Decisions with multiple attributes

A brief introduction

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Introduction

Aims

mainly pedagogical

- present elements of the classical theory
- position some extensions w.r.t. this classical theory

Introduction

Comparing holiday packages

	cost	# of days	travel time	category of hotel	distance to beach	Wifi	cultural interest
A	200 €	15	12 h	***	45 km	Y	++
B	425 €	18	15 h	****	0 km	N	−−
C	150 €	4	7 h	**	250 km	N	+
D	300 €	5	10 h	***	5 km	Y	−

Central problems

- helping a DM choose between these packages
- helping a DM structure his preferences

Introduction

Two different contexts

1. decision aiding
    - careful analysis of objectives
    - careful analysis of attributes
    - careful selection of alternatives
    - availability of the DM
2. recommendation systems
    - no analysis of objectives
    - attributes as available
    - alternatives as available
    - limited access to the user

Introduction

Basic model

- additive value function model

$$x \succsim y \Leftrightarrow \sum_{i=1}^{n} v_i(x_i) \geq \sum_{i=1}^{n} v_i(y_i)$$

$$x, y: \text{ alternatives}$$

$$x_i: \text{ evaluation of alternative } x \text{ on attribute } i$$

$$v_i(x_i): \text{ number}$$

- underlies most existing MCDM techniques

Underlying theory: conjoint measurement

- Economics (Debreu, 1960)
- Psychology (Luce & Tukey, 1964)
- tools to help structure preferences

Outline: Classical theory

1. An aside: measurement in Physics
2. An example: even swaps
3. Notation
4. Additive value functions: outline of theory
5. Additive value functions: implementation

Outline: Extensions

Part I

Classical theory: conjoint measurement

Aside: measurement of physical quantities

Lonely individual on a desert island

- no tools, no books, no knowledge of Physics
- wants to rebuild a system of physical measures

A collection a rigid straight rods

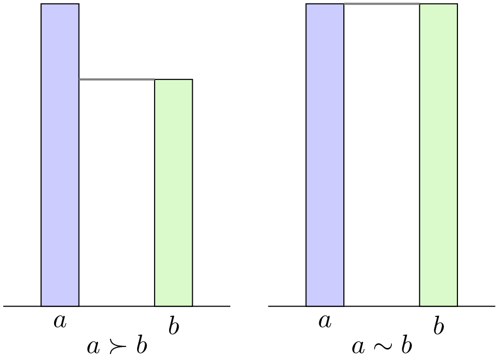
- problem: measuring the length of these rods
  - pre-theoretical intuition
    - length
    - softness, beauty

3 main steps

- comparing objects
- creating and comparing new objects
- creating standard sequences

Step 1: comparing objects

- experimental to conclude which rod has “more length”
- rods side by side on the same horizontal plane



Comparing objects

Results

- $a \succ b$: extremity of rod a is higher than extremity of rod b
- $a \sim b$: extremity of rod a is as high as extremity of rod b

Expected properties

- $a \succ b$, $a \sim b$ or $b \succ a$
- $\succ$ is asymmetric
- $\sim$ is symmetric
- $\succ$ is transitive
- $\sim$ is transitive
- $\succ$ and $\sim$ combine "nicely"
  - $a \succ b$ and $b \sim c \Rightarrow a \succ c$
  - $a \sim b$ and $b \succ c \Rightarrow a \succ c$

Comparing objects

Summary of experiments

- binary relation $\succsim \; = \; \succ \cup \sim$ that is a weak order
  - complete ($a \succsim b$ or $b \succsim a$)
  - transitive ($a \succsim b$ and $b \succsim c \Rightarrow a \succsim c$)

Consequences

- associate a real number $\Phi(a)$ to each object a
- the comparison of numbers faithfully reflects the results of experiments

$$a \succ b \Leftrightarrow \Phi(a) > \Phi(b) \qquad a \sim b \Leftrightarrow \Phi(a) = \Phi(b)$$

- the function Φ defines an ordinal scale
  - applying an increasing transformation to Φ leads to a scale that has the same properties
  - any two scales having the same properties are related by an increasing transformation

Comments

Nature of the scale

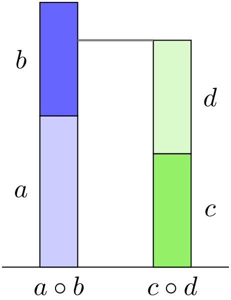
- Φ is quite far from a full-blown measure of length...
- useful though since it allows the experiments to be done only once

Hypotheses are stringent

- highly precise comparisons
- several practical problems
  - any two objects can be compared
  - connections between experiments
  - comparisons may vary in time
- idealization of the measurement process

Step 2: creating and comparing new objects

- use the available objects to create new ones
- concatenate objects by placing two or more rods "in a row"



$$a \circ b \succ c \circ d$$

Concatenation

- we want to be able to deduce $\Phi(a \circ b)$ from $\Phi(a)$ and $\Phi(b)$
- simplest requirement

$$\Phi(a \circ b) = \Phi(a) + \Phi(b)$$

- monotonicity constraints

$$a \succ b \text{ and } c \sim d \Rightarrow a \circ c \succ b \circ d$$

Example

- five rods: $r_1, r_2, \ldots, r_5$
- we may only concatenate two rods (space reasons)
- we may only experiment with different rods
- data:

$$r_1 \circ r_5 \succ r_3 \circ r_4 \succ r_1 \circ r_2 \succ r_5 \succ r_4 \succ r_3 \succ r_2 \succ r_1$$

- all constraints are satisfied: weak ordering and monotonicity

Example

$$r_1 \circ r_5 \succ r_3 \circ r_4 \succ r_1 \circ r_2 \succ r_5 \succ r_4 \succ r_3 \succ r_2 \succ r_1$$

	Φ	Φ'	Φ''
r_1	14	10	14
r_2	15	91	16
r_3	20	92	17
r_4	21	93	18
r_5	28	100	29

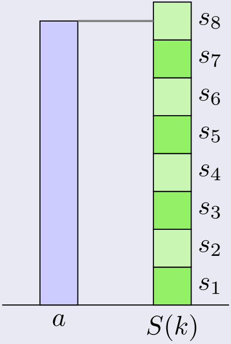
- Φ, Φ' and Φ'' are equally good to compare simple rods
- only Φ and Φ'' capture the comparison of concatenated rods
- going from Φ to Φ'' does not involve a "change of units"

- it is tempting to use Φ or Φ'' to infer comparisons that have not been performed...
- disappointing

$$\Phi : r_2 \circ r_3 \sim r_1 \circ r_4 \quad \Phi'' : r_2 \circ r_3 \succ r_1 \circ r_4$$

Step 3: creating and using standard sequences

- choose a standard rod
- be able to build perfect copies of the standard
- concatenate the standard rod with its perfects copies



$$S(8) \succ a \succ S(7)$$
$$\Phi(s) = 1 \Rightarrow 7 < \Phi(a) < 8$$

Convergence

First method

- choose a smaller standard rod
- repeat the process

Second method

- prepare a perfect copy of the object
- concatenate the object with its perfect copy
- compare the "doubled" object to the original standard sequence
- repeat the process

Summary

Extensive measurement

- Krantz, Luce, Suppes & Tversky (1971, chap. 3)

4 Ingredients

1. well-behaved relations $\succ$ and $\sim$
2. concatenation operation $\circ$
3. consistency requirements linking $\succ$, $\sim$ and $\circ$
4. ability to prepare perfect copies of some objects in order to build standard sequences

Neglected problems

- many!

Question

Can this be applied outside Physics?

- no concatenation operation (intelligence!)

What is conjoint measurement?

