relation is transitive; and

– a complete and transitive relation is negatively transitive.

Whatever the properties of T, it is clear that

– P_T is always asymmetric;

– I_T is always symmetric; and

– E_T is always reflexive, symmetric and transitive.

Remark 2.4. It is possible to reformulate the above properties in a variety of ways. For instance, observe that:

– T is complete $\Leftrightarrow T \cup T^{-} = A \times A$;

– T is asymmetric $\Leftrightarrow T \cap T^{-} = \varnothing$;

– T is transitive $\Leftrightarrow T^2 \subset T$;

– T is Ferrers $\Leftrightarrow T \cdot T^d \cdot T \subset T$; and

– T is semi-transitive $\Leftrightarrow T \cdot T \cdot T^d \subset T$.

An *equivalence* is a reflexive, symmetric and transitive binary relation (hence, the binary relation E_T defined earlier is an equivalence whatever the properties of T). Let

E be an equivalence on A. Given an element $a \in A$, the equivalence class associated to a, denoted by $[a]_E$, is the set $\{b \in A : a \ E \ b\}$. It is always true that $a \in [a]_E$. It is easy to show that $\forall a, b \in A$, either $[a]_E = [b]_E$ or $[a]_E \cap [b]_E = \varnothing$. An equivalence therefore partitions A into *equivalence classes*. The set of all these equivalence classes is called the quotient of A for E and is denoted A/E.

2.2.3. *Graphical representation of a binary relation*

A binary relation T on A can be represented as a directed graph (A, T) where A is the set of vertices of the graph and T is the set of the arcs of the graph (i.e. ordered pair of vertices). The particular properties of a binary relation can easily be interpreted using the sagittal representation of the graph (A, T). The reflexivity of T implies the presence of a loop on each vertex. The symmetry of T means that when there is an arc going from a to b, there is also an arc going from b to a. The transitivity of T means that as soon as there is a path of length 2 going from a to b, there is an arc from a to b. Taking the inverse relation is tantamount to inverting the orientation of all arcs in the graph. Taking the complement consists of adding all missing arcs and deleting all existing ones.

Observe that a symmetric relation can be more conveniently represented using a non-oriented graph, in which the ordered pairs (a, b) and (b, a) of the relation are represented using a single edge between the vertices a and b.

2.2.4. *Matrix representation of a binary relation*

Another way to represent a binary relation T on A is to associate to each element of A a row and a column of a square matrix M^T of dimension $|A|$. The element M^T_{ab} of this matrix, being at the intersection of the row associated to a and at the intersection of the column associated to b, is 1 if $a \ T \ b$ and 0 otherwise.

With such a representation, the reflexivity of T implies the presence of 1 on the diagonal of the matrix, provided that the elements of A have been associated in the order of the row and columns of the matrix. Under this hypothesis, the symmetry of T is equivalent to the fact that M^T is equal to its transpose. Taking the inverse relation consists of transposing the matrix M^T. The matrix associated to the product of two binary relations is the boolean product of the two corresponding matrices.

2.2.5. *Example*

Let $A = \{a, b, c, d, e\}$. Consider the binary relation

$$T = \{(a, b), (b, a), (b, c), (d, b), (d, d)\}.$$

A matrix representation of T is the following:

$\circlearrowleft$	a	b	c	d	e
a	0	1	0	0	0
b	1	0	1	0	0
c	0	0	0	0	0
d	0	1	0	1	0
e	0	0	0	0	0

A sagittal representation of the graph (A, T) is depicted in Figure 2.1.

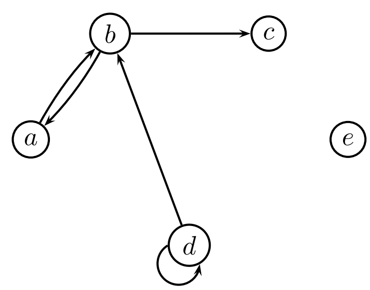

Figure 2.1. *Sagittal representation of the graph* (A, T)

2.3. Binary relations and preference structures

Consider an ordered pair (a, b) of objects. It is classically supposed that there can only be two answers to the question 'is object a at least as good as object b?': yes or no, these two answers being exclusive. Asking such a question for all ordered pais of objects leads to the definition of a *binary relation* S on the set A of all objects letting $a\ S\ b$ if and only if the answer to the question 'is a at least as good as b?' is yes. In view of its definition, it is natural to consider that S is reflexive; we will do so in all that follows.

Definition 2.1. A preference structure on A is a reflexive binary relation S on A.

Remark 2.5. The preceding definition raises a question of *observability*. If the idea of preference is to be based on observable behavior, the primitive may be taken to be choices made on various subsets of objects. This change of primitive is at the heart of 'revealed preference' theory in which the relation S is inferred from choices that are observable. Such an inference requires that choices are essentially 'binary', i.e. that choices made on pairs of objects are sufficient to infer choice made on larger sets of objects. The conditions allowing such a rationalization of a choice function through a binary relation are classical [e.g. SEN 70, SEN 77]. They have recently been severely questioned [MAL 93, SEN 93, SUG 85].

Remark 2.6. In some cases, one may envisage answers other than yes or no to the question 'is a at least as good as b?', e.g.

– answers such as 'I do not know';

– answers including information on the *intensity* of the preference, e.g. 'a is strongly/weakly/moderately preferred to b';

– answers including information on the *credibility* of the proposition 'a is at least as good as b', e.g. 'the credibility of the 'a is at least as good as b' is greater than the credibility of the proposition 'c is at least as good as d' ' or even 'the credibility of the proposition 'a is at least as good as b' is $\alpha \in [0; 1]$'.

Admitting such answers implies using a language that is richer than that of binary relations, e.g.

– the language of *fuzzy relations* [DOI 86, FOD 94, PER 92], each assertion of the type $a\ S\ b$ having a *degree of credibility*;

– languages tolerating hesitation [e.g. ROY 87];

– languages using the idea of *intensity of preference* [COS 94, DOI 87], an assertion such that $a\ S\ b$ and $b\ \neg S\ a$ being further qualified (weak, strong or extreme preference, for instance); or

– languages making use of *non-classical logics* [TSO 92, TSO 95, TSO 97] allowing the capture of the absence of information or, on the contrary, the existence of contradictory information (with such languages, the truth value of the assertion $a\ S\ b$ can take values different from just 'true' or 'false' and include 'unknown' and 'contradictory').

We do not consider such extensions in this paper.

