Bi-semiorders with frontiers on finite sets[1]

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Abstract

This paper studies an extension of bi-semiorders in which a "frontier" is added between the various relations used. This extension is motivated by the study of additive representations of ordered partitions and coverings defined on product sets of two components.

Keywords: bi-semiorder, biorder, interval order, semiorder, frontier, conjoint measurement.

1 Introduction

Let $\mathcal{T}$ be a relation between two sets A and Z, i.e., a subset of $A \times Z$. *Biorders* are relations between two sets that lead to a numerical representation in which there are real-valued functions f on A and g on Z such that, for all $a \in A$ and all $p \in Z$,

$$a \mathcal{T} p \Leftrightarrow f(a) > g(p).$$

The name "biorder" comes from Doignon, Ducamp, and Falmagne (1984) and has gained wide acceptance (see Doignon, Ducamp, and Falmagne, 1987, and Nakamura, 2002). This structure was introduced in the literature by Riguet (1951) who used the term "Ferrers relation". It was studied by Ducamp and Falmagne (1969) under the name "bi-quasi-series". Early work on biorders include Bouchet (1971) and Cogis (1976, 1982a,b) (see Monjardet, 1978, and Doignon and Falmagne, 1999, p. 60, for a detailed historical account).

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Biorders are useful to model Guttman scales (Guttman, 1944, 1950). They are also an important tool to study various classes of binary relations, most notably *interval orders* and *semiorders* (Aleskerov, Bouyssou, and Monjardet, 2007, Fishburn, 1985, Pirlot and Vincke, 1992). Indeed, when $A = Z$, an irreflexive biorder is nothing but an interval order, as defined in Fishburn (1970). Adding semitransitivity to irreflexivity leads to semiorders (Luce, 1956, Scott and Suppes, 1958).

In Bouyssou and Marchant (2011) (henceforth, BM11), we have studied an extension of biorders in which there are *two* relations $\mathcal{T}$ and $\mathcal{F}$ between the sets A and Z, leading to what we called *biorders with frontier*. They lead to a numerical representation in which there are real-valued functions f on A and g on Z such that, for all $a \in A$ and all $p \in Z$,

$$\begin{aligned} a\, \mathcal{T}\, p &\Leftrightarrow f(a) > g(p), \\ a\, \mathcal{F}\, p &\Leftrightarrow f(a) = g(p). \end{aligned}$$

With *bi-semiorders*, we have two relations $\mathcal{T}$ and $\mathcal{P}$ between the sets A and Z. The numerical representation involves a real-valued function f on A and a real-valued function g on Z such that, for all $a \in A$ and $p \in Z$,

$$\begin{aligned} a\, \mathcal{P}\, p &\Leftrightarrow f(a) > g(p) + 1, \\ a\, \mathcal{T}\, p &\Leftrightarrow f(a) > g(p). \end{aligned}$$

Necessary and sufficient conditions for the above model were given in Ducamp and Falmagne (1969, Th. 5) when both A and Z are finite sets (note that the term bi-semiorder is used in Fishburn, 1997, with a different meaning)[1].

Bi-semiorders with frontiers will use four relations $\mathcal{P}$, $\mathcal{J}$, $\mathcal{T}$ and $\mathcal{F}$ between the sets A and Z. The numerical representation involves a real-valued function f on A and a real-valued function g on Z such that, for all $a \in A$ and $p \in Z$,

$$\begin{aligned} a\, \mathcal{P}\, p &\Leftrightarrow f(a) > g(p) + 1, \\ a\, \mathcal{J}\, p &\Leftrightarrow f(a) = g(p) + 1, \\ a\, \mathcal{T}\, p &\Leftrightarrow f(a) > g(p), \\ a\, \mathcal{F}\, p &\Leftrightarrow f(a) = g(p). \end{aligned}$$

[1]The fact that two thresholds, the first one at 1 and the other one at 0, are used in the numerical representation of bi-semiorders may lead one to think that there is a link with the study of families of semiorders having a constant threshold representation (see Cozzens and Roberts, 1982, Roubens and Vincke, 1985, ch. 6, Roy and Vincke, 1987 for the case of a family of two semiorders and Doignon, 1987, for the general case). This is misleading. Indeed, Ducamp and Falmagne (1969) have shown that a bi-semiorder is the natural counterpart of a structure involving a *single* semiorder when studying relations between two different sets.

The purpose of this paper is to establish necessary and sufficient conditions for the above model when both A and Z are finite sets.

The paper is organized as follows. Section 2 briefly presents our initial motivation for studying structures with frontiers. Section 3 presents our setting. Results on biorders, biorders with frontier and bi-semiorders are recalled in Section 4. Section 5 presents our results on bi-semiorders with frontiers that are proved in Section 6.

2 Relation to conjoint measurement

2.1 Additive representations of ordered coverings

Our initial motivation for studying biorders and bi-semiorders with frontiers is linked to the following problem. Let $X = X_1 \times X_2 \times \cdots \times X_n$ be a set of objects evaluated on n attributes.

Suppose that we are given an ordered *covering* $\langle C^1, C^2, \ldots, C^r \rangle$ of the set of objects. In such a setting, we know that objects belonging to C^{k+1} are better than objects belonging to C^k but we have no information on the way two objects belonging to the same category compare in terms of preference. The category C^k can have a nonempty intersection with C^{k+1} and C^{k-1}. Its intersection with other categories is always empty, reflecting the ordered nature of the covering.

Consider first an ordered *partition* $\langle C^1, C^2, \ldots, C^r \rangle$. In this case, we are interested in finding real-valued functions u_i on X_i such that, for all $x \in X$ and all $k \in \{1, 2, \ldots, r\}$,

$$x \in C^k \Leftrightarrow \sigma^{k-1} < \sum_{i=1}^{n} u_i(x_i) \leq \sigma^k, \tag{1}$$

with the convention that $\sigma^0 = -\infty$, $\sigma^r = +\infty$ and where $\sigma^1, \sigma^2, \ldots, \sigma^{r-1}$ are real numbers such that $\sigma^1 < \sigma^2 < \cdots < \sigma^{r-1}$. In the case of an ordered *covering* $\langle C^1, C^2, \ldots, C^r \rangle$, the model becomes

$$x \in C^k \Leftrightarrow \sigma^{k-1} \leq \sum_{i=1}^{n} u_i(x_i) \leq \sigma^k, \tag{2}$$

so that, if $\sum_{i=1}^{n} u_i(x_i) = \sigma^{k-1}$, the object x belongs at the same time to C^{k-1} and to C^k, i.e., is at the frontier between these two categories.

The analysis of the above models in the general case requires the use of conjoint measurement techniques (see Bouyssou and Marchant, 2009, 2010, following initial results by Fishburn, Lagarias, Reeds, and Shepp, 1991 and Vind, 1991, 2003).

However, as suggested by the results of Levine (1970), there are some particular cases that can be dealt with in a simpler way. Biorders are useful to study the case of a product set with *two* components and an ordered *partition* with *two* categories. Biorders with frontiers are useful to deal with the case of a product set with *two* components and an *ordered covering* with *two* categories. We mentioned in BM11, Sect. 7, that the case of *three* ordered categories and a product set with *two* components was also quite particular. When the three ordered categories partition the product set, we can indeed use the results on *bi-semiorders* presented in Ducamp and Falmagne (1969, Sect. IV) (see also Ducamp, 1978). The results presented in this paper allows us to deal with the case in which the three ordered categories are a covering, instead of a partition, of the product set.

2.2 Particular cases with two attributes

Consider first the case of ordered partitions of $X = X_1 \times X_2$.

When there are only two attributes and two categories, the additive representation (1) relates more to ordinal than to conjoint measurement. Indeed, in such a case, the problem clearly reduces to finding real-valued functions u_1 on X_1 and u_2 on X_2 such that, for all $x = (x_1, x_2) \in X$,

$$x \in C^2 \Leftrightarrow u_1(x_1) + u_2(x_2) > \sigma. \tag{3}$$

It is easy to see that it is not restrictive to suppose that $\sigma = 0$. Define the relation $\mathcal{T}$ between the sets X_1 and X_2 letting, for all $x_1 \in X_1$ and all $x_2 \in X_2$,

$$x_1 \mathcal{T} x_2 \Leftrightarrow (x_1, x_2) \in C^2.$$

It is clear that asking for a representation in model (3) is equivalent to asking for the existence of two functions f on X_1 and g on X_2 such that

$$x_1 \mathcal{T} x_2 \Leftrightarrow f(x_1) > g(x_2).$$

This explains the link with *biorders*.

Similarly, when there are only two attributes and three categories, building an additive representation (1) reduces to finding real-valued functions u_1 on X_1 and u_2 on X_2 such that, for all $x \in X$,

$$\begin{aligned} x \in C^3 &\Leftrightarrow \lambda < u_1(x_1) + u_2(x_2), \\ x \in C^2 \cup C^3 &\Leftrightarrow \rho < u_1(x_1) + u_2(x_2), \end{aligned} \tag{4}$$

where ρ, λ are two thresholds such that $\rho < \lambda$. As detailed in Ducamp and Falmagne (1969), it is not restrictive to suppose that $\rho = 0$ and $\lambda = 1$.

Define the relations $\mathcal{P}$ and $\mathcal{T}$ between the sets X_1 and X_2 letting, for all $x_1 \in X_1$ and all $x_2 \in X_2$,

$$x_1 \mathcal{P} x_2 \Leftrightarrow (x_1, x_2) \in C^3.$$
$$x_1 \mathcal{T} x_2 \Leftrightarrow (x_1, x_2) \in C^2 \cup C^3.$$

It is clear that asking for a representation in model (4) is equivalent to asking for the existence of two functions f on X_1 and g on X_2 such that

$$x_1 \mathcal{P} x_2 \Leftrightarrow f(x_1) > g(x_2) + 1,$$
$$x_1 \mathcal{T} x_2 \Leftrightarrow f(x_1) > g(x_2).$$

This explains the links with *bi-semiorders*.

We now turn to the case of ordered coverings of $X = X_1 \times X_2$.

Suppose first that there are only two categories C^2 and C^1. Allowing for an hesitation between C^2 and C^1 leads to a model in which it is no more true that $C^2 \cap C^1 = \varnothing$. Objects belonging to $C^2 \cap C^1$ are at the frontier between C^2 and C^1. Define $C^2_> = C^2 \setminus C^1$.

The additive representation (2) can be written as

$$\begin{aligned} x \in C^2_> &\Leftrightarrow u_1(x_1) + u_2(x_2) > \sigma, \\ x \in C^2 \cap C^1 &\Leftrightarrow u_1(x_1) + u_2(x_2) = \sigma, \end{aligned} \tag{5}$$

for all $x \in X$. As before, it is not restrictive to suppose that $\sigma = 0$.

Define the relations $\mathcal{T}$ and $\mathcal{F}$ between the sets X_1 and X_2 letting, for all $x_1 \in X_1$ and all $x_2 \in X_2$,

$$x_1 \mathcal{T} x_2 \Leftrightarrow (x_1, x_2) \in C^2_>,$$
$$x_1 \mathcal{F} x_2 \Leftrightarrow (x_1, x_2) \in C^2 \cap C^1.$$

It is clear that asking for a representation in model (5) is equivalent to asking for the existence of two functions f on X_1 and g on X_2 such that

$$x_1 \mathcal{T} x_2 \Leftrightarrow f(x_1) > g(x_2),$$
$$x_1 \mathcal{F} x_2 \Leftrightarrow f(x_1) = g(x_2).$$

This explains the link with *biorders with frontier*.

Suppose finally that there are three ordered categories C^3, C^2 and C^1 and that we allow hesitations between two consecutive categories. Define $C^3_> = C^3 \setminus C^2$ and $C^2_> = C^2 \setminus C^1$.

The additive representation (2) can be written as

$$\begin{aligned} x \in C^3_> &\Leftrightarrow u_1(x_1) + u_2(x_2) > \lambda, \\ x \in C^3 \cap C^2 &\Leftrightarrow u_1(x_1) + u_2(x_2) = \lambda. \\ x \in C^2_> &\Leftrightarrow \rho < u_1(x_1) + u_2(x_2) < \lambda, \\ x \in C^2 \cap C^1 &\Leftrightarrow u_1(x_1) + u_2(x_2) = \rho, \\ x \in C^1 \setminus C^2 &\Leftrightarrow u_1(x_1) + u_2(x_2) < \rho, \end{aligned} \tag{6}$$

for all $x \in X$, where ρ, λ are two thresholds such that $\rho < \lambda$. Again, it is not restrictive to suppose that $\rho = 0$ and $\lambda = 1$.