Conjoint measurement

- mimicking the operations of extensive measurement
  - when there are no concatenation operation readily available
  - when several dimensions are involved

Seems overly ambitious

- let us start with a simple example

Example: Hammond, Keeney & Raiffa

Choice of an office to rent

- five locations have been identified
- five attributes are being considered
  - *Commute* time (minutes)
  - *Clients*: percentage of clients living close to the office
  - *Services*: ad hoc scale
    - A (all facilities), B (telephone and fax), C (no facility)
  - *Size*: square feet ($\simeq 0.1\ \mathrm{m}^2$)
  - *Cost*: \$ per month

Attributes

- *Commute*, *Size* and *Cost* are natural attributes
- *Clients* is a proxy attribute
- *Services* is a constructed attribute

Data

	a	b	c	d	e
Commute	45	25	20	25	30
Clients	50	80	70	85	75
Services	*A*	*B*	*C*	*A*	*C*
Size	800	700	500	950	700
Cost	1850	1700	1500	1900	1750

Hypotheses and context

- a single cooperative DM
- choice of a single office
- ceteris paribus reasoning seems possible
  Commute: decreasing *Clients*: increasing
  Services: increasing *Size*: increasing
  Cost: decreasing
- dominance has meaning

	a	b	c	d	e
Commute	45	25	20	25	30
Clients	50	80	70	85	75
Services	A	B	C	A	C
Size	800	700	500	950	700
Cost	1850	1700	1500	1900	1750

- b dominates alternative e
- d is "close" to dominating a
- divide and conquer: dropping alternatives
  - drop a and e

	b	c	d
Commute	25	20	25
Clients	80	70	85
Services	B	C	A
Size	700	500	950
Cost	1700	1500	1900

- no more dominance
- assessing tradeoffs
- all alternatives except c have a common evaluation on *Commute*
- modify c in order to bring it to this level
  - starting with c, what is the gain on *Clients* that would exactly compensate a loss of 5 min on *Commute*?
  - difficult but central question

	c	c'
Commute	20	**25**
Clients	70	$\mathbf{70} + \delta$
Services	C	C
Size	500	500
Cost	1500	1500

find δ such that $c' \sim c$

Answer

- for $\delta = 8$, I am indifferent between c and c'
- replace c with c'

	b	c'	d
Commute	25	25	25
Clients	80	78	85
Services	B	C	A
Size	700	500	950
Cost	1700	1500	1900

- all alternatives have a common evaluation on *Commute*
- divide and conquer: dropping attributes
    - drop attribute *Commute*

	b	c'	d
Clients	80	78	85
Services	B	C	A
Size	700	500	950
Cost	1700	1500	1900

	b	c'	d
Clients	80	78	85
Services	B	C	A
Size	700	500	950
Cost	1700	1500	1900

- check again for dominance
- unfruitful
- assess new tradeoffs
  - neutralize Service using *Cost* as reference

	b	c'	d
Clients	80	78	85
Services	B	C	A
Size	700	500	950
Cost	1700	1500	1900

Questions

- what maximal increase in monthly cost would you be prepared to pay to go from C to B on service for c'?
  - answer: 250 \$
- what minimal decrease in monthly cost would you ask if we go from A to B on service for d?
  - answer: 100 \$

	b	c'	c''	d	d'
Clients	80	78	78	85	85
Services	B	C	**B**	A	**B**
Size	700	500	500	950	950
Cost	1700	1500	$1500 + \mathbf{250}$	1900	$1900 - \mathbf{100}$

- replacing c' with c''
- replacing d with d'
- dropping Service

	b	c''	d'
Clients	80	78	85
Size	700	500	950
Cost	1700	1750	1800

- checking for dominance: c'' is dominated by b
- c'' can be dropped

- dropping c''

	b	d'
Clients	80	85
Size	700	950
Cost	1700	1800

- no dominance
- question: starting with b what is the additional cost that you would be prepared to pay to increase size by 250?
  - answer: 250 $

	b	b'	d'
Clients	80	80	85
Size	700	**950**	950
Cost	1700	1700 + **250**	1800

- replace b with b'
- drop *Size*

	b'	d'
Clients	80	85
Size	950	950
Cost	1950	1800

	b'	d'
Clients	80	85
Cost	1950	1800

- check for dominance
- d' dominates b'

Conclusion

- Recommend d as the final choice

Summary

Remarks

- very simple process
- process entirely governed by $\succ$ and $\sim$
- no question on "intensity of preference"
- notice that importance plays absolutely no rôle
- why be interested in something more complex?

Problems

- set of alternative is small
  - many questions otherwise
- output is not a preference model
  - if new alternatives appear, the process should be restarted
- what are the underlying hypotheses?

Monsieur Jourdain doing conjoint measurement

Similarity with extensive measurement

- $\succ$: preference, $\sim$: indifference
- we have implicitly supposed that they combine nicely

Recommendation: d

- we should be able to prove that $d \succ a$, $d \succ b$, $d \succ c$ and $d \succ e$
- dominance: $b \succ e$ and $d \succ a$
- tradeoffs + dominance: $b \succ c''$, $c \sim c'$, $c' \sim c$, $d' \sim d$, $b' \sim b$, $d' \succ b'$

$$d \succ a, b \succ e$$
$$c'' \sim c', c' \sim c, b \succ c''$$
$$\Rightarrow b \succ c$$
$$d \sim d', b \sim b', d' \succ b'$$
$$\Rightarrow d \succ b$$

Monsieur Jourdain doing conjoint measurement

OK... but where are the standard sequences?

- hidden... but really there!
- standard sequence for length: objects that have exactly the same length
- tradeoffs: preference intervals on distinct attributes that have the same length
  - $c \sim c'$
  - $[25, 20]$ on *Commute* has the same length as $[70, 78]$ on *Client*

	c	c'	f	f'
Commute	20	**25**	*20*	**25**
Clients	70	**78**	*78*	**82**
Services	*C*	*C*	*C*	*C*
Size	500	500	500	500
Cost	1500	1500	1500	1500

$[70, 78]$ has the same length $[78, 82]$ on *Client*

Setting

- $N = \{1, 2, \ldots, n\}$ set of attributes
- X_i: set of possible levels on the ith attribute
- $X = \prod_{i=1}^{n} X_i$: set of all conceivable alternatives
    - X include the alternatives under study... and many others

- $J \subseteq N$: subset of attributes
- $X_J = \prod_{j \in J} X_j$, $X_{-J} = \prod_{j \notin J} X_j$
- $(x_J, y_{-J}) \in X$
- $(x_i, y_{-i}) \in X$

- $\succsim$: binary relation on X: "at least as good as"
- $x \succ y \Leftrightarrow x \succsim y$ and $Not[y \succsim x]$
- $x \sim y \Leftrightarrow x \succsim y$ and $y \succsim x$

Preference relations on Cartesian products

Applications

- Economics: consumers comparing bundles of goods
- Decision under uncertainty: consequences in several states
- Inter-temporal decision making: consequences at several moments in time
- Inequality measurement: distribution of wealth across individuals
- Decision making with multiple attributes
  - in all other cases, the Cartesian product is homogeneous