Let us consider a preference S on a set A. For all pairs of objects $\{a, b\}$, we are in one of the following four situations (see Figure 2.2):

1) $[a\ S\ b$ and $b\ S\ a]$, denoted by $a\ I_S\ b$, interpreted as 'a is *indifferent* to b';

2) $[a\ \neg S\ b$ and $b\ \neg S\ a]$, denoted by $a\ J_S\ b$, interpreted as 'a is *incomparable* to b';

3) $[a\ S\ b$ and $b\ \neg S\ a]$, denoted by $a\ P_S\ b$, interpreted as 'a is *strictly preferred* to b'; and

4) $[a\ \neg S\ b$ and $b\ S\ a]$, denoted by $b\ P_S\ a$, interpreted as 'b is *strictly preferred* to a'.

	$b\ S\ a$	$b\ \neg S\ a$
$a\ S\ b$	$a\ I\ b$	$a\ P\ b$
$a\ \neg S\ b$	$b\ P\ a$	$a\ J\ b$

Figure 2.2. *Four exhaustive and mutually exclusive situations*

When there is no risk of ambiguity, we use I, J and P instead of I_S, J_S and P_S.

By construction, I and J are symmetric and P is asymmetric. Since S is reflexive, I is reflexive and J is irreflexive. The three relations P, I and J are:

– mutually exclusive, i.e. $P \cap I = P \cap J = I \cap J = \varnothing$ and

– exhaustive, i.e. $P \cup P^- \cup I \cup J = A^2$.

Remark 2.7. Many works use $\succsim$ instead of S, $\succ$ instead of P and $\sim$ instead of I.

Remark 2.8. Given a preference structure of S on A, it may be useful to consider the relation induced by S on the quotient set A/E_S, where E_S denotes the equivalence associated to S. This allows the simplification of many results.

Remark 2.9. Since a preference structure is a reflexive binary relation, we can use the graphical and matrix representations introduced earlier to represent it. In order to simplify graphical representations, we will systematically omit reflexivity loops and will use the conventions introduced in Figure 2.3.

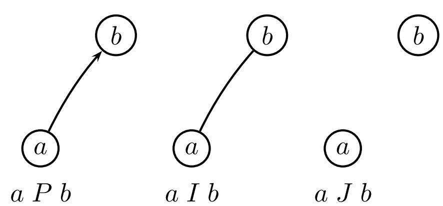

Figure 2.3. *Graphical conventions*

Example 2.1. Let $A = \{a, b, c, d, e\}$ and the preference structure $S = \{\ (a, a), (a, b), (a, c), (a, e), (b, a), (b, b), (b, c), (c, b), (c, c), (d, a), (d, b), (d, c), (d, d), (e, a), (e, c), (e, e)\ \}$. We have:

$$P = \{(a, c), (d, a), (d, b), (d, c), (e, c)\},$$

$$I = \{(a, a), (a, b), (a, e), (b, a), (b, b), (b, c), (c, b), (c, c), (d, d), (e, a), (e, e)\},$$

$$J = \{(b, e), (d, e), (e, b), (e, d)\}.$$

Using the above conventions, we obtain the matrix representation (Figure 2.4) and the graphical representation (Figure 2.5) of T.

$\circlearrowleft$	a	b	c	d	e
a	1	1	1	0	1
b	1	1	1	0	0
c	0	1	1	0	0
d	1	1	1	1	0
e	1	0	1	0	1

Figure 2.4. *Matrix representation*

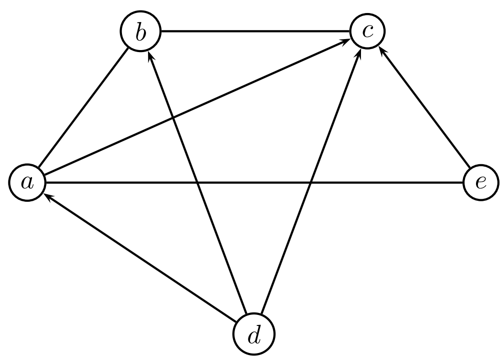

Figure 2.5. *Graphical representation*

2.4. Classical preference structures

2.4.1. *Total order*

2.4.1.1. *Definition*

A preference structure S is a total order if:

– S is complete;

– S is transitive; and

– S is antisymmetric.

In a total order, the incomparability relation is empty ($J = \varnothing$) and the indifference relation I is limited to pairs of identical objects ($I = \{(a, a) : a \in A\}$). The strict preference is P is weakly complete and transitive. A total order therefore consists of a ranking of the objects from A from best to worst (using the relation P) without the possibility of *ex aequo*.

Remark 2.10. It is easy to check that an equivalent definition of a total order consists of saying that S is complete and the only circuits in this relation are loops.

It is clear that, if S is a total order,

– P is weakly complete and transitive;

– I is transitive;

– $I \cdot P \subset P$; and

– $P \cdot I \subset P$.

Remark 2.11. Checking if a preference structure is a total order is quite simple using the matric representation of S. Indeed, labeling rows and columns of the matrix according to P, we obtain a matrix that has only 0 below the diagonal and 1 elsewhere. The relation P corresponds to off-diagonal 1's. In the graphical representation, if vertices are ranked according to P, all arcs are going from left to right.

Example 2.2. Let $A = \{a, b, c, d, e\}$. Consider the preference structure $S = \{(a, a), (a, b), (a, c), (a, d), (a, e), (b, b), (b, c), (b, d), (b, e), (c, c), (c, d), (c, e), (d, d), (d, e), (e, e)\}$.

It is easy to check that it is a total order using the matrix representation shown on Figure 2.6 or its graphical representation shown on Figure 2.7.

$\circlearrowleft$	a	b	c	d	e
a	1	1	1	1	1
b	0	1	1	1	1
c	0	0	1	1	1
d	0	0	0	1	1
e	0	0	0	0	1

Figure 2.6. *Matrix representation of a total order*

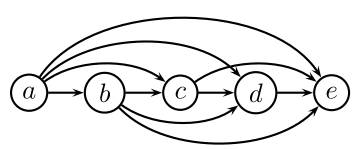

Figure 2.7. *Graphical representation of a total order*

2.4.1.2. *Numerical representation*

Let S be a total order on A. One may associate a number to each object in such a way that this number reflects the position of the object in the relation S. We leave the easy proof of the following result to the reader.