Define the relations $\mathcal{P}$, $\mathcal{J}$, $\mathcal{T}$ and $\mathcal{F}$ between the sets X_1 and X_2 letting, for all $x_1 \in X_1$ and all $x_2 \in X_2$,

$$\begin{aligned}
x_1 \, \mathcal{P} \, x_2 &\Leftrightarrow (x_1, x_2) \in C^3_>, \\
x_1 \, \mathcal{J} \, x_2 &\Leftrightarrow (x_1, x_2) \in C^3 \cap C^2, \\
x_1 \, \mathcal{T} \, x_2 &\Leftrightarrow (x_1, x_2) \in C^3 \cup C^2_>, \\
x_1 \, \mathcal{F} \, x_2 &\Leftrightarrow (x_1, x_2) \in C^2 \cap C^1,
\end{aligned}$$

It is clear that asking for a representation in model (6) is equivalent to asking for the existence of two functions f on X_1 and g on X_2 such that

$$\begin{aligned}
x_1 \, \mathcal{P} \, x_2 &\Leftrightarrow f(x_1) > g(x_2) + 1, \\
x_1 \, \mathcal{J} \, x_2 &\Leftrightarrow f(x_1) = g(x_2) + 1, \\
x_1 \, \mathcal{T} \, x_2 &\Leftrightarrow f(x_1) > g(x_2), \\
x_1 \, \mathcal{F} \, x_2 &\Leftrightarrow f(x_1) = g(x_2).
\end{aligned}$$

This explains the link with *bi-semiorders with frontiers*, which are the subject of this paper.

3 Definitions and Notation

We follow the definitions and notation presented in BM11.

3.1 Binary relations between two sets

Let $A = \{a, b, \dots\}$ and $Z = \{p, q, \dots\}$ be two sets. Following Ducamp and Falmagne (1969), we define a binary relation $\mathcal{V}$ between A and Z to be a subset of $A \times Z$. We often write $a \, \mathcal{V} \, p$ instead of $(a, p) \in \mathcal{V}$. A binary relation on a set X is a binary relation between X and X.

Let $\mathcal{V}$ be a relation between A and Z. Define the *left trace* of $\mathcal{V}$ as the binary relation $\succsim^\ell_\mathcal{V}$ on A defined letting, for all $a, b \in A$,

$$a \succsim^\ell_\mathcal{V} b \Leftrightarrow [b \, \mathcal{V} \, p \Rightarrow a \, \mathcal{V} \, p, \text{ for all } p \in Z].$$

Similarly, define the *right trace* of $\mathcal{V}$ as the binary relation $\succsim^r_\mathcal{V}$ on Z defined letting, for all $p, q \in Z$,

$$p \succsim^r_\mathcal{V} q \Leftrightarrow [a \, \mathcal{V} \, p \Rightarrow a \, \mathcal{V} \, q, \text{ for all } a \in A].$$

By construction, the relations $\succsim^\ell_\mathcal{V}$ and $\succsim^r_\mathcal{V}$ are reflexive and transitive.

A binary relation $\mathcal{V}$ between A and Z is said to be a *biorder* if it is Ferrers, i.e., for all $a, b \in A$ and all $p, q \in Z$, we have:

$$[a \mathcal{V} p \text{ and } b \mathcal{V} q] \Rightarrow [a \mathcal{V} q \text{ or } b \mathcal{V} p].$$

A simple check shows that $\mathcal{V}$ is Ferrers if and only if (iff) $\succsim^{\ell}_{\mathcal{V}}$ is complete iff $\succsim^{r}_{\mathcal{V}}$ is complete (see Doignon et al., 1984, Proposition 2, p. 78).

Let $\mathcal{V}$ be a relation between A and Z. Its *complement* is the relation $\mathcal{V}^c$ between A and Z such that for all $a \in A$ and $p \in Z$, $a\ \mathcal{V}^c\ p \Leftrightarrow Not[a\ \mathcal{V}\ p]$. The *dual* of $\mathcal{V}$ is the relation $\mathcal{V}^d$ between Z and A such that, for all $a \in A$ and $p \in Z$, $p\ \mathcal{V}^d\ a \Leftrightarrow a\ \mathcal{V}\ p$. Its *codual* $\mathcal{V}^{cd}$ is the relation between Z and A such that, for all $a \in A$ and $p \in Z$, $p\ \mathcal{V}^{cd}\ a \Leftrightarrow Not[a\ \mathcal{V}\ p]$. It is easy to check that if $\mathcal{V}$ is Ferrers iff $\mathcal{V}^{cd}$ (or $\mathcal{V}^c$, or $\mathcal{V}^d$) is Ferrers.

Suppose that $\mathcal{V}$ is a relation between A and Z and that $\mathcal{W}$ is a binary relation between Z and K. We define the *product* of $\mathcal{V}$ and $\mathcal{W}$ as the binary relation $\mathcal{V}\mathcal{W}$ between A and K such that, for all $a \in A$ and all $k \in K$, $a\ \mathcal{V}\mathcal{W}\ k$ iff $[a\ \mathcal{V}\ p$ and $p\ \mathcal{W}\ k$, for some $p \in Z]$. The Ferrers property can therefore be expressed compactly as $\mathcal{V}\mathcal{V}^{cd}\mathcal{V} \subseteq \mathcal{V}$.

For our purposes, when studying a relation between A and Z, it is not restrictive to suppose that the sets A and Z are disjoint: if they are not, we may build a disjoint duplication of these sets as done in Doignon et al. (1984, Definition 4, p. 79). We will suppose so, without explicit mention, whenever needed.

3.2 Binary relations on a set

Let V be a binary relation[2] on X.

The *asymmetric part* (resp. *symmetric part*, *symmetric complement*) of V is the binary relation V^a (resp. V^s, V^{sc}) on X that is equal to $V \cap V^{cd}$ (resp. $V \cap V^d$, $V^c \cap V^{cd}$). For instance, we have $x\ V^{sc}\ y \Leftrightarrow [x\ V^c\ y$ and $x\ V^{cd}\ y] \Leftrightarrow [Not[x\ V\ y]$ and $Not[y\ V\ x]]$.

Whenever we use the symbol $\succsim$ to denote a binary relation on a set X, it will be understood that $\succ$ (resp. $\sim$) denotes its asymmetric part (resp. symmetric part). The same convention will hold if $\succsim$ is subscripted and/or superscripted.

A binary relation that is complete (for all $x, y \in X$, $x\ V\ y$ or $y\ V\ x$) and transitive is said to be a *weak order*. If V and W are two weak orders on X, we say that V *refines* W if $V \subseteq W$. This implies $V^s \subseteq W^s$ and $W^a \subseteq V^a$.

The *trace* of a binary relation V on X is the binary relation $\succsim_V$ on X that is equal to $\succsim^{\ell}_V \cap \succsim^{r}_V$. The relations $\sim_V$, $\sim^{\ell}_V$ and $\sim^{r}_V$ are clearly reflexive, symmetric

[2]We use the following typographic convention. Relations between two sets will be denoted using a calligraphic symbol. Relations on a set will be denoted using a non-calligraphic symbol.

and transitive, i.e., are equivalences. Whenever E is an equivalence on a set X, we denote by X/E the set of equivalence classes of X under E.

A binary relation V on X is said to be *semitransitive* if, for all $x, y, z, w \in X$,

$$[x\ V\ y \text{ and } y\ V\ z] \Rightarrow [x\ V\ w \text{ or } w\ V\ z],$$

which can be written more compactly as $V^{cd}\, V\, V \subseteq V$ (or, equivalently, as $V\, V\, V^{cd} \subseteq V$).

A simple check shows that the trace $\succsim_V$ of a relation V is complete iff V is Ferrers and semitransitive. In this case the left and right traces are not contradictory, i.e., it is never true that $x \succ_V^{\ell} y$ and $y \succ_V^{r} x$, for some $x, y \in X$ (for more details, see, e.g., Fishburn, 1985, Monjardet, 1978, Pirlot and Vincke, 1997, or Suppes, Krantz, Luce, and Tversky, 1989, Ch. 16).

A binary relation V on X is an *interval order* if it is irreflexive and Ferrers. A *semiorder* is a semitransitive interval order.

4 Biorders and Bi-semiorders

This section recalls a number of useful results on the numerical representation of biorders, biorders with frontier, and bi-semiorders. We follow Ducamp and Falmagne (1969), Doignon et al. (1984) (for biorders), BM11 (for biorders with frontier), and Ducamp and Falmagne (1969) (for bi-semiorders).

4.1 Biorders

The main result on the numerical representation of biorders on finite sets is the following.

Proposition 1 (Prop. 4, p. 79 in Doignon et al., 1984)
Let A and Z be finite *sets and $\mathcal{T}$ be a relation between A and Z. The following statements are equivalent.*

1. *$\mathcal{T}$ is Ferrers.*
2. *There are a real-valued function f on A and a real-valued function g on Z such that, for all $a \in A$ and $p \in Z$,*

$$a\ \mathcal{T}\ p \Leftrightarrow f(a) \geq g(p).$$

3. *There are a real-valued function f on A and a real-valued function g on Z such that, for all $a \in A$ and $p \in Z$,*

$$a\ \mathcal{T}\ p \Leftrightarrow f(a) > g(p).$$

Furthermore, the functions f and g used in statements 2 or 3 above can always be chosen in such a way that, for all $a, b \in A$ and $p, q \in Z$,

$$\begin{aligned} a \succsim^{\ell}_{\mathcal{T}} b &\Leftrightarrow f(a) \geq f(b), \\ p \succsim^{r}_{\mathcal{T}} q &\Leftrightarrow g(p) \geq g(q). \end{aligned}$$

This result holds, without modification, when both A and Z are countably infinite. Doignon et al. (1984) and Nakamura (2002) have presented necessary order-denseness conditions allowing to extend the result to the general case.

4.2 Biorders with frontier

Consider now two disjoint relations $\mathcal{T}$ and $\mathcal{F}$ between the sets A and Z. Let $\mathcal{R} = \mathcal{T} \cup \mathcal{F}$. We investigate below the conditions on $\mathcal{T}$ and $\mathcal{F}$ such that there are a real-valued function f on A and a real-valued function g on Z such that, for all $a \in A$ and $p \in Z$,

$$a \,\mathcal{T}\, p \Leftrightarrow f(a) > g(p), \tag{7a}$$

$$a \,\mathcal{F}\, p \Leftrightarrow f(a) = g(p). \tag{7b}$$

Notice that relations $\mathcal{F}$ satisfying (7b) have been studied by Ducamp and Falmagne (1969) under the name "bi-classificatory systems" and by Doignon and Falmagne (1984) who call them "matching relations".

As above, let $\succsim^{\ell}_{\mathcal{T}}$ (resp. $\succsim^{r}_{\mathcal{T}}$) be the trace of $\mathcal{T}$ on A (resp. on Z). Similarly, let $\succsim^{\ell}_{\mathcal{R}}$ (resp. $\succsim^{r}_{\mathcal{R}}$) be the trace of $\mathcal{R}$ on A (resp. on Z). Define

$$\succsim^{\ell}_{\star} = \succsim^{\ell}_{\mathcal{T}} \cap \succsim^{\ell}_{\mathcal{R}} \quad \text{and} \quad \succsim^{r}_{\star} = \succsim^{r}_{\mathcal{T}} \cap \succsim^{r}_{\mathcal{R}}.$$

The relations $\succsim^{\ell}_{\mathcal{T}}$, $\succsim^{\ell}_{\mathcal{R}}$, $\succsim^{\ell}_{\star}$, $\succsim^{r}_{\mathcal{T}}$, $\succsim^{r}_{\mathcal{R}}$, $\succsim^{r}_{\star}$ are always reflexive and transitive. We know that $\succsim^{\ell}_{\mathcal{T}}$ is complete iff $\succsim^{r}_{\mathcal{T}}$ is complete iff $\mathcal{T}$ is a biorder. Similarly, $\succsim^{\ell}_{\mathcal{R}}$ is complete iff $\succsim^{r}_{\mathcal{R}}$ is complete iff $\mathcal{R}$ is a biorder.