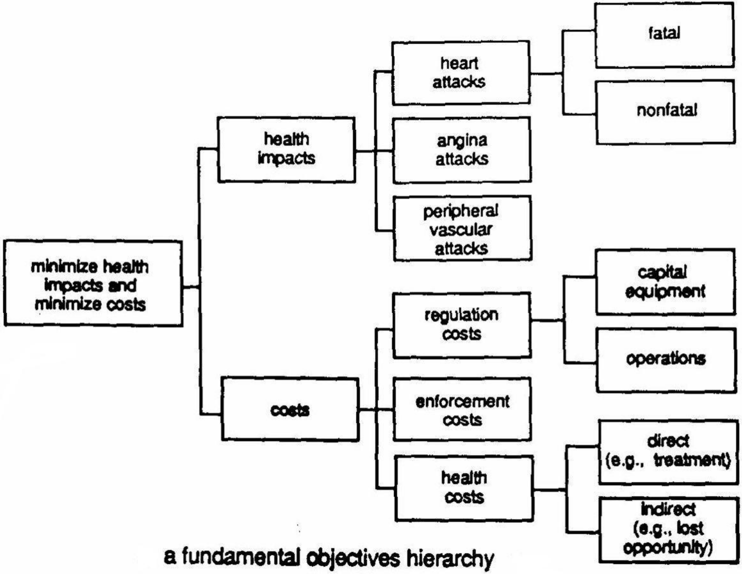
What will be ignored today

Ignored

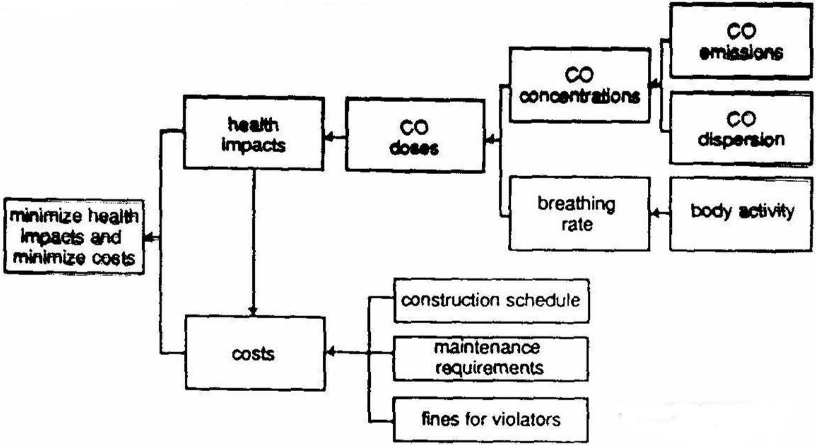
- structuring of objectives
- from objectives to attributes
- adequate family of attributes
- risk, uncertainty, imprecision

Keeney's view

- fundamental objectives: why?
- means objectives: how?



a fundamental objectives hierarchy



a means-ends objectives network

Table I. Preclosure Objectives and Performance Measures

Objective	Performance measure
Health-and-safety impacts	
1. Minimize worker health effects from radiation exposure at the repository	X_1: repository-worker radiological fatalities
2. Minimize public health effects from radiation exposure at the repository	X_2: public radiological fatalities from repository
3. Minimize worker fatalities from nonradiological causes at the repository	X_3: repository-worker nonradiological fatalities
4. Minimize public fatalities from nonradiological causes at the repository	X_4: public nonradiological fatalities from repository
5. Minimize worker health effects from radiation exposure in waste transportation	X_5: transportation-worker radiological fatalities
6. Minimize public health effects from radiation exposure in waste transportation	X_6: public radiological fatalities from transportation
7. Minimize worker fatalities from nonradiological causes in waste transportation	X_7: transportation-worker nonradiological fatalities
8. Minimize public fatalities from nonradiological causes in waste transportation	X_8: public nonradiological fatalities from transportation
Environmental impacts	
9. Minimize aesthetic degradation	X_9: constructed scale[a]
10. Minimize the degradation of archaeological, historical, and cultural properties	X_{10}: constructed scale[a]
11. Minimize biological degradation	X_{11} constructed scale[a]
Socioeconomic impacts	
12. Minimize adverse socioeconomic impacts	X_{12}: constructed scale[a]
Economic impacts	
13. Minimize repository costs	X_{13}: millions of dollars
14. Minimize waste-transportation costs	X_{14}: millions of dollars

Table 4.1. A constructed attribute for public attitudes

Attribute level	Description of attribute level
1	*Support:* No groups are opposed to the facility and at least one group has organized support for the facility.
0	*Neutrality:* All groups are indifferent or uninterested.
−1	*Controversy:* One or more groups have organized opposition, although no groups have action-oriented opposition. Other groups may either be neutral or support the facility.
−2	*Action-oriented opposition:* Exactly one group has action-oriented opposition. The other groups have organized support, indifference or organized opposition.
−3	*Strong action-oriented opposition:* Two or more groups have action-oriented opposition.

Scale to Measure Biological Impact

0. Loss of 1.0 mi^2 of entirely agricultural or urban "habitat" with no loss of any "native" communities.
1. Loss of 1.0 mi^2 of primarily (75%) agricultural habitat with loss of 25% of second growth; no measurable loss of wetlands or endangered species habitat.
2. Loss of 1.0 mi^2 of farmed (50%) and disturbed (i.e., logged or new second-growth) (50%) habitat; no measurable loss of wetlands or endangered species habitat.
3. Loss of 1.0 mi^2 of recently disturbed (logged, plowed) habitat with disturbance to surrounding (within 1.0 mi of site border) previously disturbed habitat; 15% loss of wetlands and/or endangered species habitat.
4. Loss of 1.0 mi^2 of farmed or disturbed area (50%) and mature second-growth or other undisturbed community (50%); 15% loss of wetlands and/or endangered species.
5. Loss of 1.0 mi^2 of primarily (75%) undisturbed mature desert community (i.e., sagebrush); 15% loss of wetlands and/or endangered species habitat.
6. Loss of 1.0 mi^2 of mature second-growth (but not virgin) forest community; 50% loss of big game and upland game birds; 50% loss of local wetlands and local endangered species habitat.
7. Loss of 1.0 mi^2 of mature second-growth forest community; 90% loss of local productive wetlands and local endangered species habitat.
8. Complete loss of 1.0 mi^2 of mature virgin forest; 100% loss of local wetlands and local endangered species habitat.

Impact level	Impacts on historical properties in the effected area[a]
0	There are no impacts on any significant historical properties
1	One historical property of major significance or 5 historical properties of minor significance are subjected to minimal adverse impacts
2	Two historical properties of major significance or 10 historical properties of minor significance are subjected to minimal adverse impacts
3	Two historical properties of major significance or 10 historical properties of minor significance are subjected to major adverse impacts
4	Three historical properties of major significance or 15 historical properties of minor significance are subjected to major adverse impacts
5	Four historical properties of major significance or 20 historical properties of minor significance are subjected to major adverse impacts

Marginal preference and independence

Marginal preferences

- $J \subseteq N$: subset of attributes
- $\succsim_J$ marginal preference relation induced by $\succsim$ on X_J

$$x_J \succsim_J y_J \Leftrightarrow (x_J, z_{-J}) \succsim (y_J, z_{-J}), \text{ for all } z_{-J} \in X_{-J}$$

Independence

- J is independent for $\succsim$ if

$$[(x_J, z_{-J}) \succsim (y_J, z_{-J}), \text{ for some } z_{-J} \in X_{-J}] \Rightarrow x_J \succsim_J y_J$$

- common levels on attributes other than J do not affect preference

Separability

- J is separable for $\succsim$ if

$$[(x_J, z_{-J}) \succ (y_J, z_{-J}), \text{ for some } z_{-J} \in X_{-J}] \Rightarrow x_J \succsim_J y_J$$

- varying common levels on attributes other than J do reverse strict preference

Independence

Definition

- for all $i \in N$, $\{i\}$ is independent, $\succsim$ is weakly independent
- for all $J \subseteq N$, J is independent, $\succsim$ is independent

Proposition

Let $\succsim$ be a weakly independent weak order on $X = \prod_{i=1}^{n} X_i$. Then:

- $\succsim_i$ is a weak order on X_i
- $[x_i \succsim_i y_i$, for all $i \in N] \Rightarrow x \succsim y$
- $[x_i \succsim_i y_i$, for all $i \in N$ and $x_j \succ_j y_j$ for some $j \in N] \Rightarrow x \succ y$

for all $x, y \in X$

Dominance

- as soon as I have a weakly independent weak order
- dominance arguments apply

Independence in practice

Independence

- it is easy to imagine examples in which independence is violated
  - Main course and Wine example
- it is nearly hopeless to try to work if weak independence (at least weak separability) is not satisfied
- some (e.g., R. L. Keeney) think that the same is true for independence
- in all cases if independence is violated, things get complicated
  - decision aiding vs AI

May be excessive

- much more on independence this afternoon

Outline of theory: 2 attributes

Question

- suppose I can "observe" $\succsim$ on $X = X_1 \times X_2$
- what must be supposed to guarantee that I can represent $\succsim$ in the additive value function model

$$v_1 : X_1 \to \mathbb{R}$$
$$v_2 : X_2 \to \mathbb{R}$$
$$(x_1, x_2) \succsim (y_1, y_2) \Leftrightarrow v_1(x_1) + v_2(x_2) \geq v_1(y_1) + v_2(y_2)$$

- $\succsim$ must be an independent weak order

Method

- try building standard sequences and see if it works!

Why an additive model?

Answer

- v_1 and v_2 will be built so that additivity holds
- equivalent multiplicative model

$$(x_1, x_2) \succsim (y_1, y_2) \Leftrightarrow w_1(x_1)w_2(x_2) \geq w_1(y_1)w_2(y_2)$$

$$w_1 = \exp(v_1)$$

$$w_2 = \exp(v_2)$$

Uniqueness

Important observation

Suppose that there are v_1 and v_2 such that

$$(x_1, x_2) \succsim (y_1, y_2) \Leftrightarrow v_1(x_1) + v_2(x_2) \geq v_1(y_1) + v_2(y_2)$$

If $\alpha > 0$

$$w_1 = \alpha v_1 + \beta_1 \quad w_2 = \alpha v_2 + \beta_2$$

is also a valid representation

Consequences

- fixing $v_1(x_1) = v_2(x_2) = 0$ is harmless
- fixing $v_1(y_1) = 1$ is harmless if $y_1 \succ_1 x_1$

Standard sequences

Preliminaries

- choose arbitrarily two levels $x_1^0, x_1^1 \in X_1$
- make sure that $x_1^1 \succ_1 x_1^0$
- choose arbitrarily one level $x_2^0 \in X_2$
- $(x_1^0, x_2^0) \in X$ is the reference point (origin)
- the preference interval $[x_1^0, x_1^1]$ is the unit

Building a standard sequence on X_2

- find a "preference interval" on X_2 that has the same "length" as the reference interval $[x_1^0, x_1^1]$
- find x_2^1 such that

$$(x_1^0, x_2^1) \sim (x_1^1, x_2^0)$$

$$v_1(x_1^0) + v_2(x_2^1) = v_1(x_1^1) + v_2(x_2^0) \text{ so that}$$
$$v_2(x_2^1) - v_2(x_2^0) = v_1(x_1^1) - v_1(x_1^0)$$

- the structure of X_2 has to be "rich enough"

Standard sequences

Consequences

$$(x_1^0, x_2^1) \sim (x_1^1, x_2^0)$$
$$v_2(x_2^1) - v_2(x_2^0) = v_1(x_1^1) - v_1(x_1^0)$$

- it can be supposed that

$$v_1(x_1^0) = v_2(x_2^0) = 0$$
$$v_1(x_1^1) = 1$$

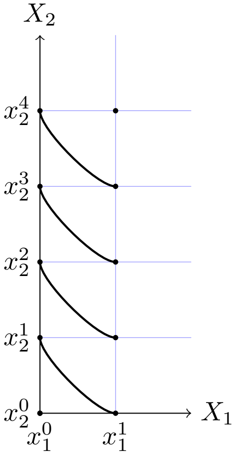
$$\Rightarrow v_2(x_2^1) = 1$$

Going on

$$(x_1^0, x_2^1) \sim (x_1^1, x_2^0)$$
$$(x_1^0, x_2^2) \sim (x_1^1, x_2^1)$$
$$(x_1^0, x_2^3) \sim (x_1^1, x_2^2)$$
$$\dots$$
$$(x_1^0, x_2^k) \sim (x_1^1, x_2^{k-1})$$

$$v_2(x_2^1) - v_2(x_2^0) = v_1(x_1^1) - v_1(x_1^0) = 1$$
$$v_2(x_2^2) - v_2(x_2^1) = v_1(x_1^1) - v_1(x_1^0) = 1$$
$$v_2(x_2^3) - v_2(x_2^2) = v_1(x_1^1) - v_1(x_1^0) = 1$$
$$\dots$$
$$v_2(x_2^k) - v_2(x_2^{k-1}) = v_1(x_1^1) - v_1(x_1^0) = 1$$

$$\Rightarrow v_2(x_2^2) = 2, v_2(x_2^3) = 3, \dots, v_2(x_2^k) = k$$


X_2
x_2^4
x_2^3
x_2^2
x_2^1
x_2^0
X_1
x_1^0
x_1^1

Standard sequence

Archimedean

- implicit hypothesis for length
  - the standard sequence can reach any the length of any object

$$\forall x, y \in \mathbb{R}, \exists n \in \mathbb{N} : ny > x$$

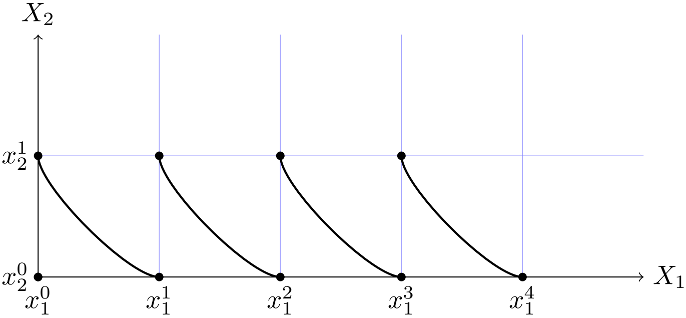
- a similar hypothesis has to hold here
- rough interpretation
  - there are not "infinitely" liked or disliked consequences

Building a standard sequence on X_1

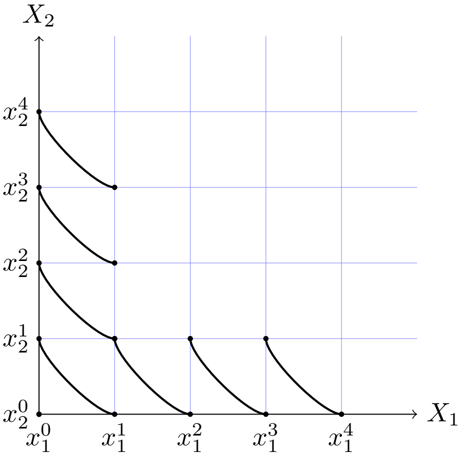
$$(x_1^2, x_2^0) \sim (x_1^1, x_2^1)$$
$$(x_1^3, x_2^0) \sim (x_1^2, x_2^1)$$
$$\dots$$
$$(x_1^k, x_2^0) \sim (x_1^{k-1}, x_2^1)$$