Theorem 2.1. *A preference structure S on a finite set A is a total order if and only if there is a function $g : A \to \mathbb{R}$ such that $\forall a, b \in A$:*

$$\begin{cases} a\ S\ b \Leftrightarrow g(a) \geq g(b), \\ g(a) = g(b) \Rightarrow a = b. \end{cases}$$

Remark 2.12. The numerical representation of a total order is not unique. It is easy to show that given a numerical representation g satisfying the conditions of Theorem 2.1, any increasing transformation applied to g leads to another admissible representation. Conversely, if g and h are two numerical representations of the same total order in the sense of Theorem 2.1, there is an increasing function ϕ such that $g = \phi \circ h$. The scale g is said to be an *ordinal scale*.

Let g be a function satisfying the condition of the above theorem. It is possible to compare differences such as $g(a) - g(b)$ and $g(c) - g(d)$. These comparisons are nevertheless clearly dependent upon the choice of the particular function g: another legitimate choice can lead to other comparisons of differences. Hence, in general, it is impossible to give a particular meaning to these comparisons.

Remark 2.13. Theorem 2.1 remains true if A is countably infinite (g is defined by an easy induction argument). It is clear that the result is no more true in the general case. Let us illustrate this fact by two examples.

1) It is well know that the cardinality of $\mathcal{P}(\mathbb{R})$ (i.e. the set of subsets of $\mathbb{R}$) is strictly greater than that of $\mathbb{R}$. Any total order on $\mathcal{P}(\mathbb{R})$ cannot have a numerical representation in the sense of Theorem 2.1. A natural question arises: is Theorem 2.1 true when attention is restricted to sets A, having at most the cardinality of $\mathbb{R}$? This is not so, as shown by the following famous example.

2) Let $A = \mathbb{R} \times \{0, 1\}$. It is easy to show that A has the same cardinality as $\mathbb{R}$. Consider the lexicographic order defined, letting:

$$(x, y)\ P\ (z, w) \Leftrightarrow \begin{cases} x > z \text{ or} \\ x = z \text{ and } y > w, \end{cases}$$

and

$$(x, y)\ I\ (z, w) \Leftrightarrow x = z \text{ and } y = w.$$

It is easy to show that the structure $S = P \cup I$ is a total order. It does not have a numerical representation in the sense of Theorem 2.1. Indeed, suppose that g is such a representation. We would have $\forall x \in \mathbb{R}$, $(x,1)\ P\ (x,0)$ so that $g(x,1) > g(x,0)$. There exists a rational number $\mu(x)$ such that $g(x,1) > \mu(x) > g(x,0)$. We have $(y,1)\ P\ (y,0)\ P\ (x,1)\ P\ (x,0) \Leftrightarrow y > x$. Hence, $y > x$ implies $\mu(y) > \mu(x)$. The function μ built above is therefore a bijection between $\mathbb{R}$ and $\mathbb{Q}$, a contradiction.

Beardon *et al.* [BEA 02] propose a detailed analysis of the various situations in which a total order does not have a numerical representation. The necessary and sufficient conditions ensuring that a total order has a numerical representation are known [BRI 95, DEB 54, FIS 70, KRA 71]. They amount to supposing that S on A has a behavior that is 'close' to that of $\geq$ in $\mathbb{R}$.

2.4.2. *Weak orders*

2.4.2.1. *Definition*

A preference structure S is a weak order if:

– S is complete; and

– S is transitive.

Weak orders generalize total orders since they do not have to be antisymmetric. Hence, indifference between distinct elements is allowed in weak orders.

Remark 2.14. An equivalent definition of a weak order is that S is complete and any circuit of S has no P arc.

It is clear that, if S is a weak order,

– P is transitive;

– P is negatively transitive;

– I is transitive (I is therefore an equivalence);

– $I \cdot P \subset P$;

– $P \cdot I \subset P$; and

– the relation S induces a total order on the quotient set A/I.

Remark 2.15. Let T be an asymmetric and negatively transitive binary relation on A. Let $S = T \cup (T^- \cap T^d)$. It is easy to show that S is a weak order.

Remark 2.16. If the rows and columns of the matrix representation of a weak order are ordered according to a relation that is compatible with P (the ordering of the rows and columns for indifferent elements being unimportant), we obtain a matrix in which the 1's are separated from the 0's by a stepped frontier that is below the diagonal and touches the diagonal. In a similar way, the graphical representation of a weak order generalizes that of a total order.

Example 2.3. Let $A = \{a, b, c, d, e\}$. Consider the preference structure S = (a, a), (a, b), (a, c), (a, d), (a, e), (b, a), (b, b), (b, c), (b, d), (b, e), (c, c), (c, d), (c, e), (d, c), (d, d), (d, e), $(e, e)\}$. It is easy to check that this is a weak order, considering the matrix representation depicted in Figure 2.8 or the graphical representation depicted in Figure 2.9.

$\circlearrowleft$	a	b	c	d	e
a	1	1	1	1	1
b	1	1	1	1	1
c	0	0	1	1	1
d	0	0	1	1	1
e	0	0	0	0	1

Figure 2.8. *Matrix representation of a weak order*

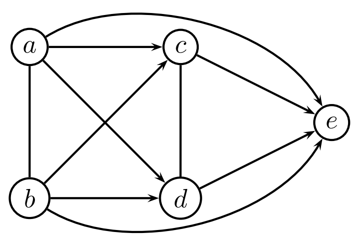

Figure 2.9. *Graphical representation of a weak order*

2.4.2.2. *Numerical representation*

Remembering that weak order induces a total order on the quotient set A/I, it is easy to prove the following result.

Theorem 2.2. *A preference structure S on a finite set A is a weak order if and only if (iff) there is a function $g : A \to \mathbb{R}$ such that $\forall a, b \in A$*

$$a \; S \; b \Leftrightarrow g(a) \geq g(b).$$

Remark 2.17. As above, the numerical representation of a weak order is defined up to an increasing transformation. The function g is an ordinal scale and most of the assertions that can be obtained using arithmetic operations on the values of g have a truth value that depends on the function g that was chosen: they are not meaningful in the sense of [ROB 79].

Remark 2.18. It is clear that the above result remain true when A is countably infinite (since in this case a total order structure always has a numerical representation). As was the case with total orders, extending this result to arbitrary sets implies the introduction of additional conditions.

2.4.3. *Classical problems*

In most studies involving preferences, the weak order model is used: the function g representing the weak order is the function that should be maximized. Depending on the context, it is referred to as the value function, objective function, criterion or value function. It is striking that decision problems have been dealt with so often in this way without much investigation on the adequateness of g as a model of preference.

We discuss here a few classical questions that have been dealt with using the weak order model.