It is clear that (7) implies that both $\mathcal{T}$ and $\mathcal{R}$ are biorders. The next two conditions capture the fact that the relation $\mathcal{F}$ is "thin" in model (7). Indeed, suppose that $a \,\mathcal{F}\, p$ and $b \,\mathcal{F}\, p$. This implies $f(a) = g(p)$ and $f(b) = g(p)$, so that $f(a) = f(b)$. Hence, for all $q \in Z$, we have $a \,\mathcal{F}\, q \Leftrightarrow b \,\mathcal{F}\, q$ and $a \,\mathcal{T}\, q \Leftrightarrow b \,\mathcal{T}\, q$.

We say that the pair of relations $\mathcal{T}$ and $\mathcal{F}$ is *left thin* if, for all $a, b \in A$ and $p \in Z$,

$$[a \,\mathcal{F}\, p \text{ and } b \,\mathcal{F}\, p] \Rightarrow a \sim^{\ell}_{\star} b.$$

Similarly, we say that the pair of relations $\mathcal{T}$ and $\mathcal{F}$ is *right thin* if, for all $a \in A$ and $p, q \in Z$,

$$[a \,\mathcal{F}\, p \text{ and } a \,\mathcal{F}\, q] \Rightarrow p \sim^{r}_{\star} q.$$

Observe that left thinness (resp. right thinness) may be formulated as $\mathcal{F}\mathcal{F}^d \subseteq \sim^\ell_\star$ (resp. $\mathcal{F}^d\mathcal{F} \subseteq \sim^r_\star$). We say that *thinness* holds if left and right thinness are satisfied.

The central result on the numerical representation of biorders with frontier on finite sets is as follows.

Proposition 2 (Prop. 11 in BM11)
Let A and Z be finite *sets and let $\mathcal{T}$ and $\mathcal{F}$ be a pair of disjoint relations between A and Z. There are real-valued functions f on A and g on Z such that* (7) *holds if and only if $\mathcal{T}$ is a biorder, $\mathcal{R} = \mathcal{T} \cup \mathcal{F}$ is a biorder and thinness holds.*

Furthermore, the functions f and g can always be chosen in such a way that, for all $a, b \in A$ and $p, q \in Z$,

$$\begin{aligned} a \succsim^\ell_\star b \Leftrightarrow f(a) \geq f(b), \\ p \succsim^r_\star q \Leftrightarrow g(p) \geq g(q). \end{aligned} \tag{8}$$

BM11 have shown that the conditions used in the above result are independent. As with biorders, the result holds without modification for countably infinite sets. Adding appropriate order-denseness conditions, it can be extended to cover the general case (see BM11).

4.3 Bi-semiorders

Let $\mathcal{T}$ and $\mathcal{P}$ be two relations between the sets A and Z. We consider a model in which there are a real-valued function f on A and a real-valued function g on Z, such that, for all $a \in A$ and $p \in Z$,

$$a \, \mathcal{P} \, p \Leftrightarrow f(a) > g(p) + 1, \tag{9a}$$

$$a \, \mathcal{T} \, p \Leftrightarrow f(a) > g(p). \tag{9b}$$

Pairs of relations $\mathcal{T}$ and $\mathcal{P}$ admitting such a representation are called *bi-semiorders* in Ducamp and Falmagne (1969), who consider the case in which both A and Z are finite.

An obvious necessary condition for (9) is that $\mathcal{P} \subseteq \mathcal{T}$. As before, the left (resp. right) trace of $\mathcal{T}$ is denoted by $\succsim^\ell_\mathcal{T}$ (resp. $\succsim^r_\mathcal{T}$). Similarly the left (resp. right) trace of $\mathcal{P}$ is denoted by $\succsim^\ell_\mathcal{P}$ (resp. $\succsim^r_\mathcal{P}$). Define $\succsim^\ell_\circ = \succsim^\ell_\mathcal{T} \cap \succsim^\ell_\mathcal{P}$ and $\succsim^r_\circ = \succsim^r_\mathcal{T} \cap \succsim^r_\mathcal{P}$. By construction, the six relations $\succsim^\ell_\mathcal{T}$, $\succsim^r_\mathcal{T}$, $\succsim^\ell_\mathcal{P}$, $\succsim^r_\mathcal{P}$, $\succsim^\ell_\circ$ and $\succsim^r_\circ$ are reflexive and transitive. We know that imposing that $\mathcal{T}$ and $\mathcal{P}$ are Ferrers will imply the completeness of $\succsim^\ell_\mathcal{T}$, $\succsim^r_\mathcal{T}$, $\succsim^\ell_\mathcal{P}$ and $\succsim^r_\mathcal{P}$. It remains to impose conditions that will ensure that $\succsim^\ell_\mathcal{T}$ and $\succsim^\ell_\mathcal{P}$ (resp. $\succsim^r_\mathcal{T}$ and $\succsim^r_\mathcal{P}$) are compatible.

In order to do so, Ducamp and Falmagne (1969) introduce the following pair of conditions, for all $a, b \in A$ and all $p, q \in Z$,

$$\left.\begin{array}{c} a \mathcal{P} p \\ \text{and} \\ b \mathcal{T} q \end{array}\right\} \Rightarrow \left\{\begin{array}{c} b \mathcal{P} p \\ \text{or} \\ a \mathcal{T} q. \end{array}\right. \tag{10}$$

$$\left.\begin{array}{c} a \mathcal{P} p \\ \text{and} \\ b \mathcal{T} q \end{array}\right\} \Rightarrow \left\{\begin{array}{c} a \mathcal{P} q \\ \text{or} \\ b \mathcal{T} p. \end{array}\right. \tag{11}$$

It is easy to check that (9) implies that the pair of relations $\mathcal{T}$ and $\mathcal{P}$ satisfies (10) and (11). Moreover, when $\mathcal{T}$ and $\mathcal{P}$ are Ferrers and (10) and (11) hold, the two relations $\succsim_\circ^\ell$ and $\succsim_\circ^r$ are complete.

Ducamp and Falmagne (1969, p. 377) have given examples showing that the following four conditions are independent: $\mathcal{T}$ is a biorder, $\mathcal{P}$ is a biorder, (10) and (11). This leads to the central result on the numerical representation of bi-semiorders on finite sets.

Proposition 3 (Th. 5, p. 377 in Ducamp and Falmagne, 1969)
Let A and Z be finite *sets. Let $\mathcal{T}$ and $\mathcal{P}$ be two relations between A and Z. There are real-valued functions f on A and g on Z such that* (9) *holds if and only if $\mathcal{P}$ and $\mathcal{T}$ are biorders satisfying conditions* (10) *and* (11) *and such that $\mathcal{P} \subseteq \mathcal{T}$.*

Moreover, the functions f and g can always be chosen in such a way that, for all $a, b \in A$ and $p, q \in Z$,

$$\begin{array}{l} a \succsim_\circ^\ell b \Leftrightarrow f(a) \geq f(b), \\ p \succsim_\circ^r q \Leftrightarrow g(p) \geq g(q). \end{array} \tag{12}$$

Our results below for bi-semiorders with frontiers will heavily rely on the method of proof used by Ducamp and Falmagne (1969).

5 Bi-semiorders with frontiers

5.1 Definitions

Consider now four relations $\mathcal{P}$, $\mathcal{J}$, $\mathcal{T}$ and $\mathcal{F}$ between the sets A and Z. We are interested in a model in which there are a real-valued function f on A and a real-valued function g on Z such that, for all $a \in A$ and $p \in Z$,

$$a \mathcal{P} p \Leftrightarrow f(a) > g(p) + 1, \tag{13a}$$

$$a \mathcal{J} p \Leftrightarrow f(a) = g(p) + 1, \tag{13b}$$

$$a \mathcal{T} p \Leftrightarrow f(a) > g(p), \tag{13c}$$

$$a \mathcal{F} p \Leftrightarrow f(a) = g(p). \tag{13d}$$

This model implies that $\mathcal{P} \cap \mathcal{J} = \varnothing$. We define $\mathcal{S} = \mathcal{P} \cup \mathcal{J}$, so that

$$a \mathrel{\mathcal{S}} p \Leftrightarrow f(a) \geq g(p) + 1.$$

Similarly, this model implies that $\mathcal{T} \cap \mathcal{F} = \varnothing$. We define $\mathcal{R} = \mathcal{T} \cup \mathcal{F}$, so that

$$a \mathrel{\mathcal{R}} p \Leftrightarrow f(a) \geq g(p).$$

Clearly, we also have that $\mathcal{J} \cap \mathcal{F} = \varnothing$ and $\mathcal{P} \cup \mathcal{J} = \mathcal{S} \subseteq \mathcal{T}$.

Our primitives consist in four relations $\mathcal{P}$, $\mathcal{J}$, $\mathcal{T}$ and $\mathcal{F}$ between the sets A and Z. Our aim is to establish conditions on these four relations leading to the existence of a representation defined by (13). We will suppose throughout that $\mathcal{P} \cap \mathcal{J} = \varnothing$, $\mathcal{T} \cap \mathcal{F} = \varnothing$, $\mathcal{J} \cap \mathcal{F} = \varnothing$, and $\mathcal{P} \cup \mathcal{J} = \mathcal{S} \subseteq \mathcal{T}$. In particular, we have $\mathcal{P} \subseteq \mathcal{S} \subseteq \mathcal{T} \subseteq \mathcal{R}$. This is summarized below:

$$\begin{array}{ccccccc} \mathcal{P} & \subseteq & \mathcal{S} & \subseteq & \mathcal{T} & \subseteq & \mathcal{R} \\ & & \parallel & & & & \parallel \\ & & \mathcal{P} \cup \mathcal{J} & & & & \mathcal{T} \cup \mathcal{F} \end{array}$$

The interpretation is that $\mathcal{J}$ is at the frontier of $\mathcal{P}$. Similarly, $\mathcal{F}$ is at the frontier of $\mathcal{T}$.

We define the relation $\succsim^{\ell}_{\diamond}$ on A letting, for all $a, b \in A$,

$$a \succsim^{\ell}_{\diamond} b \Leftrightarrow \left\{ \begin{array}{c} b \mathrel{\mathcal{P}} r \Rightarrow a \mathrel{\mathcal{P}} r, \\ b \mathrel{\mathcal{S}} r \Rightarrow a \mathrel{\mathcal{S}} r, \\ b \mathrel{\mathcal{T}} r \Rightarrow a \mathrel{\mathcal{T}} r, \\ b \mathrel{\mathcal{R}} r \Rightarrow a \mathrel{\mathcal{R}} r, \end{array} \right\} \text{ for all } r \in Z,$$

Similarly, we define the relation $\succsim^{r}_{\diamond}$ on A letting, for all $p, q \in Z$,

$$p \succsim^{r}_{\diamond} q \Leftrightarrow \left\{ \begin{array}{c} c \mathrel{\mathcal{P}} p \Rightarrow c \mathrel{\mathcal{P}} q, \\ c \mathrel{\mathcal{S}} p \Rightarrow c \mathrel{\mathcal{S}} q, \\ c \mathrel{\mathcal{T}} p \Rightarrow c \mathrel{\mathcal{T}} q, \\ c \mathrel{\mathcal{R}} p \Rightarrow c \mathrel{\mathcal{R}} q, \end{array} \right\} \text{ for all } c \in A,$$

By construction, both $\succsim^{\ell}_{\diamond}$ and $\succsim^{r}_{\diamond}$ are reflexive and transitive.