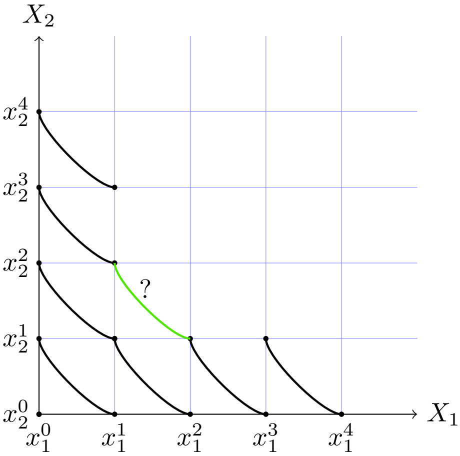
$$v_1(x_1^2) - v_1(x_1^1) = v_2(x_2^1) - v_2(x_2^0) = 1$$
$$v_1(x_1^3) - v_1(x_1^2) = v_2(x_2^1) - v_2(x_2^0) = 1$$
$$\dots$$
$$v_1(x_1^k) - v_1(x_1^{k-1}) = v_2(x_2^1) - v_2(x_2^0) = 1$$

$$v_1(x_1^2) = 2, v_1(x_1^3) = 3, \dots, v_1(x_1^k) = k$$



X_2
x_2^1
x_2^0
X_1
x_1^0
x_1^1
x_1^2
x_1^3
x_1^4

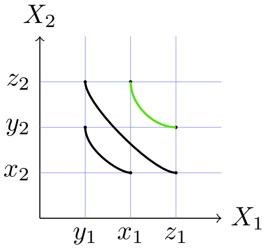

X_2
x_2^4
x_2^3
x_2^2
x_2^1
x_2^0
X_1
x_1^0
x_1^1
x_1^2
x_1^3
x_1^4



X_2
x_2^4
x_2^3
x_2^2
?
x_2^1
x_2^0
X_1
x_1^0
x_1^1
x_1^2
x_1^3
x_1^4

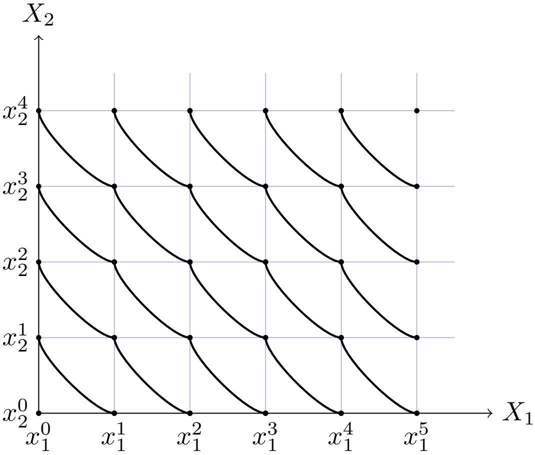
Thomsen condition

$$\begin{array}{c}(x_1, x_2) \sim (y_1, y_2) \\ \text{and} \\ (y_1, z_2) \sim (z_1, x_2)\end{array} \Rightarrow (x_1, z_2) \sim (z_1, y_2)$$



Consequence

- there is an additive value function on the grid



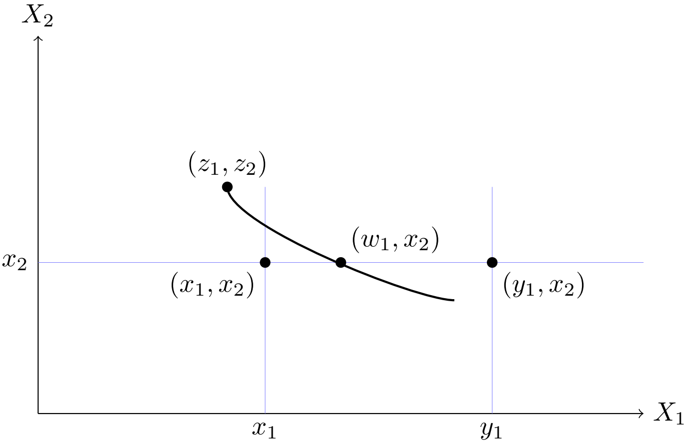
X_2
X_1
x_2^4
x_2^3
x_2^2
x_2^1
x_2^0
x_1^0
x_1^1
x_1^2
x_1^3
x_1^4
x_1^5

Summary

- we have defined a "grid"
- there is an additive value function on the grid
- iterate the whole process with a "denser grid"

Hypotheses

- Archimedean: every strictly bounded standard sequence is finite
- essentiality: both $\succ_1$ and $\succ_2$ are nontrivial
- restricted solvability



$$\left.\begin{array}{r} (y_1, x_2) \succ (z_1, z_2) \\ (z_1, z_2) \succ (x_1, x_2) \end{array}\right\} \Rightarrow \exists w_1 \text{ such that } (z_1, z_2) \sim (w_1, x_2)$$

Basic result

Theorem (2 attributes)

If

- restricted solvability holds
- each attribute is essential

then
the additive value function model holds
if and only if
$\succsim$ is an independent weak order satisfying the Thomsen and the Archimedean conditions

The representation is unique up to scale and location

General case

Good news

- entirely similar...
- with a very nice surprise: Thomsen can be forgotten
  - if $n = 2$, independence is identical with weak independence
  - if $n > 3$, independence is much stronger than weak independence

	X_1	X_2	X_3
a	75	10	0
b	100	2	0
c	75	10	40
d	100	2	40

X_1: % of nights at home
X_2: attractiveness of city
X_3: salary increase
weak independence holds
$a \succ b$ and $d \succ c$ is reasonable

Basic result

Theorem (more than 2 attributes)

If

- restricted solvability holds
- at least three attributes are essential

then
the additive value function model holds
if and only if
$\succsim$ is an independent weak order satisfying the Archimedean condition

The representation is unique up to scale and location

Independence and even swaps

Even swaps technique

- assessing tradeoffs. . .
- after having suppressed attributes

Implicit hypothesis

- what happens on these attributes do not influence tradeoffs
- this is another way to formulate independence

Assessing value functions

Standard technique

- check independence
- build standard sequences
  - importance has no rôle
  - do not even pronounce the word!!

Problems

- many questions
- questions on fictitious alternatives
- rests on indifference judgments
- discrete attributes
- propagation of "errors"

UTA: outline

Principle

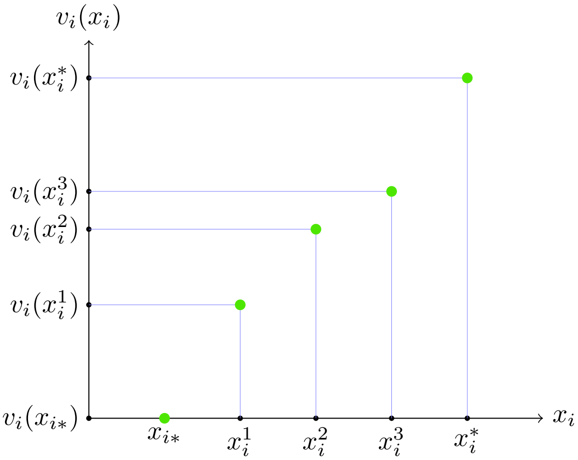
- select a number of reference alternatives that the DM knows well
- rank order these alternatives
- test, using LP, if this information is compatible with an additive value function
  - if yes, present a central one
    - interact with the DM
    - apply the resulting function to the whole set of alternatives
  - if not
    - interact with the DM