2.4.3.1. *Choosing on the basis of binary relation*

Suppose that we have a weak order S on a set A and consider the situation (common in Economics) in which a choice must be made in a subset $B \subseteq A$. How should the information contained in S be used to guide such a choice? A natural way to define the set $C(B, S)$ of chosen objects (note that since we do not require $C(B, S)$ to be a singleton, it would be more adequate to speak of objects that are susceptible to be chosen) in B on the basis of S is to let

$$C(B, S) = \{b \in B : Not[\, a\ P\ b\,] \text{ for all } a \in B\}.$$

An object a belongs to the choice set as soon as there is no other object that is strictly preferred to a. It is not difficult to show that $C(B, S)$ is always non-empty as soon as B is finite (the general case raises difficult questions, see [BER 75]) and S is a weak order. Let us observe that, when B is finite, imposing that S is a weak order is only a sufficient condition for the non-emptyness of $C(B, S)$.

A classic result [SEN 70] states that, when B is finite, $C(B, S)$ is non-empty as soon as P is acyclic in B (it is never true that, for all $a_1, a_2, \ldots, a_k$ in B, $a_1\ P\ a_2, a_2\ P\ a_3, \ldots, a_{k-1}\ P\ a_k$ and $a_k\ P\ a_1$). The use of structures that are more general than the weak order also allows a simple answer to the problem to be derived.

We note that there are situations (e.g. a competitive exam) in which it is desirable to rank order all elements in a subset $B \subseteq A$ and also to define the choice set $C(B, S)$. The weak order model allows a trivial answer to this problem to be derived since the restriction of a weak order on A to a subset $B \subseteq A$ is a weak order on B.

2.4.3.2. *Aggregating preferences*

Suppose that you have collected $n \geq 2$ preference structures on A, for example because the objects are evaluated according to various points of view (voters, criteria or experts). In such a situation, it is natural to try to build a 'collective' preference structure S that aggregates the information contained in $(S_1, S_2, \ldots, S_n)$.

In general, one looks for a mechanism (e.g. an electoral system or an aggregation method) that is able to aggregate *any* n-tuple of preference structures on A into a collective preference structure. When the weak order model is used, defining such a mechanism amounts to defining an aggregation function F from $\mathcal{WO}(A)^n$ in $\mathcal{WO}(A)$, where $\mathcal{WO}(A)$ is the set of all weak orders on A.

The work of Arrow [ARR 63] has clearly shown the difficulty of such a problem. Imposing a small number of apparently reasonable conditions on F (unanimity, independence with respect to irrelevant alternatives and absence of dictator) leads to a logical impossibility: it is impossible to simultaneously satisfy all these principles (for a rich synthesis of such results, see [CAM 02, SEN 86]). The simple majority method can be used to illustrate the problem uncovered by Arrow's result. This method consists of declaring that 'a is collectively at least as good as b' if there are more weak orders in which 'a is at least as good as b' than weak orders for which 'b is at least as good as a'. Such a method seems highly reasonable and in line with our intuitive conception of democracy.

It does not always lead to a collective weak order; it may even lead to a collective relation having a cycle in its asymmetric part. This is the famous Condorcet paradox: $A = \{a, b, c\}$, $n = 3$, $a \ P_1 \ b \ P_1 \ c, c \ P_2 \ a \ P_2 \ b$ and $b \ P_3 \ c \ P_3 \ a$ gives the simplest example of such a situation. Using a collective preference structure in which strict preference may be cyclic in order to choose and/or to rank order is far from being an easy task. Many works have investigated the question [e.g. LAS 97, MOU 86, SCH 86].

2.4.3.3. *Particular structure of the set of objects*

In many situations, it is natural to suppose that the set of objects A has a particular structure. This will be the case in:

– decision with multiple criteria in which the elements of A are vectors of evaluations on several dimensions, attributes or criteria ($A \subseteq A_1 \times A_2 \times \cdots \times A_n$ where A_i is the set of possible evaluations of the objects on the ith dimension);

– decision under risk in which the elements on A are viewed as probability distribution on a set of consequences ($A \subseteq \mathcal{P}(C)$ where $\mathcal{P}(C)$ is a set of probability distributions on a set of consequences C); or

– decision under uncertainty in which the elements of A are characterized by consequences occurring contingently upon the occurrence of 'several states of nature' ($A \subseteq C^n$ where C is a set of consequences, supposing that n distinct states of nature are distinguished).

In all these cases it is tempting to add to the weak order model additional conditions that will allow us to take advantage of the particular structure of the set A. Among these condition, let us mention the following.

1) *Preference independence* [KEE 76, KRA 71, WAK 89]: In the case of decision-making with multiple criteria, this implies that the comparison of two objects differing only on a subset of criteria is independent from their common evaluations:

$$(a_I, c_{-I})\ S\ (b_I, c_{-I}) \Leftrightarrow (a_I, d_{-I})\ S\ (b_I, d_{-I})$$

where I is a subset of criteria $\{1, 2, \ldots, n\}$ and where (a_I, c_{-I}) denotes the object $e \in A$ such that $e_i = a_i$ if $i \in I$ and $e_i = c_i$ otherwise.

2) *Independence with respect to probabilistic mixing*[FIS 70, FIS 88]: In the case of decision-making under risk, this implies that the preference relation between two probability distributions is not altered when they are both mixed with a common probability distribution:

$$a\ S\ b \Leftrightarrow (a\alpha c)\ S\ (b\alpha c)$$

where $(a\alpha b)$ denotes the convex combination of the probability distributions a and b with the coefficient $\alpha \in (0; 1)$.

3) *The sure-thing principle* [FIS 70, SAV 54, WAK 89]: In the case of decision-making under uncertainty, this implies that the preference between two acts does not depend on common consequences obtained in some states of nature, i.e.

$$(a_I, c_{-I})\ S\ (b_I, c_{-I}) \Leftrightarrow (a_I, d_{-I})\ S\ (b_I, d_{-I})$$

where I is a subset of states of nature and (a_I, c_{-I}) denotes the act $e \in A$ such that $e_i = a_i$ if $i \in I$ and $e_i = c_i$ otherwise.

When these conditions are applied to sets of objects that are sufficiently ‘rich’ (and when it is required that S behaves coherently with this richness [FIS 70, WAK 89]), we obtain some famous models based on that of the classical theory.

– The model of *additive value functions* in the case of decision with multiple criteria:

$$a\ S\ b \Leftrightarrow \sum_{i=1}^{n} u_i(a_i) \geq \sum_{i=1}^{n} u_i(b_i)$$

where u_i is a real-valued function on A_i and the evaluation of object a on the ith criterion is denoted by a_i.