5.2 Traces

It is clear that (13) implies that that $\mathcal{P}$, $\mathcal{S}$, $\mathcal{T}$, and $\mathcal{R}$ must be biorders. As above, we also have to impose conditions that ensure that the various traces of these relations are compatible, so that both $\succsim^{\ell}_{\diamond}$ on A and $\succsim^{r}_{\diamond}$ on Z are complete. We have to suppose that, for all $a, b \in A$ and all $p, q \in Z$,

$$\left.\begin{array}{c} a \,\mathcal{P}\, p \\ \text{and} \\ b \,\mathcal{S}\, q \end{array}\right\} \Rightarrow \left\{\begin{array}{c} b \,\mathcal{P}\, p \\ \text{or} \\ a \,\mathcal{S}\, q, \end{array}\right. \tag{14}$$

$$\left.\begin{array}{c} a \,\mathcal{P}\, p \\ \text{and} \\ b \,\mathcal{S}\, q \end{array}\right\} \Rightarrow \left\{\begin{array}{c} a \,\mathcal{P}\, q \\ \text{or} \\ b \,\mathcal{S}\, p, \end{array}\right. \tag{15}$$

$$\left.\begin{array}{c} a \,\mathcal{T}\, p \\ \text{and} \\ b \,\mathcal{R}\, q \end{array}\right\} \Rightarrow \left\{\begin{array}{c} b \,\mathcal{T}\, p \\ \text{or} \\ a \,\mathcal{R}\, q, \end{array}\right. \tag{16}$$

$$\left.\begin{array}{c} a \,\mathcal{T}\, p \\ \text{and} \\ b \,\mathcal{R}\, q \end{array}\right\} \Rightarrow \left\{\begin{array}{c} a \,\mathcal{T}\, q \\ \text{or} \\ b \,\mathcal{R}\, p, \end{array}\right. \tag{17}$$

$$\left.\begin{array}{c} a \,\mathcal{P}\, p \\ \text{and} \\ b \,\mathcal{T}\, q \end{array}\right\} \Rightarrow \left\{\begin{array}{c} b \,\mathcal{P}\, p \\ \text{or} \\ a \,\mathcal{T}\, q, \end{array}\right. \tag{18}$$

$$\left.\begin{array}{c} a \,\mathcal{P}\, p \\ \text{and} \\ b \,\mathcal{T}\, q \end{array}\right\} \Rightarrow \left\{\begin{array}{c} a \,\mathcal{P}\, q \\ \text{or} \\ b \,\mathcal{T}\, p, \end{array}\right. \tag{19}$$

$$\left.\begin{array}{c} a \,\mathcal{P}\, p \\ \text{and} \\ b \,\mathcal{R}\, q \end{array}\right\} \Rightarrow \left\{\begin{array}{c} b \,\mathcal{P}\, p \\ \text{or} \\ a \,\mathcal{R}\, q, \end{array}\right. \tag{20}$$

$$\left.\begin{array}{c} a \,\mathcal{P}\, p \\ \text{and} \\ b \,\mathcal{R}\, q \end{array}\right\} \Rightarrow \left\{\begin{array}{c} a \,\mathcal{P}\, q \\ \text{or} \\ b \,\mathcal{R}\, p, \end{array}\right. \tag{21}$$

$$\left.\begin{array}{c} a \,\mathcal{S}\, p \\ \text{and} \\ b \,\mathcal{T}\, q \end{array}\right\} \Rightarrow \left\{\begin{array}{c} b \,\mathcal{S}\, p \\ \text{or} \\ a \,\mathcal{T}\, q, \end{array}\right. \tag{22}$$

$$\left.\begin{array}{c} a \,\mathcal{S}\, p \\ \text{and} \\ b \,\mathcal{T}\, q \end{array}\right\} \Rightarrow \left\{\begin{array}{c} a \,\mathcal{S}\, q \\ \text{or} \\ b \,\mathcal{T}\, p, \end{array}\right. \tag{23}$$

$$\left.\begin{array}{c} a \,\mathcal{S}\, p \\ \text{and} \\ b \,\mathcal{R}\, q \end{array}\right\} \Rightarrow \left\{\begin{array}{c} b \,\mathcal{S}\, p \\ \text{or} \\ a \,\mathcal{R}\, q, \end{array}\right. \tag{24}$$

$$\left.\begin{array}{c} a \,\mathcal{S}\, p \\ \text{and} \\ b \,\mathcal{R}\, q \end{array}\right\} \Rightarrow \left\{\begin{array}{c} a \,\mathcal{S}\, q \\ \text{or} \\ b \,\mathcal{R}\, p. \end{array}\right. \tag{25}$$

We summarize the consequences of the conditions introduced so far in the following:

Lemma 4

1. *Suppose that there are a real-valued function f on A and a real-valued function g on Z such that* (13) *holds. Then conditions* (14–25) *hold.*

2. *Suppose that the four relations $\mathcal{P}$, $\mathcal{J}$, $\mathcal{T}$ and $\mathcal{F}$ are such that, $\mathcal{P} \cap \mathcal{J} = \varnothing$, $\mathcal{T} \cap \mathcal{F} = \varnothing$, $\mathcal{J} \cap \mathcal{F} = \varnothing$, and $\mathcal{S} \subseteq \mathcal{T}$. Suppose furthermore that $\mathcal{P}$, $\mathcal{S}$, $\mathcal{T}$, and $\mathcal{R}$ are biorders. If conditions* (14), (16), (18), (20), (22), *and* (24) *hold, then the relation $\succsim^{\ell}_{\diamond}$ is a weak order.*

3. *Suppose that the four relations $\mathcal{P}$, $\mathcal{J}$, $\mathcal{T}$ and $\mathcal{F}$ are such that, $\mathcal{P} \cap \mathcal{J} = \varnothing$, $\mathcal{T} \cap \mathcal{F} = \varnothing$, $\mathcal{J} \cap \mathcal{F} = \varnothing$, and $\mathcal{S} \subseteq \mathcal{T}$. Suppose furthermore that $\mathcal{P}$, $\mathcal{S}$, $\mathcal{T}$, and $\mathcal{R}$ are biorders. If conditions* (15), (17), (19), (21), (23), *and* (25) *hold, then the relation $\succsim^{r}_{\diamond}$ is a weak order.*

Proof

Part 1 is easily shown. Let us prove Part 2, the proof of Part 3 being similar.

Suppose that $Not[a \succsim^{\ell}_{\diamond} b]$. This implies that, for some $p \in Z$, we have:

$$b \,\mathcal{P}\, p \text{ and } Not[a \,\mathcal{P}\, p] \text{ or} \tag{26a}$$

$$b \,\mathcal{S}\, p \text{ and } Not[a \,\mathcal{S}\, p] \text{ or} \tag{26b}$$

$$b \,\mathcal{T}\, p \text{ and } Not[a \,\mathcal{T}\, p] \text{ or} \tag{26c}$$

$$b \,\mathcal{R}\, p \text{ and } Not[a \,\mathcal{R}\, p]. \tag{26d}$$

Similarly, $Not[b \succsim^{\ell}_{\diamond} a]$ imply that, for some $q \in Z$, we have:

$$a \,\mathcal{P}\, q \text{ and } Not[b \,\mathcal{P}\, q] \text{ or} \tag{26e}$$

$$a \,\mathcal{S}\, q \text{ and } Not[b \,\mathcal{S}\, q] \text{ or} \tag{26f}$$

$$a \,\mathcal{T}\, q \text{ and } Not[b \,\mathcal{T}\, q] \text{ or} \tag{26g}$$

$$a \,\mathcal{R}\, q \text{ and } Not[b \,\mathcal{R}\, q]. \tag{26h}$$

There are 16 cases to examine. It is clear that (26a) and (26e) violates the fact that $\mathcal{P}$ is a biorder. Similarly, (26b) and (26f) violates the fact that $\mathcal{S}$ is a biorder, (26c) and (26g) violates the fact that $\mathcal{T}$ is a biorder, and (26d) and (26h) violates the fact that $\mathcal{R}$ is a biorder.

This leaves 12 cases that are dealt with as follows.

$$\left.\begin{array}{l}[\text{(26a) and (26f)}]\\ [\text{(26b) and (26e)}]\end{array}\right\}\quad \text{are identical and violate (14)}$$

$$\left.\begin{array}{l}[\text{(26a) and (26g)}]\\ [\text{(26c) and (26e)}]\end{array}\right\}\quad \text{are identical and violate (18)}$$

$$\left.\begin{array}{l}[\text{(26a) and (26h)}]\\ [\text{(26d) and (26e)}]\end{array}\right\}\quad \text{are identical and violate (20)}$$

$$\left.\begin{array}{l}[\text{(26b) and (26g)}]\\ [\text{(26c) and (26f)}]\end{array}\right\}\quad \text{are identical and violate (22)}$$

$$\left.\begin{array}{l}[\text{(26b) and (26h)}]\\ [\text{(26d) and (26f)}]\end{array}\right\}\quad \text{are identical and violate (24)}$$

$$\left.\begin{array}{l}[\text{(26c) and (26h)}]\\ [\text{(26d) and (26g)}]\end{array}\right\}\quad \text{are identical and violate (16)}$$

□

Because we are now manipulating four relations, we need a notion of thinness that is stronger than the one used above for biorders with frontier.

We say that the relations $\mathcal{P}$, $\mathcal{S}$, $\mathcal{T}$, and $\mathcal{R}$ satisfy *strong thinness* for $\mathcal{F}$ on A if, for all $a, b \in A$ and $p, q \in Z$,

$$[a \,\mathcal{F}\, p \text{ and } b \,\mathcal{F}\, p] \Rightarrow a \sim^{\ell}_{\diamond} b.$$

Similarly, we say that $\mathcal{P}$, $\mathcal{S}$, $\mathcal{T}$, and $\mathcal{R}$ satisfy *strong thinness* for $\mathcal{F}$ on Z if, for all $a, b \in A$ and $p, q \in Z$,

$$[a \mathcal{F} p \text{ and } a \mathcal{F} q] \Rightarrow p \sim_{\diamond}^{r} q.$$

We say that $\mathcal{P}$, $\mathcal{S}$, $\mathcal{T}$, and $\mathcal{R}$ satisfy *strong thinness* for $\mathcal{J}$ on A if, for all $a, b \in A$ and $p, q \in Z$,

$$[a \mathcal{J} p \text{ and } b \mathcal{J} p] \Rightarrow a \sim_{\diamond}^{\ell} b.$$

Similarly, we say $\mathcal{P}$, $\mathcal{S}$, $\mathcal{T}$, and $\mathcal{R}$ satisfy *strong thinness* for $\mathcal{J}$ on Z if, for all $a, b \in A$ and $p, q \in Z$,

$$[a \mathcal{J} p \text{ and } a \mathcal{J} q] \Rightarrow p \sim_{\diamond}^{r} q.$$

The main consequences of these conditions are summarized in the following:

Lemma 5

1. *The existence of a representation* (13) *implies that strong thinness for both $\mathcal{J}$ and $\mathcal{F}$ holds on both A and Z.*

2. *If $\mathcal{P}$, $\mathcal{S}$, $\mathcal{T}$, and $\mathcal{R}$ are biorders and strong thinness for $\mathcal{F}$ and $\mathcal{J}$ holds on Z then* (14) *and* (16) *hold.*

3. *If $\mathcal{P}$, $\mathcal{S}$, $\mathcal{T}$, and $\mathcal{R}$ are biorders and strong thinness for $\mathcal{F}$ and $\mathcal{J}$ holds on A then* (15) *and* (17) *hold.*

4. *If strong thinness for $\mathcal{F}$ and $\mathcal{J}$ holds on A then, for all $a, b, c \in A$ and all $p \in Z$,*

$$[a \mathcal{F} p \textit{ and } b \succ_{\diamond}^{\ell} a] \Rightarrow b \mathcal{T} p, \tag{27a}$$
$$[a \mathcal{J} p \textit{ and } b \succ_{\diamond}^{\ell} a] \Rightarrow b \mathcal{P} p, \tag{27b}$$
$$[a \mathcal{F} p \textit{ and } a \succ_{\diamond}^{\ell} c] \Rightarrow Not[c \mathcal{R} p], \tag{27c}$$
$$[a \mathcal{J} p \textit{ and } a \succ_{\diamond}^{\ell} c] \Rightarrow Not[c \mathcal{S} p]. \tag{27d}$$

5. *If strong thinness for $\mathcal{F}$ and $\mathcal{J}$ holds on Z then, for all $a \in A$ and all $p, q, r \in Z$,*

$$[a \mathcal{F} p \textit{ and } p \succ_{\diamond}^{r} q] \Rightarrow a \mathcal{T} q, \tag{28a}$$
$$[a \mathcal{J} p \textit{ and } p \succ_{\diamond}^{r} q] \Rightarrow a \mathcal{P} q, \tag{28b}$$
$$[a \mathcal{F} p \textit{ and } r \succ_{\diamond}^{r} p] \Rightarrow Not[a \mathcal{R} r], \tag{28c}$$
$$[a \mathcal{J} p \textit{ and } r \succ_{\diamond}^{r} p] \Rightarrow Not[a \mathcal{S} r]. \tag{28d}$$

6. *In the set of all relations $\mathcal{P}$, $\mathcal{J}$, $\mathcal{T}$ and $\mathcal{F}$ between the sets A and Z such that $\mathcal{P} \cap \mathcal{J} = \varnothing$, $\mathcal{T} \cap \mathcal{F} = \varnothing$, $\mathcal{J} \cap \mathcal{F} = \varnothing$, and $\mathcal{S} \subseteq \mathcal{T}$, the following* 16 *conditions are independent: $\mathcal{P}$ is Ferrers, $\mathcal{S}$ is Ferrers, $\mathcal{T}$ is Ferrers, $\mathcal{R}$ is*

Ferrers, strong thinness for $\mathcal{J}$ holds on A, strong thinness for $\mathcal{J}$ holds on Z, strong thinness for $\mathcal{F}$ holds on A, strong thinness for $\mathcal{F}$ holds on Z, and (18) *to* (25).