UTA: decision variables

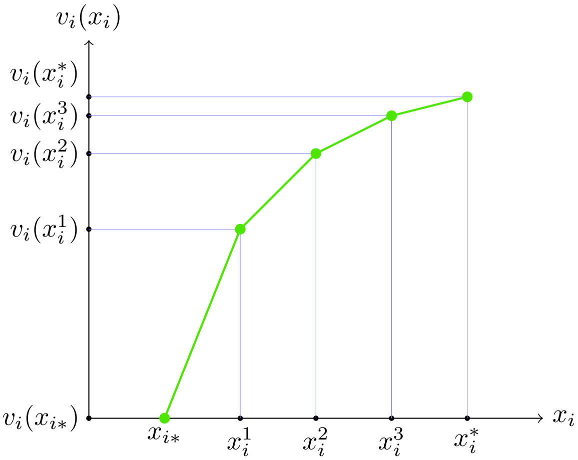
Aim

- assess $v_1, v_2, \ldots, v_n$
- normalization
  - x_{i*}: worst level on attribute i
  - x_i^*: best level on attribute i
  - $v_1(x_{1*}) = v_2(x_{2*}) = \ldots = v_n(x_{n*}) = 0$
  - $\sum_{i=1}^{n} v_i(x_i^*) = 1$
- if the attribute is discrete
  - take as many variables as there are levels
- if the attribute is not discrete
  - consider a piecewise linear approximation

- discrete attribute
  - $X_i = \{x_{i*}, x_i^1, x_i^2, \ldots, x_i^{r_i}, x_i^*\}$
- continuous attribute
  - choose the number of linear pieces $r_i + 1$
  - $[x_{i*}, x_i^1]$, $[x_i^1, x_i^2], \ldots, [x_i^{r_i-1}, x_i^{r_i}], [x_i^{r_i}, x_i^*]$



$v_i(x_i)$
$v_i(x_i^*)$
$v_i(x_i^3)$
$v_i(x_i^2)$
$v_i(x_i^1)$
$v_i(x_{i*})$
x_{i*}
x_i^1
x_i^2
x_i^3
x_i^*
x_i



$v_i(x_i)$
$v_i(x_i^*)$
$v_i(x_i^3)$
$v_i(x_i^2)$
$v_i(x_i^1)$
$v_i(x_{i*})$
x_{i*}
x_i^1
x_i^2
x_i^3
x_i^*
x_i

UTA: constraints

Using these conventions

- for all x, $v(x) = \sum_{i=1}^{n} v_i(x_i)$ can be expressed as a linear combination of the $n \sum_i^n (r_i + 1)$ variables

$$
\begin{aligned}
x \succ y \Leftrightarrow & v(x) > v(y) \\
& v(x) - v(y) + \sigma^+(xy) - \sigma^-(xy) \geq \varepsilon \\
x \sim y \Leftrightarrow & v(x) = v(y) \\
& v(x) - v(y) + \sigma^+(xy) - \sigma^-(xy) = 0
\end{aligned}
$$

$$\text{minimize } Z = \sum_{\text{constraints}} \sigma^{+}(xy) + \sigma^{-}(xy)$$

s.t.

one constraint per pair of compared alternatives

normalization constraints

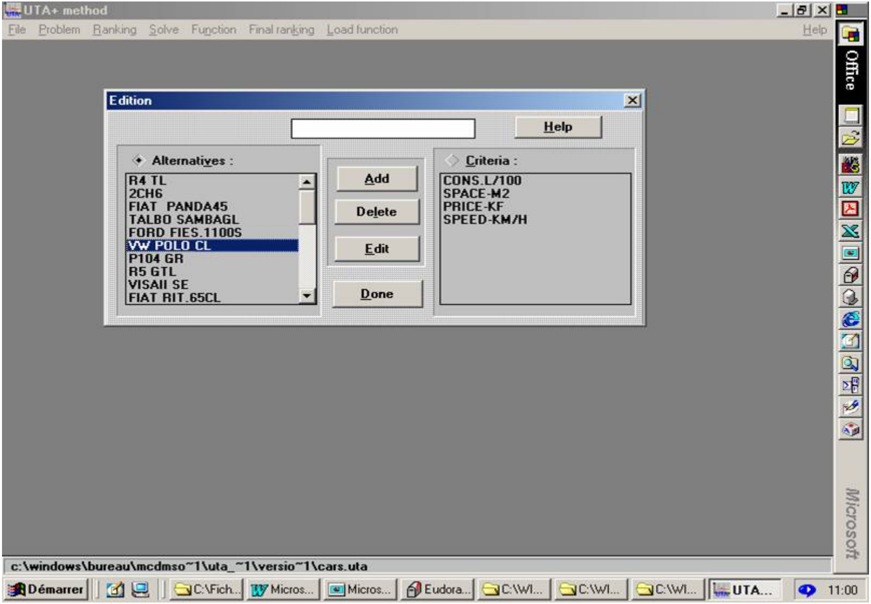
UTA: analyzing results

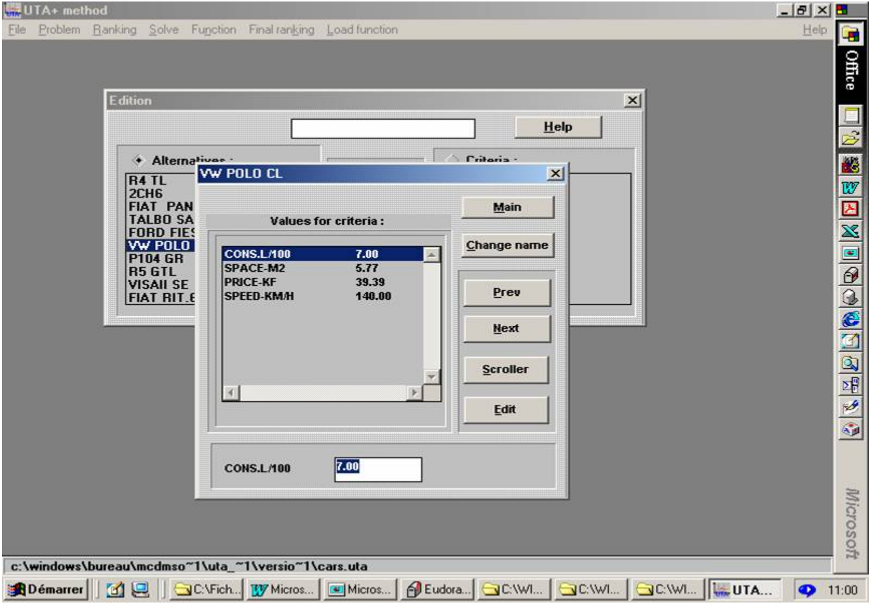
If $Z^* = 0$

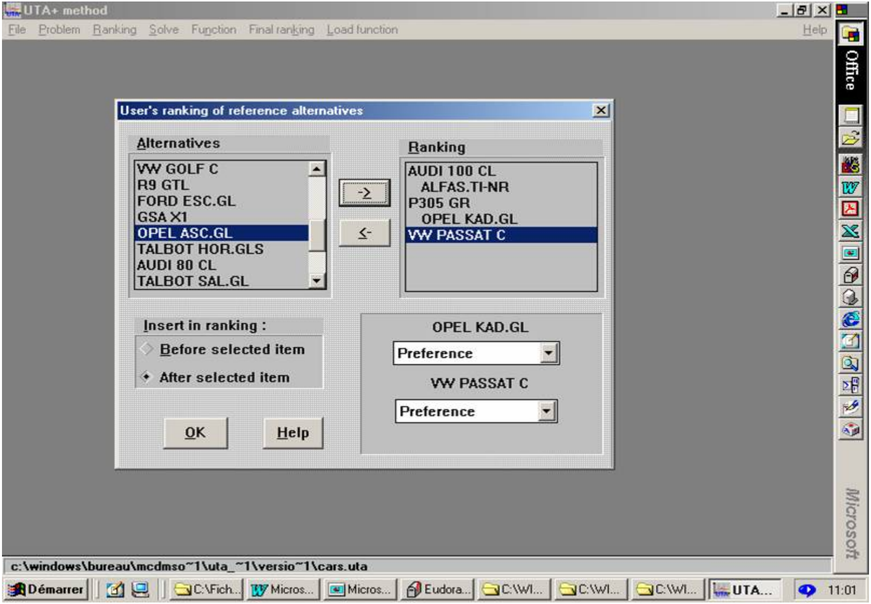
- there is one additive value function compatible with the given information
- there are infinitely many (identically normalized) compatible additive value functions $v \in \mathcal{V}$
- use post-optimality analysis and/or interaction to explore $\mathcal{V}$