– The *expected utility* model in the case of decision making under risk:

$$a\ S\ b \Leftrightarrow \sum_{c \in C} p_a(c)u(c) \geq \sum_{c \in C} p_b(c)u(c)$$

where u is a real-valued function on C and $p_a(c)$ is the probability to obtain consequence $c \in C$ with object a.

– The *subjective expected utility* model in the case of decision-making under uncertainty:

$$a\ S\ b \Leftrightarrow \sum_{i=1}^{n} p_i u(a_i) \geq \sum_{i=1}^{n} p_i u(b_i)$$

where u is a real-valued function on C and the p_i's are non-negative numbers summing to 1 that can be interpreted as the subjective probabilities of the various states of nature.

One of the major aims of these models is to allow a numerical representation g of S that is much more specific than that given by Theorem 2.2. The additional conditions mentioned above imply that, when A is adequately rich (e.g. that $A = A_1 \times A_2 \times \cdots \times A_n$ in the case of decision making with multiple criteria, and that each A_i has a rich structure [WAK 89]), g can be additively decomposed. The numerical representation obtained is an interval scale (unique up to the choice of origin and unit). It is then possible to use sophisticated elicitation techniques to assess g and, therefore, structure a preference model [KEE 76, KRA 71, WAK 89].

These additional conditions were subjected to many empirical tests. In the fields of decision making under risk and uncertainty, it was show that the conditions at the heart of the expected utility model (independence axiom and sure-thing principle) were falsified in a predictable and reproducible way [ALL 53, ELL 61, KAH 79, MCC 79]. This has generated numerous studies investigating models using only weakening of these additional conditions (see [FIS 88, MAC 82, QUI 82, QUI 93, YAA 87] for decision under risk and [DUB 01, GIL 87, GIL 89, SCH 89, WAK 89] for decision under uncertainty).

Dutch book-like arguments (adhering to these generalized models may transform an individual into a 'money pump') have often been used to criticise these models [RAI 70]. The validity of such arguments nevertheless raises difficult questions (see [MAC 89, MCC 90] for a criticism of such arguments for decision making under risk).

Finally, let us mention that other structures for A can be usefully studied. For instance, when A is endowed with a topological structure, it is natural to investigate numerical representation having continuity properties [BOS 02a, BRI 95, JAF 75]. Similarly, if A is endowed with a binary operation allowing the combination of its elements (this is the case in decision under risk using 'probabilistic mixing' of two objects), a numerical representation is sought that is somehow compatible (most often through addition) with this operation [KRA 71].

2.5. Semi-orders and interval orders

In weak orders, the indifference relation I is transitive. This hypothesis is sometimes inadequate since it amounts to supposing a perfect discrimination between close but distinct objects. Luce [LUC 56] was the first to suggest a preference structure in which indifference may be intransitive [PIR 97]. He suggested the following example.

Example 2.4. Consider a set A consisting of 101 cups of coffee numbered from 0–100 and identical except that there are i grains of sugar in the ith cup. It is likely that an individual comparing these cups will not be able to detect a difference between two consecutive cups. Hence, it is likely that we obtain:

$$a_0 \ I \ a_1, a_1 \ I \ a_2, \ldots, a_{99} \ I \ a_{100}.$$

If the relation I is supposed to be transitive, we should have $a_0 \ I \ a_{100}$, which seems unlikely as the individual is supposed to prefer sugared coffee.

The two preference structures introduced in this section aim to model situations in which indifference is not transitive, while maintaining our other hypotheses (transitivity of P, no incomparability) made so far.

2.5.1. *Semi-order*

2.5.1.1. *Definition*

A preference structure S is a semi-order if:

– S is complete;

– S is Ferrers; and

– S is semitransitive.

Remark 2.19. It is easy to check that an equivalent definition of a semi-order is to suppose that S is complete and all circuits of S have more I arcs than P arcs.

Moreover, it is easy to prove that if S is a semi-order:

– P is transitive;

– P is Ferrers;

– P is semi-transitive;

– $P \cdot I \cdot P \subset P$;

– $P \cdot P \cdot I \subset P$;

– $I \cdot P \cdot P \subset P$; and

– $P^2 \cap I^2 = \varnothing$.

As will become apparent later, semi-orders arise when an indifference threshold is introduced when comparing objects evaluated on a numerical scale. As an easy exercise, the reader may wish to check that any weak order is a semi-order.

Remark 2.20. The graphical representation of a semi-order is characterized by the fact that the four configurations depicted in Figure 2.10 are forbidden (whatever appears on the diagonal and with the possibility that two indifferent elements may be identical).

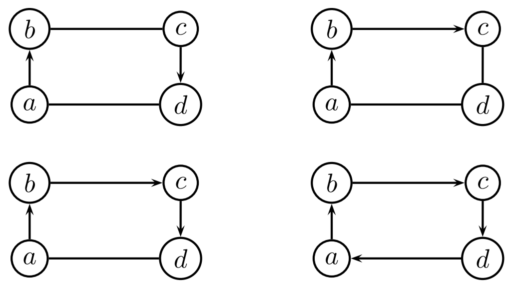

Figure 2.10. *Forbidden configurations in a semi-order*

2.5.1.2. *Weak order associated with a semi-order*

Let S is be a binary relation on A. The binary relation $S^{\pm}$ on A defined by

$$a\ S^{\pm}\ b \Leftrightarrow \left\{ \begin{array}{l} b\ S\ c \Rightarrow a\ S\ c, \\ c\ S\ a \Rightarrow c\ S\ b, \end{array} \right\} \forall c \in A$$

is called the trace of S. It is clear that the trace of a relation is always reflexive and transitive. We leave the easy proof of the following result to the reader.

Theorem 2.3. *Let S be a reflexive binary relation on A. S is a semi-order if and only if its trace $S^{\pm}$ is complete.*

Remark 2.21. When S is a semi-order, the weak order $S^{\pm}$ is obtained by ranking the elements of A according to their degree in S (i.e. number of arcs leaving a vertex minus the number of arcs entering it). One can check that a weak order is always identical to its trace.

2.5.1.3. *Matrix representation [JAC 78]*

By ordering the row and columns of the matrix representation of a semi-order, by using an order that is compatible with the trace of the relation, we obtain a matrix in which the 1's are separated from the 0's by frontiers that are stepped and located below the diagonal. This follows immediately from the definition of the trace. In contrast with what happens with weak orders, the frontier separating the 1's and the 0's does not necessarily touch the diagonal.