Proof

Part 1 is easily shown. Let us first prove Part 2, the proof of Part 3 being similar. Suppose that condition (16) is violated, so that, $a \mathcal{T} p$, $b \mathcal{R} q$, $Not[b \mathcal{T} p]$ and $Not[a \mathcal{R} q]$, for some $a, b \in A$ and some $p, q \in Z$. Because $\mathcal{R}$ is a biorder and $\mathcal{T} \subseteq \mathcal{R}$, we know that we must have $b \mathcal{R} p$. Since $Not[b \mathcal{T} p]$, this implies $b \mathcal{F} p$. If $b \mathcal{F} q$, then $a \mathcal{T} p$ and $Not[a \mathcal{R} q]$ violates strong thinness on Z. Therefore, we must have $b \mathcal{T} q$, so that, using the fact that $\mathcal{T}$ is a biorder, we have either $b \mathcal{T} p$ or $a \mathcal{T} q$, a contradiction.

Similarly, suppose that condition (14) is violated, so that, $a \mathcal{P} p$, $b \mathcal{S} q$, $Not[b \mathcal{P} p]$ and $Not[a \mathcal{S} q]$, for some $a, b \in A$ and some $p, q \in Z$. Because $\mathcal{S}$ is a biorder and $\mathcal{P} \subseteq \mathcal{S}$, we know that we must have $b \mathcal{S} p$. Since $Not[b \mathcal{P} p]$, this implies $b \mathcal{J} p$. If $b \mathcal{J} q$, then $a \mathcal{P} p$ and $Not[a \mathcal{P} q]$ violates strong thinness on Z. Therefore, we must have $b \mathcal{P} q$, so that, using the fact that $\mathcal{P}$ is a biorder, we have either $b \mathcal{P} p$ or $a \mathcal{P} q$, a contradiction.

Let us now prove Part 4, the proof of Part 5 being similar. Let us show that (27a) holds. Suppose that $a \mathcal{F} p$ and $b \succ^{\ell}_{\diamond} a$. Since $b \succ^{\ell}_{\diamond} a$ implies $b \succsim^{\ell}_{\diamond} a$, we know that $b \mathcal{R} p$. Suppose that $b \mathcal{F} p$. Using strong thinness on A, it is easy to see that $a \mathcal{F} p$ and $b \mathcal{F} p$ imply $b \sim^{\ell}_{\diamond} a$, a contradiction. Hence, we must have $b \mathcal{T} p$.

Let us now show that (27b) holds. Suppose that $a \mathcal{J} p$ and $b \succ^{\ell}_{\diamond} a$. Since $b \succ^{\ell}_{\diamond} a$ implies $b \succsim^{\ell}_{\diamond} a$, we know that $b \mathcal{S} p$. Suppose that $b \mathcal{J} p$. Using strong thinness on A, it is easy to see that $a \mathcal{J} p$ and $b \mathcal{J} p$ imply $b \sim^{\ell}_{\diamond} a$, a contradiction. Hence, we must have $b \mathcal{P} p$. The proof of (27c) and (27d) is similar.

Part 6. We provide below the required 16 examples. We indicate, for each example, which condition among the set of 16 conditions is the only one to be violated. In each of the matrices below, we only indicate the weakest relation that is satisfied.

$\mathcal{P}$ biorder
$$\begin{array}{ccc} & p & q \\ a & \mathcal{P} & \mathcal{T} \\ b & \mathcal{J} & \mathcal{P} \end{array}$$

$\mathcal{S}$ biorder
$$\begin{array}{ccc} & p & q \\ a & \mathcal{P} & \mathcal{T} \\ b & \mathcal{T} & \mathcal{J} \end{array}$$

$\mathcal{T}$ biorder
$$\begin{array}{ccc} & p & q \\ a & \mathcal{T} & \mathcal{F} \\ b & \mathcal{F} & \mathcal{T} \end{array}$$

$\mathcal{R}$ biorder
$$\begin{array}{ccc} & p & q \\ a & \mathcal{F} & - \\ b & - & \mathcal{F} \end{array}$$

$\mathcal{F}$ thin on Z
$$\begin{array}{ccc} & p & q \\ a & \mathcal{F} & \mathcal{F} \\ b & - & \mathcal{T} \end{array}$$

$\mathcal{F}$ thin on A
$$\begin{array}{ccc} & p & q \\ a & \mathcal{F} & \mathcal{T} \\ b & \mathcal{F} & - \end{array}$$

$\mathcal{J}$ thin on Z
$$\begin{array}{ccc} & p & q \\ a & \mathcal{J} & \mathcal{J} \\ b & - & \mathcal{T} \end{array}$$

$\mathcal{J}$ thin on A
$$\begin{array}{ccc} & p & q \\ a & \mathcal{J} & \mathcal{T} \\ b & \mathcal{J} & - \end{array}$$

$$
\begin{array}{cccc}
(18) & (19) & (22) & (23) \\
\begin{array}{ccc} & p & q \\ a & \mathcal{T} & \mathcal{T} \\ b & \mathcal{F} & \mathcal{P} \end{array} &
\begin{array}{ccc} & p & q \\ a & \mathcal{P} & \mathcal{T} \\ b & \mathcal{F} & \mathcal{T} \end{array} &
\begin{array}{ccc} & p & q \\ a & \mathcal{T} & \mathcal{T} \\ b & \mathcal{F} & \mathcal{J} \end{array} &
\begin{array}{ccc} & p & q \\ a & \mathcal{J} & \mathcal{T} \\ b & \mathcal{F} & \mathcal{T} \end{array} \\
(24) & (25) & (20) & (21) \\
\begin{array}{ccc} & p & q \\ a & \mathcal{F} & \mathcal{T} \\ b & - & \mathcal{J} \end{array} &
\begin{array}{ccc} & p & q \\ a & \mathcal{J} & \mathcal{T} \\ b & - & \mathcal{F} \end{array} &
\begin{array}{ccc} & p & q \\ a & \mathcal{F} & \mathcal{J} \\ b & - & \mathcal{P} \end{array} &
\begin{array}{ccc} & p & q \\ a & \mathcal{P} & \mathcal{J} \\ b & - & \mathcal{F} \end{array}
\end{array}
$$

□

5.3 Results

Our main result is the following.

Theorem 6
Let A and Z be finite *sets. Let $\mathcal{P}$, $\mathcal{J}$, $\mathcal{T}$, and $\mathcal{F}$ be four relations between the sets A and Z such that $\mathcal{P} \cap \mathcal{J} = \varnothing$, $\mathcal{T} \cap \mathcal{F} = \varnothing$, $\mathcal{J} \cap \mathcal{F} = \varnothing$, and $\mathcal{P} \cup \mathcal{J} = \mathcal{S} \subseteq \mathcal{T}$.*

There are real-valued functions f on A and g on Z such that (13) *holds if and only if $\mathcal{P}$, $\mathcal{S} = \mathcal{P} \cup \mathcal{J}$, $\mathcal{T}$, $\mathcal{R} = \mathcal{T} \cup \mathcal{F}$ are biorders satisfying conditions* (18) *to* (25) *and such that strong thinness holds for both $\mathcal{J}$ and $\mathcal{F}$ on both A and Z.*

Furthermore, these conditions are independent and the functions f and g in (13) *can always be chosen so that, for all $a, b \in A$ and $p, q \in Z$,*

$$
\begin{aligned}
a \succsim^{\ell}_{\diamond} b &\Leftrightarrow f(a) \geq f(b), \\
p \succsim^{r}_{\diamond} q &\Leftrightarrow g(p) \geq g(q).
\end{aligned} \tag{29}
$$

Theorem 6 is proved in Section 6.

An important limitation of the above result is that it only covers the case of *finite* sets A and Z. Extending them to possibly countably infinite sets and to possibly uncountable sets is an important open problem. The recent breakthrough on the constant threshold representation of semiorders on general sets (Candeal and Induráin, 2010) gives some hope to obtain interpretable results. This will require a proof strategy that is different from the one used here.

6 Proof of Theorem 6

6.1 Lemmas on semiorders and semiorders with frontier

We begin by a simple lemma on semiorders that is almost identical to Ducamp and Falmagne (1969, Lemma 6, page 380).

Lemma 7
Let T be a binary relation on a set X.

1. *If there is a weak order $\succsim$ on X such that, for all $x, y, z \in X$,*
$$\begin{aligned} &x \; T \; y \text{ and } y \succsim z \Rightarrow x \; T \; z, \\ &x \succsim y \text{ and } y \; T \; z \Rightarrow x \; T \; z, \end{aligned} \tag{30}$$
then T is Ferrers and semitransitive and $\succsim$ refines $\succsim_T$.

2. *If T is irreflexive and there is a weak order $\succsim$ on X such that* (30) *holds, then, when X is* finite, *there are a real-valued function u on X such that, for all $x, y \in X$,*
$$\begin{aligned} &x \; T \; y \Leftrightarrow u(x) > u(y) + 1, \\ &x \succsim y \Leftrightarrow u(x) \geq u(y). \end{aligned}$$

PROOF
Part 1 is straightforward. Part 2 is Ducamp and Falmagne (1969, Lemma 6, page 380). Let us simply outline the proof. An irreflexive, semitransitive Ferrers relation is a semiorder. Scott and Suppes (1958) have shown that a semiorder on a finite set always has a constant threshold representation. The classical proof (Scott and Suppes, 1958) of the existence of a constant threshold representation for finite semiorders leads to a function u that represents the weak order $\succsim_T$ and is such that it is never true that $u(x) - u(y) = 1$. Using the fact that X is finite, we can therefore modify u in such a way that any two elements x and y such that $x \sim_T y$ will be assigned distinct (but close) values in the modified numerical representation. Since $\succsim$ refines $\succsim_T$, we can modify u in such a way that it will represent $\succsim$. □

The proof of Proposition 3 given in Ducamp and Falmagne (1969) consists, starting with the relations $\mathcal{T}$ and $\mathcal{P}$ between A and Z, in building two relations T and $\succsim$ on $A \cup Z$ such that T and $\succsim$ satisfy the conditions of Lemma 7 and the restriction of T (resp. $\succsim$) to $A \times Z$ is $\mathcal{P}$ (resp. $\mathcal{T}$).

Let us now consider a pair of disjoint relations T and F on a set X. Let $R = T \cup F$. As before, let $\succsim_T = \succsim_T^\ell \cap \succsim_T^r$ and $\succsim_R = \succsim_R^\ell \cap \succsim_R^r$. Let $\succsim_\star = \succsim_T \cap \succsim_R$.