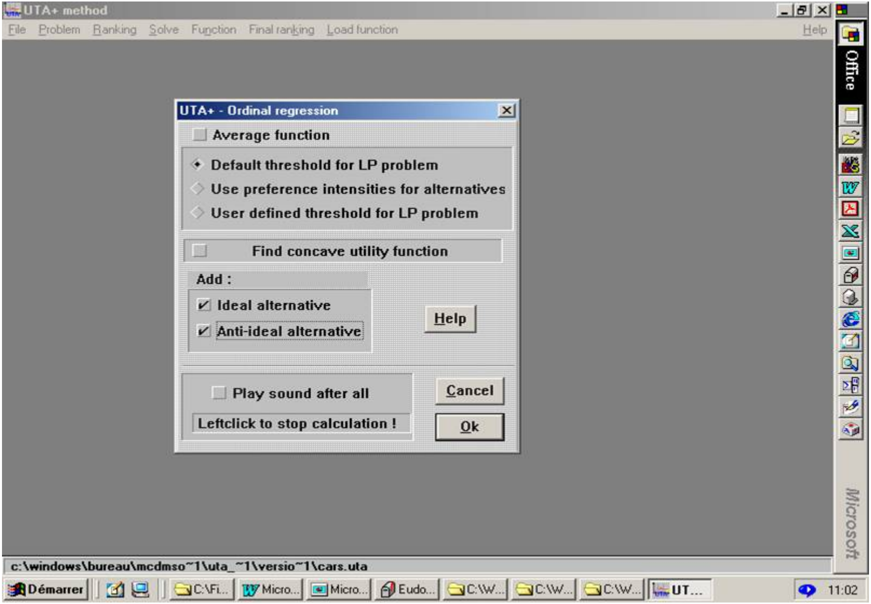
If $Z^* > 0$

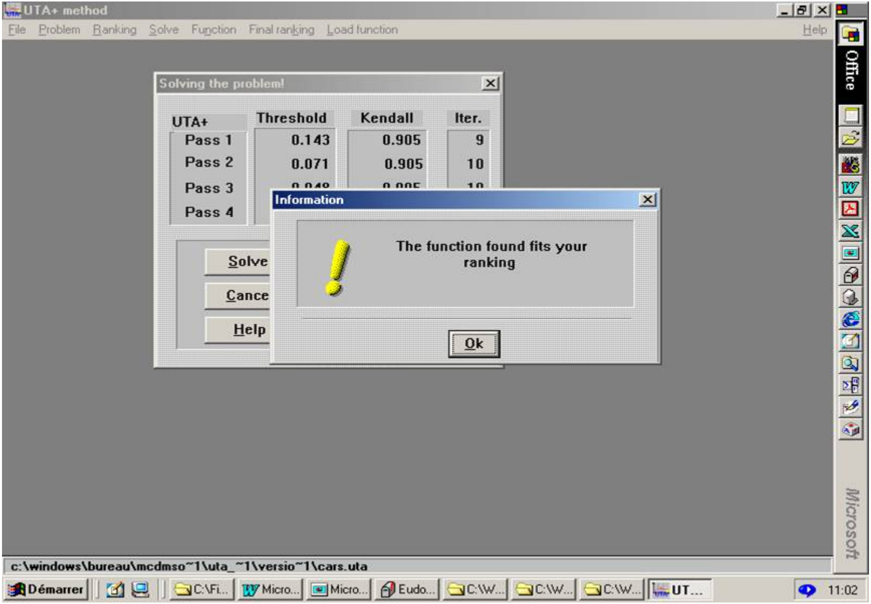
- there is no additive value function compatible with the given information
- interact
  - increase the number of linear pieces
  - decrease ε
  - modify ranking
  - diagnostic a failure of independence
  - use approximate function

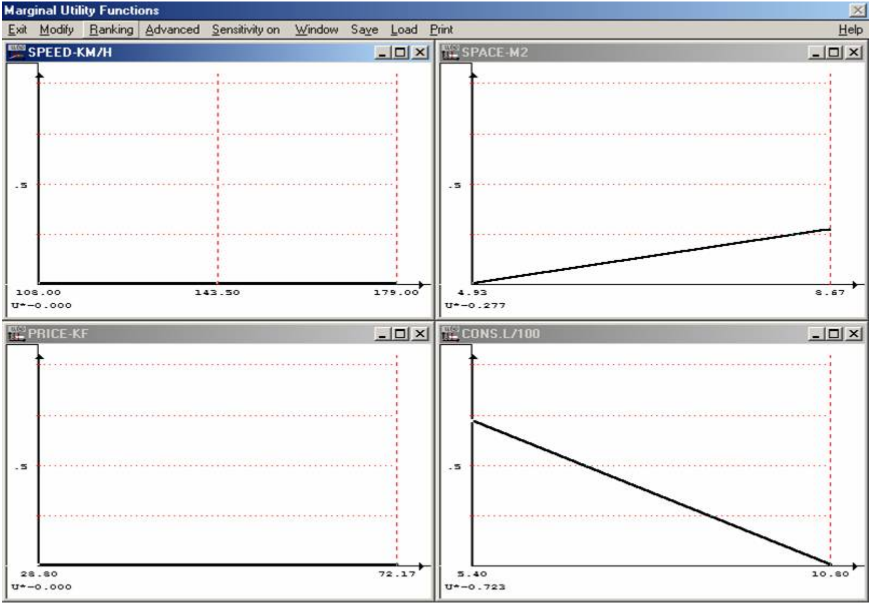

UTA+ method
File
Problem
Ranking
Solve
Function
Final ranking
Load function
Help
Office
Edition
Help
Alternatives :
R4 TL
2CH6
FIAT PANDA45
TALBO SAMBAGL
FORD FIES.1100S
VW POLO CL
P104 GR
R5 GTL
VISAII SE
FIAT RIT.65CL
Add
Delete
Edit
Done
Criteria :
CONS.L/100
SPACE-M2
PRICE-KF
SPEED-KM/H
Microsoft
c:\windows\bureau\mcdmso~1\uta_~1\versio~1\cars.uta
Démarrer
C:\Fich...
Micros...
Micros...
Eudora...
C:\WI...
C:\WI...
C:\WI...
UTA...
11:00


UTA+ method
File
Problem
Ranking
Solve
Function
Final ranking
Load function
Help
Edition
Help
Alternatives :
Criteria :
R4 TL
2CH6
FIAT PAN
TALBO SA
FORD FIES
VW POLO
P104 GR
R5 GTL
VISAII SE
FIAT RIT.
VW POLO CL
Main
Values for criteria :
Change name
CONS.L/100 7.00
SPACE-M2 5.77
PRICE-KF 39.39
SPEED-KM/H 140.00
Prev
Next
Scroller
Edit
CONS.L/100
7.00
c:\windows\bureau\mcdmso~1\uta_~1\versio~1\cars.uta
Démarrer
C:\Fich...
Micros...
Micros...
Eudora...
C:\WI...
C:\WI...
C:\WI...
UTA...
11:00
Office
Microsoft