Example 2.5. Let $A = \{a, b, c, d, e, f\}$. Consider the preference structure $S = \{(a, a),\ (a, b),\ (a, c),\ (a, d),\ (a, e),\ (a, f)\ (b, a),\ (b, b),\ (b, c),\ (b, d), (b, e),\ (b, f),\ (c, b),\ (c, c),\ (c, d),\ (c, e),\ (c, f),\ (d, c),\ (d, d),\ (d, e),\ (d, f),\ (e, c),\ (e, d),\ (e, e),\ (e, f),\ (f, e),\ (f, f)\ \}$. We obtain the matric representation shown in Figure 2.11. This relation is not a weak order: we have e.g. $e\ S\ c$ and $c\ S\ b$ but $e\ \neg S\ b$.

$\circlearrowleft$	a	b	c	d	e	f
a	1	1	1	1	1	1
b	1	1	1	1	1	1
c	0	1	1	1	1	1
d	0	0	1	1	1	1
e	0	0	1	1	1	1
f	0	0	0	0	1	1

Figure 2.11. *Matrix representation of a semi-order*

2.5.1.4. *Numerical representation*

Theorem 2.4. *Let A be a finite set. The following propositions are equivalent.*

1) S is a semi-order on A.

2) There is a function $g : A \to \mathbb{R}$ and a constant $q \geq 0$ such that $\forall a, b \in A$:

$$a\ S\ b \Leftrightarrow g(a) \geq g(b) - q.$$

3) There is function $g : A \to \mathbb{R}$ and a function $q : \mathbb{R} \to \mathbb{R}^+$ such that $\forall a, b \in A$:

$$g(a) > g(b) \Rightarrow g(a) + q(g(a)) \geq g(b) + q(g(b))$$

and

$$a\ S\ b \Leftrightarrow g(a) \geq g(b) - q(g(b)).$$

Proof: See [FIS 85], [PIR 97, theorem 3.1], [SCO 58] or [SUP 89, chapter 16].

This result shows that semi-orders naturally arise when objects evaluated on a numerical scale are compared on the basis of the scale, however, differences that are less than a constant threshold are not perceived or are not considered to be significant. The threshold is not necessarily constant provided that we never have $g(a) > g(b)$ and $g(b) + q(g(b)) > g(a) + q(g(a))$. Let us observe that the generalization of this result to arbitrary sets raises delicate problems [BEJ 92, CAN 02, FIS 73, FIS 85].

Remark 2.22. Let us build the numerical representation of the semi-order for which we gave the matrix representation earlier. Having chosen an arbitrary positive value for q, e.g. $q = 1$, the function g is built associating increasing values to the elements f, e, d, c, b, a (i.e. considering the lower elements in the weak order $S^{\pm}$ first), while satisfying the desired numerical representation. In such a way, we obtain: $g(f) = 0$, $g(e) = 0.5$, $g(d) = 1.1$, $g(c) = 1.2$, $g(b) = 2.15$ and $g(a) = 3$.

Remark 2.23. The numerical representation of a semi-order is not unique. All increasing transformation applied to g gives another acceptable representation provided that the same transformation is applied to q. However, all representations of a semi-order cannot be obtained in this way as shown by the following example. The scale that is built is more complex than an ordinal scale.

Example 2.6. Let $A = \{a, b, c, d\}$. Consider the preference structure $S = \{(a, d), (a, a), (b, b), (c, c), (d, d), (a, b), (b, a), (b, c), (c, b), (b, d), (d, b), (c, d), (d, c)\}$. It is easy to check, e.g. using a matrix representation, that this structure is a semi-order. Table 2.1 gives two numerical representations of S that cannot be obtained from one another by an increasing transformation.

	a	b	c	d	threshold
g	2	1.1	1	0	1.5
g'	2	1	1	0	1.5

Table 2.1. *Two numerical representations of a semi-order*

2.5.2. *Interval order*

2.5.2.1. *Definition*

A preference structure S is an interval order if:

– S is complete; and

– S is Ferrers.

This structure generalized all structures introduced so far. As we will later see, it arises naturally when one wishes to compare intervals on an ordinal scale.

Remark 2.24. It is easy to check that an equivalent definition of an interval order consists of saying that S is complete and that all circuits in S have at least two consecutive I arcs.

It is easily checked that, if S is an interval order,

– P is transitive;

– P is Ferrers; and

– $P \cdot I \cdot P \subset P$.

Remark 2.25. The graphical representation of an interval order is characterized by the fact that the three configurations depicted on Figure 2.12 are forbidden (anything can appear on the diagonal).

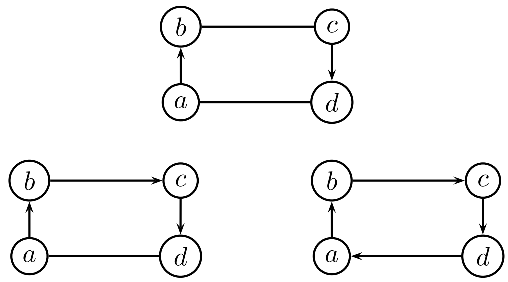

Figure 2.12. *Forbidden configurations in an interval order*

2.5.2.2. *Weak orders associated to an interval order*

Let S be a binary relation on A. Let us define a relation S^+ on A, setting

$$a\ S^+\ b \Leftrightarrow [b\ S\ c \Rightarrow a\ S\ c, \forall c \in A].$$

Similarly, we define the relation S^- setting

$$a\ S^-\ b \Leftrightarrow [c\ S\ a \Rightarrow c\ S\ b, \forall c \in A].$$

The relation S^+ (respectively S^-) is called the right trace (respectively left trace) of S. It is clear that S^+ and S^- are always reflexives and transitives.

The proof of the following result is easy and left to the reader.

Theorem 2.5. *Let S be a reflexive binary relation on A. The following three propositions are equivalent:*

1) S is an interval order;

2) S^+ is complete; and

3) S^- is complete.

Remark 2.26. When S is an interval order, the weak order S^+ (respectively S^-) can be obtained ranking the elements of A according to their out-degree (respectively in-degree) in S.

2.5.2.3. *Matrix representation*

Let us rank the rows of the matrix representation in a way that is compatible with S^+ taking care to rank indifferent elements according to S^+ using an order that is compatible with S^-. Let us perform a similar operation on the columns of the matrix, permuting the roles of S^+ and S^-. We obtain a matrix in which the 1's are separated from the 0's by a stepped frontier that is below the diagonal.