We say that F is *strongly upper thin* for the pair T and F if, for all $x, y, z, w \in X$,
$$\left.\begin{matrix} x \; F \; z \\ y \; F \; z \end{matrix}\right\} \Rightarrow \left\{\begin{matrix} x \; T \; w \Leftrightarrow y \; T \; w \\ x \; F \; w \Leftrightarrow y \; F \; w \\ w \; F \; x \Leftrightarrow w \; F \; y \\ w \; T \; x \Leftrightarrow w \; T \; y \end{matrix}\right\}.$$

Similarly, we say that F is *strongly lower thin* for the pair T and F if, for all $x, y, z, w \in X$,

$$\left.\begin{array}{c} z\ F\ x \\ z\ F\ y \end{array}\right\} \Rightarrow \left\{\begin{array}{c} x\ T\ w \Leftrightarrow y\ T\ w \\ x\ F\ w \Leftrightarrow y\ F\ w \\ w\ F\ x \Leftrightarrow w\ F\ y \\ w\ T\ x \Leftrightarrow w\ T\ y \end{array}\right\}.$$

We say that *strong thinness* holds if we have both strong lower thinness and strong upper thinness.

Lemma 8 (Prop. 19 in BM11)
Let T and F be a pair of disjoint relations on a finite *set X. Let $R = T \cup F$. There is a real-valued function u on X such that, for all $x, y \in X$,*

$$\begin{aligned} x\ T\ y &\Leftrightarrow u(x) > u(y) + 1, \\ x\ F\ y &\Leftrightarrow u(x) = u(y) + 1, \end{aligned} \tag{31}$$

iff T is a semiorder, R is a semiorder, $TFR^{sc} \subseteq T$, $R^{sc}FT \subseteq T$, and strong thinness holds. Furthermore the function u can always be chosen so that, for all $x, y \in X$,

$$x \succsim_{\star} y \Leftrightarrow u(x) \geq u(y). \tag{32}$$

The following lemma generalizes Lemma 7 to cope with a frontier.

Lemma 9
Let T and F be a pair of disjoint irreflexive relations on a set X.

1. *If there is a weak order $\succsim$ on X such that, for all $x, y, z \in X$,*

$$\begin{aligned} &x\ T\ y \text{ and } y \succsim z \Rightarrow x\ T\ z, \\ &x\ F\ y \text{ and } y \succ z \Rightarrow x\ T\ z, \\ &x\ F\ y \text{ and } y \sim z \Rightarrow x\ F\ z, \\ &x \succsim y \text{ and } y\ T\ z \Rightarrow x\ T\ z, \\ &x \succ y \text{ and } y\ F\ z \Rightarrow x\ T\ z, \\ &x \sim y \text{ and } y\ F\ z \Rightarrow x\ F\ z, \end{aligned} \tag{33}$$

   then T is a semiorder, $R = T \cup F$ is a semiorder, $TFR^{sc} \subseteq T$, $R^{sc}FT \subseteq T$, F is strongly upper thin, and F is strongly lower thin

2. *Under the conditions of Part 1, when X is finite, there is a real-valued function u on X such that, for all $x, y \in X$,*

$$\begin{aligned} x\ T\ y &\Leftrightarrow u(x) > u(y) + 1, \\ x\ F\ y &\Leftrightarrow u(x) = u(y) + 1, \\ x \succsim y &\Leftrightarrow u(x) \geq u(y). \end{aligned} \tag{34}$$

Proof

Part 1. By hypothesis, we know that T and F are disjoint. Using Lemma 7 and (33), it is clear that both T and R are Ferrers and semitransitive. Since we have supposed that T and F are irreflexive, both T and R are semiorders.

Suppose that $a\ T\ b$, $b\ F\ c$ and $c\ R^{sc}\ d$, for some $a, b, c, d \in X$. If $b \succsim d$, then $a\ T\ b$ and (33) imply $a\ T\ d$. If $d \succ b$, then $b\ F\ c$ and (33) imply $d\ T\ c$, a contradiction. Hence, we have $TFR^{sc} \subseteq T$. The proof that $R^{sc}FT \subseteq T$ is similar.

Suppose now that $a\ F\ c$ and $b\ F\ c$. If $a \succ b$ then $b\ F\ c$ and (33) imply $a\ T\ c$, a contradiction. Similarly if $b \succ a$, $a\ F\ c$ and (33) imply $b\ T\ c$, a contradiction. Hence, we must have $a \sim b$, so that strong upper thinness holds. The proof for strong lower thinness is similar.

Part 2. When X is finite, we may use Proposition 8 to obtain a numerical representation of the pair T and F in model (34), except that u is a numerical representation of the weak order $\succsim_\star$. It remains to show that it is possible to modify this numerical representation in such a way that u will represent $\succsim$. Observe that (33) implies that $\succsim$ refines $\succsim_\star$. We may therefore use here the same construction as the one used in Lemma 7, provided that it never happens that [$x\ F\ z$, $y\ F\ z$ and $x \succ y$] or [$z\ F\ x$, $z\ F\ y$ and $x \succ y$]. This is implied by (33). □

For proving Theorem 6, our strategy will be as follows. Starting with the four relations $\mathcal{P}$, $\mathcal{J}$, $\mathcal{T}$, and $\mathcal{F}$ between the sets A and Z, we will build three relations T, F and $\succsim$ on $A \cup Z$ such that T and $\succsim$ satisfy the conditions of Lemma 9. The restriction of T (resp. F, $\succ$, $\sim$) to $A \times Z$ will be $\mathcal{P}$ (resp. $\mathcal{J}$, $\mathcal{T}$, $\mathcal{F}$). This will lead to the desired representation.

6.2 Lemmas on biorders and biorders with frontier

The first lemma on biorders is taken from BM11.

Lemma 10 (Lemma 1 in BM11)
Suppose that $\mathcal{T}$ is a biorder between A and Z. Let $\succsim^\ell_\flat$ be a weak order on A and $\succsim^r_\flat$ be a weak order on Z. Suppose that, for all $\alpha, \beta \in A$ and all $\gamma, \delta \in Z$,

$$\alpha \succsim^\ell_\flat \beta \text{ and } \beta\ \mathcal{T}\ \gamma \Rightarrow \alpha\ \mathcal{T}\ \gamma,$$
$$\gamma \succsim^r_\flat \delta \text{ and } \beta\ \mathcal{T}\ \gamma \Rightarrow \beta\ \mathcal{T}\ \delta.$$

Then:

1. *$\succsim^\ell_\flat$ refines $\succsim^\ell_{\mathcal{T}}$,*
2. *$\succsim^r_\flat$ refines $\succsim^r_{\mathcal{T}}$,*

3. the binary relation Q on $A \cup Z$ that is defined letting , for all $\alpha, \beta \in A \cup Z$,

$$\alpha \; Q \; \beta \Leftrightarrow \begin{cases} \alpha \in A, \beta \in A, \text{ and } \alpha \succsim_\flat^\ell \beta, \\ \alpha \in Z, \beta \in Z, \text{ and } \alpha \succsim_\flat^r \beta, \\ \alpha \in A, \beta \in Z, \text{ and } \alpha \; \mathcal{T} \; \beta, \\ \alpha \in Z, \beta \in A, \text{ and } Not[\beta \; \mathcal{T} \; \alpha]. \end{cases}$$

is a weak order.

The following lemma generalizes Lemma 10 to cope with a frontier.

Lemma 11 (Lemma 3 in BM11)
Let $\mathcal{T}$ and $\mathcal{F}$ be a pair of disjoint relations between A and Z. Let $\mathcal{R} = \mathcal{T} \cup \mathcal{F}$ Suppose that $\succsim_\flat^\ell$ is a weak order on A and $\succsim_\flat^r$ is a weak order on Z. Suppose that, for all $\alpha, \beta \in A$ and all $\gamma \in Z$,

$$\begin{aligned} &\beta \; \mathcal{T} \; \gamma \text{ and } \alpha \succsim_\flat^\ell \beta \Rightarrow \alpha \; \mathcal{T} \; \gamma, \\ &\beta \; \mathcal{F} \; \gamma \text{ and } \alpha \succ_\flat^\ell \beta \Rightarrow \alpha \; \mathcal{T} \; \gamma, \\ &\beta \; \mathcal{F} \; \gamma \text{ and } \alpha \sim_\flat^\ell \beta \Rightarrow \alpha \; \mathcal{F} \; \gamma. \end{aligned} \tag{35}$$

Suppose furthermore that, for all $\alpha \in A$ and all $\beta, \gamma \in Z$,

$$\begin{aligned} &\alpha \; \mathcal{T} \; \beta \text{ and } \beta \succsim_\flat^r \gamma \Rightarrow \alpha \; \mathcal{T} \; \gamma, \\ &\alpha \; \mathcal{F} \; \beta \text{ and } \beta \succ_\flat^\ell \gamma \Rightarrow \alpha \; \mathcal{T} \; \gamma, \\ &\alpha \; \mathcal{F} \; \beta \text{ and } \beta \sim_\flat^\ell \gamma \Rightarrow \alpha \; \mathcal{F} \; \gamma. \end{aligned} \tag{36}$$

Then $\succsim_\flat^\ell$ refines $\succsim_\star^\ell = \succsim_\mathcal{T}^\ell \cap \succsim_\mathcal{R}^\ell$ and $\succsim_\flat^r$ refines $\succsim_\star^r = \succsim_\mathcal{T}^r \cap \succsim_\mathcal{R}^r$. Furthermore, the binary relation L on $A \cup Z$ that is defined letting , for all $\alpha, \beta \in A \cup Z$,

$$\alpha \; L \; \beta \Leftrightarrow \begin{cases} \alpha \in A, \beta \in A, \text{ and } \alpha \succsim_\flat^\ell \beta, \\ \alpha \in Z, \beta \in Z, \text{ and } \alpha \succsim_\flat^r \beta, \\ \alpha \in A, \beta \in Z, \text{ and } \alpha \; \mathcal{R} \; \beta, \\ \alpha \in Z, \beta \in A, \text{ and } Not[\beta \; \mathcal{T} \; \alpha]. \end{cases}$$

is a weak order.

6.3 Proof of Theorem 6

Necessity follows from Lemmas 4 and 5, together with Proposition 1. The independence of the conditions was shown in Part 6 of Lemma 5. We show sufficiency. We know from Lemmas 4 and 5 that $\succsim_\diamond^\ell$ is a weak order on A and that $\succsim_\diamond^r$ is a weak order on Z.

Step 1.

We define the relation $Q_\diamond$ on $A \cup Z$ letting, for all $\alpha, \beta \in A \cup Z$,

$$\alpha \; Q_\diamond \; \beta \Leftrightarrow \begin{cases} \alpha, \beta \in A & \text{and } \alpha \succsim_\diamond^\ell \beta, \\ \alpha, \beta \in Z & \text{and } \alpha \succsim_\diamond^r \beta, \\ \alpha \in A, \beta \in Z & \text{and } \alpha \; \mathcal{R} \; \beta, \\ \alpha \in Z, \beta \in A & \text{and } Not[\beta \; \mathcal{T} \; \alpha]. \end{cases}$$

Using Lemmas 5 and 11, we know that $Q_\diamond$ is a weak order.

Step 2.

Define the relation $H_\diamond$ on $A \cup Z$ letting for all $\alpha, \beta \in A \cup Z$,

$$\alpha \; H_\diamond \; \beta \Leftrightarrow \begin{cases} \alpha, \beta \in A & \text{and } [\alpha \; \mathcal{J} \; \delta, \beta \; \mathcal{F} \; \delta] \text{ for some } \delta \in Z \\ \alpha, \beta \in Z & \text{and } [\gamma \; \mathcal{J} \; \beta, \gamma \; \mathcal{F} \; \alpha] \text{ for some } \gamma \in A \\ \alpha \in A, \beta \in Z & \text{and } \alpha \; \mathcal{J} \; \beta, \\ \alpha \in Z, \beta \in A & \text{and } [\beta \; \mathcal{F} \; \delta, \gamma \; \mathcal{J} \; \delta, \gamma \; \mathcal{F} \; \alpha] \text{ for some } \gamma \in A, \delta \in Z \end{cases}$$

Since $\mathcal{J}$ and $\mathcal{F}$ are disjoint, it is easy to see that $H_\diamond$ is irreflexive.

Step 3.