Example 2.7. Let $A = \{a, b, c, d, e, f\}$. Consider the following structure: $S = \{ (a, a), (a, b), (a, c), (a, d), (a, e), (a, f), (b, a), (b, b), (b, c), (b, d), (b, e), (b, f), (c, b), (c, c), (c, d), (c, e), (c, f), (d, c), (d, d), (d, e), (d, f), (e, c), (e, d), (e, e), (e, f), (f, e), (f, f) \}$.

We obtain the following matrix representation:

$\circlearrowleft$	a	b	d	c	e	f
a	1	1	1	1	1	1
b	1	1	1	1	1	1
c	0	1	1	1	1	1
d	0	0	1	1	1	1
e	0	0	1	1	1	1
f	0	0	0	1	1	1

This structure is an interval order. It is not a semi-order since $f\ S\ c$ and $c\ S\ b$ but $f\ \neg S\ d$ and $d\ \neg S\ b$. It is therefore impossible to represent this structure using a stepped matrix with a similar order on rows and columns.

2.5.2.4. *Numerical representation*

The proof of the following result can be found in [PIR 97, theorem 3.11] or [FIS 85].

Theorem 2.6. *Let A be a finite set. The following propositions are equivalent:*

1) S is an interval order on A; and

2) there are two functions $g : A \to \mathbb{R}$ and $q : \mathbb{R} \to \mathbb{R}^+$ such that $\forall a, b \in A$:

$$a\ S\ b \Leftrightarrow g(a) + q(g(a)) \geq g(b).$$

We refer to [BRI 95, CHA 87, FIS 73, FIS 85, NAK 02, OLO 98] for the problems involved in generalizing this result to arbitrary sets.

Remark 2.27. For instance, it is possible to build the numerical representation of the interval order presented earlier as follows. The values of g are arbitrarily chosen provided they increase from the first to the last row of the matrix. The values of $g + q$ are then defined in such a way that they increase from the first to the last column of the matrix and they satisfy the desired representation. For instance, we successively obtain:

$$g(f) = 0, g(e) = 5, g(c) = 10, g(d) = 15, g(b) = 20, g(a) = 25,$$

$$(g + q)(f) = 12, (g + q)(e) = 17, (g + q)(d) = 19,$$

$$(g + q)(c) = 23, (g + q)(b) = 28, (g + q)(a) = 30.$$

Letting $\underline{g} = g$ and $\overline{g} = (g + q)$, it is clear that the numerical representation of an interval order amounts to associating an interval $[\underline{g}, \overline{g}]$ with each $a \in A$ such that:

$$\begin{cases} a\ P\ b \Leftrightarrow \underline{g}(a) > \overline{g}(b), \\ \\ a\ I\ b \Leftrightarrow \begin{cases} \underline{g}(a) \leq \overline{g}(b), \\ \underline{g}(b) \leq \overline{g}(a), \end{cases} \end{cases}$$

which leads to the representation depicted in Figure 2.13.

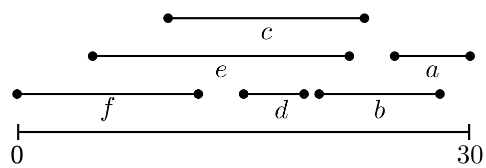

Figure 2.13. *Interval representation of an interval order*

2.5.3. *Remarks*

Remark 2.28. Interval orders may be generalized using a threshold depending on both objects compared. One then obtains a threshold representation of all relations for which the asymmetric part is acyclic [ABB 93, ABB 95, AGA 93, ALE 06, DIA 99, SUB 94]. We do not tackle such models here.

Remark 2.29. In an interval order, the relation P is transitive and hence is acyclic. For all non-empty finite subsets $B \subset A$, $C(B, S)$ is therefore always non-empty. Using one of the structures introduced in this section does not raise major problems when it comes to linking preferences and choices.

Remark 2.30. We saw that when A has a particular structure and that S is a weak order, it is interesting to use such a structure to try to arrive at a numerical representation that is more constrained than an ordinal scale. These extensions make central use of the transitivity of indifference in order to build these numerical representations. It is therefore not simple to do similar things on the basis of a semi-order or an interval order [DOM 71, KRA 67, LUC 73, SUP 89].

Remark 2.31. Building a collective preference that is a semi-order or an interval order does not significantly contribute to the solution of the aggregation problem of weak orders uncovered by Arrow's theorem [SEN 86]. As soon as $|A| \geq 4$, the theorem still holds if the collective preference is required to be complete and Ferrers (or complete and semi-transitive).

2.6. Preference structures with incomparability

In all the structures envisaged so far, we supposed that S was complete. This hypothesis may seem innocuous, in particular when preferences are inferred from observed choices. It is not without problems however. Indeed, it may well happen that:

– information is poor concerning one or several of the elements of A;

– comparing elements of A implies synthesizing on several conflicting points of view; and

– the objects are not familiar to the individual.

In such cases, it may prove useful for preference modeling to use structures that explicitly include incomparability [FLA 83, ROY 85].

2.6.1. *Partial order*

A preference structure S is a partial if:

– S is reflexive;

– S is antisymmetric; and

– S is transitive.

Intuitively, a partial order is a structure in which, given two distinct objects, either object is strictly preferred to the other or the two objects are incomparable, with strict preference being transitive.

Remark 2.32. It is easy to check that, if S is a partial order,

– P is transitive; and

– I is limited to loops.

A fundamental result [DUS 41, FIS 85] shows that all partial orders on a finite set can be obtained intersecting a finite number of total orders on this set. The minimal number of total orders that are needed for this is called the *dimension* of the partial order. This easily implies the following result.

Theorem 2.7. *Let A be a finite set. The following propositions are equivalent:*

1) S is a partial order on A; and

2) there is a function $g : A \to \mathbb{R}$ such that $\forall a, b \in A$:

$$\begin{cases} a\ S\ b \Rightarrow g(a) \geq g(b), \\ g(a) = g(b) \Rightarrow a = b. \end{cases}$$

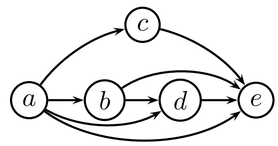

Figure 2.14. *Graphical representation of a partial order*

Example 2.8. Let $A = \{a, b, c, d, e\}$. Consider the preference structure: $S = \{ (a, a)$, (a, b), (a, c), (a, d), (b, b), (b, d), (b, e), (c, c), (c, e), (d, d), (d, e), $(e, e) \}$. A graphical representation of this structure is depicted in Figure 2.14.