Define the relation $K_\diamond$ on $A \cup Z$ letting for all $\alpha, \beta \in A \cup Z$:

$$\begin{aligned} &\alpha \; K_\diamond \; \beta \Leftrightarrow \\ &\alpha, \beta \in A \text{ and } \left\{ \begin{array}{c} \alpha \; \mathcal{S} \; \delta, Not[\beta \; \mathcal{R} \; \delta], \\ \text{or} \\ \alpha \; \mathcal{P} \; \delta, Not[\beta \; \mathcal{T} \; \delta], \end{array} \right\} \text{ for some } \delta \in Z, \\ &\alpha, \beta \in Z \text{ and } \left\{ \begin{array}{c} \gamma \; \mathcal{S} \; \beta, Not[\gamma \; \mathcal{R} \; \alpha], \\ \text{or} \\ \gamma \; \mathcal{P} \; \beta, Not[\gamma \; \mathcal{T} \; \alpha], \end{array} \right\} \text{ for some } \gamma \in A, \\ &\alpha \in A, \beta \in Z \text{ and } \alpha \; \mathcal{P} \; \beta, \\ &\alpha \in Z, \beta \in A \text{ and } \left\{ \begin{array}{c} Not[\gamma \; \mathcal{R} \; \alpha], \gamma \; \mathcal{S} \; \delta, Not[\beta \; \mathcal{T} \; \delta], \\ \text{or} \\ Not[\gamma \; \mathcal{T} \; \alpha], \gamma \; \mathcal{P} \; \delta, Not[\beta \; \mathcal{T} \; \delta], \\ \text{or} \\ Not[\gamma \; \mathcal{T} \; \alpha], \gamma \; \mathcal{S} \; \delta, Not[\beta \; \mathcal{R} \; \delta] \end{array} \right\} \text{ for some } \gamma \in A, \delta \in Z. \end{aligned}$$

Since $\mathcal{P} \subseteq \mathcal{S} \subseteq \mathcal{T} \subseteq \mathcal{R}$, it is easy to see that $K_\diamond$ is irreflexive.

Step 4.

Let us show that $K_\diamond$ and $H_\diamond$ are disjoint.

If $\alpha \in A$ and $\beta \in Z$, the conclusion follows from the fact that $\mathcal{J}$ and $\mathcal{P}$ are disjoint.

If $\alpha \in A$ and $\beta \in A$, $\alpha\ H_\diamond\ \beta$ implies that $\alpha\ \mathcal{J}\ \delta$ and $\beta\ \mathcal{F}\ \delta$, for some $\delta \in Z$. Similarly $\alpha\ K_\diamond\ \beta$ implies either $\alpha\ \mathcal{S}\ \rho$, $Not[\beta\ \mathcal{R}\ \rho]$ or $\alpha\ \mathcal{P}\ \rho$, $Not[\beta\ \mathcal{T}\ \rho]$, for some $\rho \in Z$. In the first case, $\beta\ \mathcal{F}\ \delta$ and $Not[\beta\ \mathcal{R}\ \rho]$ implies $\rho \succ^r_\diamond \delta$. Hence, $\alpha\ \mathcal{S}\ \rho$ implies $\alpha\ \mathcal{P}\ \delta$, a contradiction. In the second case, $\alpha\ \mathcal{J}\ \delta$ and $\alpha\ \mathcal{P}\ \rho$ imply $\delta \succ^r_\diamond \rho$. Hence, $\beta\ \mathcal{F}\ \delta$ implies $\beta\ \mathcal{T}\ \rho$, a contradiction.

The case $\alpha \in Z$ and $\beta \in Z$ is dealt with in a similar way.

Suppose now that $\alpha \in Z$ and $\beta \in A$. By definition, $\alpha\ H_\diamond\ \beta$ implies that $\beta\ \mathcal{F}\ \delta$, $\rho\ \mathcal{J}\ \delta$, $\rho\ \mathcal{F}\ \alpha$, for some $\rho \in A, \delta \in Z$.

Similarly $\alpha\ K_\diamond\ \beta$ implies either $Not[\omega\ \mathcal{R}\ \alpha]$, $\omega\ \mathcal{S}\ \tau$, $Not[\beta\ \mathcal{T}\ \tau]$, or $Not[\omega\ \mathcal{T}\ \alpha]$, $\omega\ \mathcal{P}\ \tau$, $Not[\beta\ \mathcal{T}\ \tau]$, or $Not[\omega\ \mathcal{T}\ \alpha]$, $\omega\ \mathcal{S}\ \tau$, $Not[\beta\ \mathcal{R}\ \tau]$ for some $\omega \in A$, $\tau \in Z$.

In the first case, because $\beta\ \mathcal{F}\ \delta$ and $Not[\beta\ \mathcal{T}\ \tau]$, we must have $\tau \succsim^r_\diamond \delta$. Because $\rho\ \mathcal{F}\ \alpha$ and $Not[\omega\ \mathcal{R}\ \alpha]$, we must have $\rho \succ^\ell_\diamond \omega$. Therefore $\omega\ \mathcal{S}\ \tau$ implies $\rho\ \mathcal{P}\ \tau$ and, hence, $\rho\ \mathcal{P}\ \delta$, a contradiction.

In the second case, $\beta\ \mathcal{F}\ \delta$ and $Not[\beta\ \mathcal{T}\ \tau]$, we must have $\tau \succsim^r_\diamond \delta$. Because $\rho\ \mathcal{F}\ \alpha$ and $Not[\omega\ \mathcal{T}\ \alpha]$, we must have $\rho \succsim^\ell_\diamond \omega$. Therefore $\omega\ \mathcal{P}\ \tau$ implies $\rho\ \mathcal{P}\ \tau$ and, hence, $\rho\ \mathcal{P}\ \delta$, a contradiction.

In the third case, $\beta\ \mathcal{F}\ \delta$ and $Not[\beta\ \mathcal{R}\ \tau]$, we must have $\tau \succ^r_\diamond \delta$. Because $\rho\ \mathcal{F}\ \alpha$ and $Not[\omega\ \mathcal{T}\ \alpha]$, we must have $\rho \succsim^\ell_\diamond \omega$. Therefore $\omega\ \mathcal{S}\ \tau$ implies $\rho\ \mathcal{S}\ \tau$ and, hence, $\rho\ \mathcal{P}\ \delta$, a contradiction.

Our plan is now to apply Lemma 9 to the relations $K_\diamond$ (playing the role of T) and $H_\diamond$ (playing the role of F) with $Q_\diamond$ playing the role of $\succsim$.

We have already observed that $Q_\diamond$ is a weak order and that $H_\diamond$ and $K_\diamond$ are disjoint and both irreflexive. It remains to show that, for all $\alpha, \beta, \gamma \in A \cup Z$,

$$\alpha\ K_\diamond\ \beta \text{ and } \beta\ Q_\diamond\ \gamma \Rightarrow \alpha\ K_\diamond\ \gamma, \tag{37a}$$

$$\alpha\ H_\diamond\ \beta \text{ and } \beta\ Q^a_\diamond\ \gamma \Rightarrow \alpha\ K_\diamond\ \gamma, \tag{37b}$$

$$\alpha\ H_\diamond\ \beta \text{ and } \beta\ Q^s_\diamond\ \gamma \Rightarrow \alpha\ H_\diamond\ \gamma, \tag{37c}$$

$$\alpha\ Q_\diamond\ \beta \text{ and } \beta\ K_\diamond\ \gamma \Rightarrow \alpha\ K_\diamond\ \gamma, \tag{37d}$$

$$\alpha\ Q^a_\diamond\ \beta \text{ and } \beta\ H_\diamond\ \gamma \Rightarrow \alpha\ K_\diamond\ \gamma, \tag{37e}$$

$$\alpha\ Q^s_\diamond\ \beta \text{ and } \beta\ H_\diamond\ \gamma \Rightarrow \alpha\ H_\diamond\ \gamma, \tag{37f}$$

where $Q^a_\diamond$ (resp. $Q^s_\diamond$) denotes the asymmetric (resp. symmetric) part of $Q_\diamond$.

Step 5.

Let us first prove (37c). Observe first that $\beta\ Q^s_\diamond\ \gamma$ means that $\beta \sim^\ell_\diamond \gamma$ if $\beta, \gamma \in A$, $\beta \sim^r_\diamond \gamma$ if $\beta, \gamma \in Z$, $\beta\ \mathcal{F}\ \gamma$ if $\beta \in A, \gamma \in Z$ and $\gamma\ \mathcal{F}\ \beta$ if $\beta \in Z, \gamma \in A$.

Suppose that $\alpha\ H_\diamond\ \beta$ and $\beta\ Q^s_\diamond\ \gamma$. There are eight cases to consider.

1. Suppose that $\alpha, \beta, \gamma \in A$. We have $[\alpha \; \mathcal{J} \; \delta, \beta \; \mathcal{F} \; \delta]$, for some $\delta \in Z$ and $\beta \sim_\diamond^\ell \gamma$. We obtain $[\alpha \; \mathcal{J} \; \delta, \gamma \; \mathcal{F} \; \delta]$, so that $\alpha \; H_\diamond \; \gamma$.

2. Suppose that $\alpha, \beta, \gamma \in Z$.

   We have $[\delta \; \mathcal{J} \; \beta, \delta \; \mathcal{F} \; \alpha]$ for some $\delta \in A$ and $\beta \sim_\diamond^r \gamma$. We have $[\delta \; \mathcal{J} \; \gamma, \delta \; \mathcal{F} \; \alpha]$, so that $\alpha \; H_\diamond \; \gamma$.

3. Suppose that $\alpha, \beta \in A, \gamma \in Z$. We have $[\alpha \; \mathcal{J} \; \delta, \beta \; \mathcal{F} \; \delta]$, for some $\delta \in Z$ and $\beta \; \mathcal{F} \; \gamma$. This implies $\gamma \sim_\diamond^r \delta$, so that $\alpha \; \mathcal{J} \; \gamma$ and $\alpha \; H_\diamond \; \gamma$.

4. Suppose that $\alpha, \gamma \in A, \beta \in Z$. We have $\alpha \; \mathcal{J} \; \beta$ and $\gamma \; \mathcal{F} \; \beta$, so that $\alpha \; H_\diamond \; \gamma$.

5. Suppose that $\beta, \gamma \in A, \alpha \in Z$. We have $[\beta \; \mathcal{F} \; \delta, \rho \; \mathcal{J} \; \delta, \rho \; \mathcal{F} \; \alpha]$, for some $\rho \in A, \delta \in Z$ and $\beta \sim_\diamond^\ell \gamma$. We obtain $[\gamma \; \mathcal{F} \; \delta, \rho \; \mathcal{J} \; \delta, \rho \; \mathcal{F} \; \alpha]$, so that $\alpha \; H_\diamond \; \gamma$.

6. Suppose that $\alpha, \beta \in Z, \gamma \in A$. We have $[\delta \; \mathcal{J} \; \beta, \delta \; \mathcal{F} \; \alpha]$ for some $\delta \in A$ and $\gamma \; \mathcal{F} \; \beta$. We obtain $[\gamma \; \mathcal{F} \; \beta, \delta \; \mathcal{J} \; \beta, \delta \; \mathcal{F} \; \alpha]$, so that $\alpha \; H_\diamond \; \gamma$.

7. Suppose that $\alpha, \gamma \in Z, \beta \in A$. We have $[\beta \; \mathcal{F} \; \delta, \rho \; \mathcal{J} \; \delta, \rho \; \mathcal{F} \; \alpha]$, for some $\rho \in A, \delta \in Z$ and $\beta \; \mathcal{F} \; \gamma$. We obtain $\gamma \sim_\diamond^r \delta$. We obtain $\rho \; \mathcal{J} \; \gamma$ and $\rho \; \mathcal{F} \; \alpha$, so that $\alpha \; H_\diamond \; \gamma$.

8. Suppose that $\beta, \gamma \in Z, \alpha \in A$. $\alpha \; \mathcal{J} \; \beta$ and $\beta \sim_\diamond^r \gamma$. We obtain $\alpha \; \mathcal{J} \; \gamma$, so that $\alpha \; H_\diamond \; \gamma$.

Step 6.

Let us now prove (37b). Observe first that $\beta \; Q_\diamond^a \; \gamma$ means that $\beta \succ_\diamond^\ell \gamma$ if $\beta, \gamma \in A$, $\beta \succ_\diamond^r \gamma$ if $\beta, \gamma \in Z$, $\beta \; \mathcal{T} \; \gamma$ if $\beta \in A, \gamma \in Z$ and $Not[\gamma \; \mathcal{R} \; \beta]$ if $\beta \in Z, \gamma \in A$.