It is easy to check that the structure is partial order with dimension 2, obtained by intersecting the two total orders (using obvious notation):

$a > b > d > c > e$ and

$a > c > b > d > e$.

Let us note that the detection of a partial order of dimension 2 can be done in polynomial time. On the contrary, the determination of the dimension of a partial order is NP-difficult [DOI 84, FIS 85].

2.6.2. *Quasi-order*

A preference structure S is a quasi-order if:
– S is reflexive; and
– S is transitive.

Quasi-orders generalize partial orders by allowing indifference between distinct elements, the indifference relation being transitive.

Remark 2.33. It is easy to check that, if S is a quasi-order,
– P is transitive;
– I is transitive;
– $P \cdot I \subset P$; and
– $I \cdot P \subset P$.

As with partial orders, it is easy to show that any quasi-order on a finite set can be obtained intersecting a finite number of weak orders [BOS 02b, DON 98]. This implies the following result.

Theorem 2.8. *Let A be a finite set. The following propositions are equivalent:*

1) S is a quasi-order on A; and

2) there is a function $g : A \to \mathbb{R}$ such that $\forall a, b \in A$:

$$a\ S\ b \Rightarrow g(a) \geq g(b).$$

Remark 2.34. Alternatively, one can build a numerical representation of a quasi-order considering a set of numerical representations of weak orders [OK 02].

Example 2.9. Let $A = \{a, b, c, d, e, f\}$. Consider the preference structure $S = \{(a, a)$, (a, b), (a, c), (a, d), (a, e), (a, f) (b, b), (b, d), (b, e), (b, f), (c, c), (c, e), (c, f), (d, b), (d, d), (d, e), (d, f), (e, e), (e, f), (f, e), $(f, f)\}$. It is easy to check that this is a quasi-order. Its graphical representation is depicted in Figure 2.15.

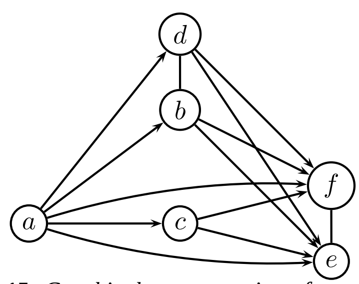

Figure 2.15. *Graphical representation of a quasi-order*

Remark 2.35. It is possible to extend classical models of decision under risk to deal with quasi-orders [AUM 62, FIS 70]. The multi-attribute case was only studied in the finite case [FIS 70, SCO 64]. Let us also mention that allowing for incomparability in the collective preference does not significantly contribute to the solution of the problem uncovered by Arrow's theorem [WEY 84].

Remark 2.36. Roubens and Vincke [ROU 85] proposed definitions of partial semi-orders and interval orders. They allow an intransitive indifference relation at the same time as incomparability situations. We do not detail this point here.

2.6.3. *Synthesis*

We summarize in Table 2.2 the properties of preference structures that have been introduced so far.

Structures	Definition
Total order	S complete S antisymmetric S transitive
Weak order	S complete S transitive
Semi-order	S complete S Ferrers S semi-transitive
Interval order	S complete S Ferrers
Partial order	S reflexive S antisymmetric S transitive
Quasi-order	S reflexive S transitive

Table 2.2. *Common preference structures*

2.7. Conclusion

2.7.1. *Other preference structures*

In all the structures introduced so far, the relation P was transitive and, hence, was acyclic. This seems a natural hypothesis. Abandoning it implies reconsidering the links existing between 'preference' and 'choice' as we already saw. Nevertheless, it is possible to obtain such preferences in experiments [MAY 54, TVE 69] when subjects are asked to compare objects evaluated on several dimensions. They are also common in social choice due to Condorcet's paradox. Indeed, a famous result [MCG 53] shows that with a simple majority, any complete preference structure can be be obtained as the result of the aggregation of individual weak orders. With other aggregation methods, all preference structures may occur [BOU 96].

The literature on Social Choice abounds with studies of adequate choice procedure on the basis of such preferences. The particular case of *tournaments* (complete and antisymmetric relations) have been explored in depth [LAS 97, MOU 86].

More recently, it was shown that it is possible to build numerical representations of such relations [BOU 86, BOU 99, BOU 02, FIS 82, FIS 88, FIS 91a, FIS 91b, FIS 92, TVE 69, VIN 91]. In the models proposed in [BOU 02], we have sets A being Cartesian products (as in decision under uncertainty or in decision with multiple attributes):

$$a \; S \; b \Leftrightarrow F(p_1(a_1, b_1), p_2(a_2, b_2), \ldots, p_n(a_n, b_n)) \geq 0$$

where p_i are functions from A_i^2 to $\mathbb{R}$, F is a function from $\prod_{i=1}^n p_i(A_i^2)$ to $\mathbb{R}$ and where, for example, F can be increasing in all its arguments. This model generalizes the classical additive difference model proposed in [TVE 69] in which:

$$a\ S\ b \Leftrightarrow \sum_{i=1}^{n} \varphi_i(u_i(a_i) - u_i(b_i)) \geq 0$$

where u_i are functions from A_i to $\mathbb{R}$ and φ_i are odd increasing functions on $\mathbb{R}$.

Similarly, in the models studied in [FIS 82, FIS 88] for the case of decision-making under risk, the numerical representation is such that:

$$a\ S\ b \Leftrightarrow \sum_{c \in C} \sum_{c' \in C} p_a(c) p_b(c') \phi(c, c') \geq 0$$

where ϕ is a function from C^2 to $\mathbb{R}$ and $p_a(c)$ is the probability to obtain the consequence $c \in C$ with object a.

A common criticism of such models is that cycles leave the door open to apparently ‘irrational’ behavior and makes an individual vulnerable to Dutch books [RAI 70]. As in the case of decision under risk mentioned earlier, it is not clear that the arguments are actually convincing [FIS 91b].

2.7.2. *Other problems*

This brief survey of classical preference structures used in preference modeling will hopefully give the reader enough clues to tackle a vast and complex literature. This chapter has neglected many important questions, including

– the question of the approximation of preference structure by another one, e.g. the search for a total order at minimal distance of a tournament [BAR 89, BAR 81, BER 72, CHA 92, HUD 96, MON 79, SLA 61];

– the way to collect and validate preference information in a given context [WIN 86];

– the links between preference modeling and the question of meaningfulness in measurement theory [ROB 79];

– the statistical analysis of preference data [COO 64, GRE 88]; and

– deeper questions on the links between value systems and preferences [BRO 91, COW 88, TSO 92, WRI 63].

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