Suppose that $\alpha \; H_\diamond \; \beta$ and $\beta \; Q_\diamond^a \; \gamma$. There are eight cases to consider.

1. Suppose that $\alpha, \beta, \gamma \in A$. We have $[\alpha \; \mathcal{J} \; \delta, \beta \; \mathcal{F} \; \delta]$, for some $\delta \in Z$ and $\beta \succ_\diamond^\ell \gamma$. We obtain $Not[\gamma \; \mathcal{R} \; \delta]$ and $\alpha \; \mathcal{J} \; \delta$, so that $\alpha \; K_\diamond \; \gamma$.

2. If $\alpha, \beta, \gamma \in Z$. We have $[\delta \; \mathcal{J} \; \beta, \delta \; \mathcal{F} \; \alpha]$ for some $\delta \in A$ and $\beta \succ_\diamond^r \gamma$. We obtain $\delta \; \mathcal{P} \; \gamma$ and $Not[\delta \; \mathcal{T} \; \alpha]$, so that $\alpha \; K_\diamond \; \gamma$.

3. Suppose that $\alpha, \beta \in A, \gamma \in Z$. We have $[\alpha \; \mathcal{J} \; \delta, \beta \; \mathcal{F} \; \delta]$, for some $\delta \in Z$ and $\beta \; \mathcal{T} \; \gamma$. Because $\beta \; \mathcal{F} \; \delta$ and $\beta \; \mathcal{T} \; \gamma$, we have $\delta \succ_\diamond^r \gamma$. Hence, $\alpha \; \mathcal{J} \; \delta$ implies $\alpha \; \mathcal{P} \; \gamma$, so that $\alpha \; K_\diamond \; \gamma$.

4. Suppose that $\alpha, \gamma \in A, \beta \in Z$. We have $\alpha \; \mathcal{J} \; \beta$ and $Not[\gamma \; \mathcal{R} \; \beta]$. Hence, we have $\alpha \; K_\diamond \; \gamma$.

5. Suppose that $\beta, \gamma \in A, \alpha \in Z$. We have $[\beta \; \mathcal{F} \; \delta, \rho \; \mathcal{J} \; \delta, \rho \; \mathcal{F} \; \alpha]$, for some $\rho \in A, \delta \in Z$ and $\beta \succ_{\diamond}^{\ell} \gamma$. We obtain $Not[\gamma \; \mathcal{R} \; \delta]$, $\rho \; \mathcal{J} \; \delta$ and $\rho \; \mathcal{F} \; \alpha$, so that $\alpha \; K_{\diamond} \; \gamma$.

6. Suppose that $\alpha, \beta \in Z, \gamma \in A$. We have $[\delta \; \mathcal{J} \; \beta, \delta \; \mathcal{F} \; \alpha]$ for some $\delta \in A$ and $Not[\gamma \; \mathcal{R} \; \beta]$. We therefore have $Not[\delta \; \mathcal{T} \; \alpha]$, $\delta \; \mathcal{J} \; \beta$ and $Not[\gamma \; \mathcal{R} \; \beta]$, so that $\alpha \; K_{\diamond} \; \gamma$.

7. Suppose that $\alpha, \gamma \in Z, \beta \in A$. We have $[\beta \; \mathcal{F} \; \delta, \rho \; \mathcal{J} \; \delta, \rho \; \mathcal{F} \; \alpha]$, for some $\rho \in A, \delta \in Z$ and $\beta \; \mathcal{T} \; \gamma$. Because $\beta \; \mathcal{F} \; \delta$ and $\beta \; \mathcal{T} \; \gamma$, we obtain $\delta \succ_{\diamond}^{r} \gamma$, so that $\rho \; \mathcal{P} \; \gamma$. Because $\rho \; \mathcal{F} \; \alpha$ implies $Not[\rho \; \mathcal{T} \; \alpha]$, we obtain $\alpha \; K_{\diamond} \; \gamma$.

8. Suppose that $\beta, \gamma \in Z, \alpha \in A$. We have $\alpha \; \mathcal{J} \; \beta$ and $\beta \succ_{\diamond}^{r} \gamma$. We obtain $\alpha \; \mathcal{P} \; \gamma$, so that $\alpha \; K_{\diamond} \; \gamma$.

Step 7.

Let us prove (37a). Suppose that $\alpha \; K_{\diamond} \; \beta$ and $\beta \; Q_{\diamond} \; \gamma$. There are eight cases to examine.

1. Suppose that $\alpha, \beta, \gamma \in A$. We have $[\alpha \; \mathcal{S} \; \delta$ and $Not[\beta \; \mathcal{R} \; \delta]]$, or $[\alpha \; \mathcal{P} \; \delta$ and $Not[\beta \; \mathcal{T} \; \delta]]$, for some $\delta \in Z$ and $\beta \succsim_{\diamond}^{\ell} \gamma$. We obtain either $\alpha \; \mathcal{S} \; \delta$, $Not[\gamma \; \mathcal{R} \; \delta]$, or $\alpha \; \mathcal{P} \; \delta$, $Not[\gamma \; \mathcal{T} \; \delta]$, so that $\alpha \; K_{\diamond} \; \gamma$.

2. Suppose that $\alpha, \beta \in A$ and $\gamma \in Z$. We have $[\alpha \; \mathcal{S} \; \delta$ and $Not[\beta \; \mathcal{R} \; \delta]]$, or $[\alpha \; \mathcal{P} \; \delta$ and $Not[\beta \; \mathcal{T} \; \delta]]$, for some $\delta \in Z$ and $\beta \; \mathcal{R} \; \gamma$. In the first case, $Not[\beta \; \mathcal{R} \; \delta]$ and $\beta \; \mathcal{R} \; \gamma$ implies $\delta \succ_{\diamond}^{r} \gamma$. Hence $\alpha \; \mathcal{S} \; \delta$ implies $\alpha \; \mathcal{P} \; \gamma$. In the second case, $Not[\beta \; \mathcal{T} \; \delta]$ and $\beta \; \mathcal{R} \; \gamma$ implies $\delta \succsim_{\diamond}^{r} \gamma$. Hence, we obtain $\alpha \; \mathcal{P} \; \gamma$, so that $\alpha \; K_{\diamond} \; \gamma$.

3. Suppose that $\alpha, \gamma \in A$ and $\beta \in Z$. We have $\alpha \; \mathcal{P} \; \beta$ and $Not[\gamma \; \mathcal{T} \; \beta]$, so that $\alpha \; K_{\diamond} \; \gamma$.

4. Suppose that $\alpha \in Z$ and $\beta, \gamma \in A$. We have $[Not[\rho \; \mathcal{R} \; \alpha], \rho \; \mathcal{S} \; \delta, Not[\beta \; \mathcal{T} \; \delta]]$ or $[Not[\rho \; \mathcal{T} \; \alpha], \rho \; \mathcal{P} \; \delta, Not[\beta \; \mathcal{T} \; \delta]]$ or $[Not[\rho \; \mathcal{T} \; \alpha], \rho \; \mathcal{S} \; \delta, Not[\beta \; \mathcal{R} \; \delta]]$ for some $\rho \in A$, $\delta \in Z$, and $\beta \succsim_{\diamond}^{\ell} \gamma$. Hence, $Not[\beta \; \mathcal{T} \; \delta]$ implies $Not[\gamma \; \mathcal{T} \; \delta]$ and $Not[\beta \; \mathcal{R} \; \delta]$ implies $Not[\gamma \; \mathcal{R} \; \delta]$. In either of these three cases, we therefore have $\alpha \; K_{\diamond} \; \gamma$.

5. Suppose that $\alpha, \beta \in Z$ and $\gamma \in A$. We have $[\delta \; \mathcal{S} \; \beta, Not[\delta \; \mathcal{R} \; \alpha]]$ or $[\delta \; \mathcal{P} \; \beta, Not[\delta \; \mathcal{T} \; \alpha]]$, for some $\delta \in A$, and $Not[\gamma \; \mathcal{T} \; \beta]$. We have either $Not[\gamma \; \mathcal{T} \; \beta]$, $\delta \; \mathcal{S} \; \beta$, $Not[\delta \; \mathcal{R} \; \alpha]$ or $Not[\gamma \; \mathcal{T} \; \beta]$, $\delta \; \mathcal{P} \; \beta$, $Not[\delta \; \mathcal{T} \; \alpha]$, so that $\alpha \; K_{\diamond} \; \gamma$.

6. Suppose that $\alpha, \gamma \in Z$ and $\beta \in A$. We have $[Not[\rho\ \mathcal{R}\ \alpha], \rho\ \mathcal{S}\ \delta, Not[\beta\ \mathcal{T}\ \delta]]$, or $[Not[\rho\ \mathcal{T}\ \alpha], \rho\ \mathcal{P}\ \delta, Not[\beta\ \mathcal{T}\ \delta]]$, or $[Not[\rho\ \mathcal{T}\ \alpha], \rho\ \mathcal{S}\ \delta, Not[\beta\ \mathcal{R}\ \delta]]$, for some $\rho \in A$, $\delta \in Z$, and $\beta\ \mathcal{R}\ \gamma$.

   In the first two cases, $\beta\ \mathcal{R}\ \gamma$ and $Not[\beta\ \mathcal{T}\ \delta]$ imply $\delta \succsim^r_\diamond \gamma$. Hence, we have either $Not[\rho\ \mathcal{R}\ \alpha]$, $\rho\ \mathcal{S}\ \gamma$ or $Not[\rho\ \mathcal{T}\ \alpha]$, $\rho\ \mathcal{P}\ \gamma$.

   In the third case, $\beta\ \mathcal{R}\ \gamma$ and $Not[\beta\ \mathcal{R}\ \delta]$ imply $\delta \succ^r_\diamond \gamma$. Hence, we have $Not[\rho\ \mathcal{T}\ \alpha]$ and $\rho\ \mathcal{P}\ \gamma$.

   In either case, we therefore have $\alpha\ K_\diamond\ \gamma$.

7. Suppose that $\beta, \gamma \in Z$ and $\alpha \in A$. We have $\alpha\ \mathcal{P}\ \beta$ and $\beta \succsim^r_\diamond \gamma$. This implies $\alpha\ \mathcal{P}\ \gamma$, so that $\alpha\ K_\diamond\ \gamma$.

8. Suppose that $\alpha, \beta, \gamma \in Z$. We have $[\delta\ \mathcal{S}\ \beta, Not[\delta\ \mathcal{R}\ \alpha]]$ or $[\delta\ \mathcal{P}\ \beta, Not[\delta\ \mathcal{T}\ \alpha]]$, for some $\gamma \in A$, and $\beta \succsim^r_\diamond \gamma$. We have either $\delta\ \mathcal{S}\ \gamma$, $Not[\delta\ \mathcal{R}\ \alpha]$ or $\delta\ \mathcal{P}\ \gamma$, $Not[\delta\ \mathcal{T}\ \alpha]$, so that $\alpha\ K_\diamond\ \gamma$.

Step 8.
The proof of (37d), (37e) and (37f) is entirely similar.

Step 9.
We are now in position to apply Lemma 9 to the relations $K_\diamond$ (playing the role of T) and $H_\diamond$ (playing the role of F) with $Q_\diamond$ playing the role of $\succsim$.

Hence, we know that there is a real-valued function F on $A \cup Z$ such that, for all $\alpha, \beta \in A \cup Z$,

$$\alpha\ K_\diamond\ \beta \Leftrightarrow F(\alpha) > F(\beta) + 1,$$
$$\alpha\ H_\diamond\ \beta \Leftrightarrow F(\alpha) = F(\beta) + 1,$$
$$\alpha\ Q_\diamond\ \beta \Leftrightarrow F(\alpha) \geq F(\beta).$$

By construction, the restriction of $K_\diamond$ to $A \times Z$ is $\mathcal{P}$. Similarly, the restriction of $H_\diamond$ to $A \times Z$ is $\mathcal{J}$. The restriction of $Q^a_\diamond$ to $A \times Z$ is $\mathcal{T}$. The restriction of $Q^s_\diamond$ to $A \times Z$ is $\mathcal{F}$. Hence, defining f as the restriction of F on A and g as the restriction of F on Z leads to a representation in model (13). Finally, in view of the definition of $Q_\diamond$, it is clear that (29) holds. The proof is complete. $\square$

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