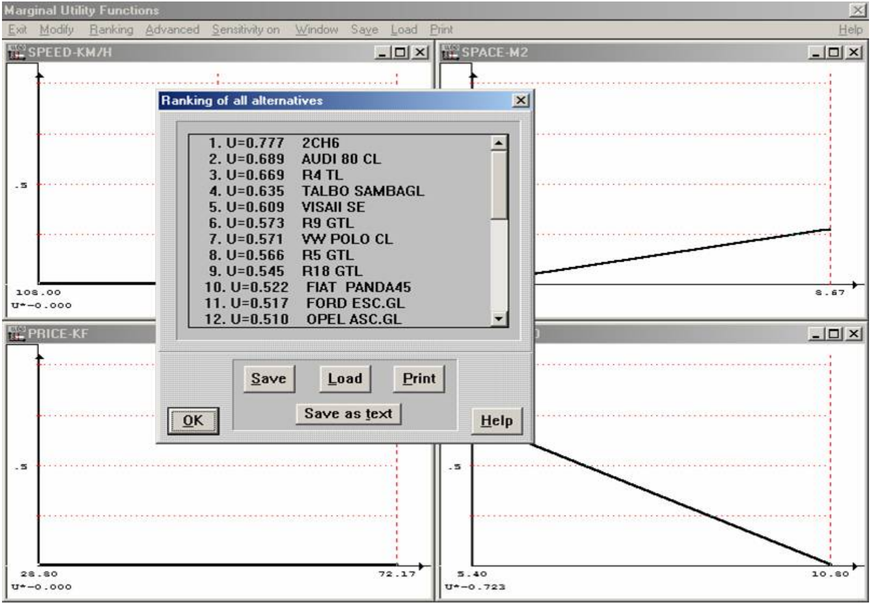

UTA+ method
File
Problem
Ranking
Solve
Function
Final ranking
Load function
Help
Office
User's ranking of reference alternatives
Alternatives
VW GOLF C
R9 GTL
FORD ESC.GL
GSA X1
OPEL ASC.GL
TALBOT HOR.GLS
AUDI 80 CL
TALBOT SAL.GL
->
<-
Ranking
AUDI 100 CL
ALFAS.TI-NR
P305 GR
OPEL KAD.GL
VW PASSAT C
Insert in ranking :
Before selected item
After selected item
OPEL KAD.GL
Preference
VW PASSAT C
Preference
OK
Help
Microsoft
c:\windows\bureau\mcdmso~1\uta_~1\versio~1\cars.uta
Démarrer
C:\Fich...
Micros...
Micros...
Eudora...
C:\WI...
C:\WI...
C:\WI...
UTA...
11:01


UTA+ method
File Problem Ranking Solve Function Final ranking Load function Help
UTA+ - Ordinal regression
Average function
Default threshold for LP problem
Use preference intensities for alternatives
User defined threshold for LP problem
Find concave utility function
Add :
Ideal alternative
Anti-ideal alternative
Help
Play sound after all
Leftclick to stop calculation !
Cancel
Ok
c:\windows\bureau\mcdmso~1\uta_~1\versio~1\cars.uta
Démarrer
11:02
Office
Microsoft


UTA+ method
File Problem Ranking Solve Function Final ranking Load function Help
Solving the problem!
UTA+
Threshold
Kendall
Iter.
Pass 1 0.143 0.905 9
Pass 2 0.071 0.905 10
Pass 3
Pass 4
Solve
Help
Information
The function found fits your ranking
Ok
Office
Microsoft
c:\windows\bureau\mcdmso~1\uta_~1\versio~1\cars.uta
Démarrer
11:02



Marginal Utility Functions
Exit Modify Ranking Advanced Sensitivity on Window Save Load Print Help
SPEED-KM/H
.5
108.00
143.50
179.00
U*-0.000
SPACE-M2
.5
4.93
8.67
U*-0.277
PRICE-KF
.5
28.80
72.17
U*-0.000
CONS.L/100
.5
5.40
10.80
U*-0.723


Marginal Utility Functions
Exit Modify Ranking Advanced Sensitivity on Window Save Load Print Help
SPEED-KM/H
SPACE-M2
PRICE-KF
Ranking of all alternatives
1. U=0.777 2CH6
2. U=0.689 AUDI 80 CL
3. U=0.669 R4 TL
4. U=0.635 TALBO SAMBAGL
5. U=0.609 VISAII SE
6. U=0.573 R9 GTL
7. U=0.571 VW POLO CL
8. U=0.566 R5 GTL
9. U=0.545 R18 GTL
10. U=0.522 FIAT PANDA45
11. U=0.517 FORD ESC.GL
12. U=0.510 OPEL ASC.GL
Save
Load
Print
Save as text
OK
Help
.5
108.00
U*-0.000
8.67
28.80
72.17
U*-0.000
5.40
10.80
U*-0.723

UTA: variants

Possible variants

- use a different formulation (e.g., minimize the maximum deviation)
- add constraints on the shape of the v_i
  - decreasing, increasing, convex, s-shaped
- post optimality analysis
- interaction with the DM
- choice of the reference alternatives
- dealing with "inconsistencies"
- admitting other type of information
  - x is "much better" then y
  - the difference between x and y is "larger" than the difference between z and w
- exploit the whole set $\mathcal{V}$ to build a recommendation

Scaling constants

$$x \succsim y \Leftrightarrow \sum_{i=1}^{n} v_i(x_i) \geq \sum_{i=1}^{n} v_i(y_i)$$

Convenient normalization

- x_{i*}, x_i^*
- $v_1(x_{1*}) = v_2(x_{2*}) = \ldots = v_n(x_{n*}) = 0$
- $\sum_{i=1}^{n} v_i(x_i^*) = 1$

Scaling constants

$$x \succsim y \Leftrightarrow \sum_{i=1}^{n} v_i(x_i) \geq \sum_{i=1}^{n} v_i(y_i)$$

$$v_1(x_{1*}) = v_2(x_{2*}) = \ldots = v_n(x_{n*}) = 0$$

$$\sum_{i=1}^{n} v_1(x_i^*) = 1$$

$$x \succsim y \Leftrightarrow \sum_{i=1}^{n} \lambda_i u_i(x_i) \geq \sum_{i=1}^{n} \lambda_i u_i(y_i)$$

$$\sum_{i=1}^{n} \lambda_i = 1$$

$$u_1(x_{1*}) = u_2(x_{2*}) = \ldots = u_n(x_{n*}) = 0$$

$$u_1(x_1^*) = u_2(x_2^*) = \ldots = u_n(x_n^*) = 1$$

$$u_i = v_i / v_i(x_i^*)$$

Scaling constants

$$x \succsim y \Leftrightarrow \sum_{i=1}^{n} \lambda_i u_i(x_i) \geq \sum_{i=1}^{n} \lambda_i u_i(y_i)$$

$$\sum_{i=1}^{n} \lambda_i = 1$$

$$u_1(x_{1*}) = u_2(x_{2*}) = \ldots = u_n(x_{n*}) = 0$$

$$u_1(x_1^*) = u_2(x_2^*) = \ldots = u_n(x_n^*) = 1$$

Most critical mistake

- the numbers λ_i do NOT reflect the importance of attribute i
- they reflect the width of the interval $[x_{i*}, x_i^*]$
- if this interval is changed, the λ_i MUST be changed

MACBETH

Conventions

$$x \succsim y \Leftrightarrow \sum_{i=1}^{n} \lambda_i u_i(x_i) \geq \sum_{i=1}^{n} \lambda_i u_i(y_i)$$

$$\sum_{i=1}^{n} \lambda_i = 1$$

$$u_1(x_{1*}) = u_2(x_{2*}) = \ldots = u_n(x_{n*}) = 0$$

$$u_1(x_1^*) = u_2(x_2^*) = \ldots = u_n(x_n^*) = 1$$

Principles

- assess the u_i independently on each attribute using "preference differences"
- assess the λ_i to fit these functions together

Assessing the u_i

- compare alternatives only differing on attribute i
- rate their difference of attractiveness on a 7-point scale

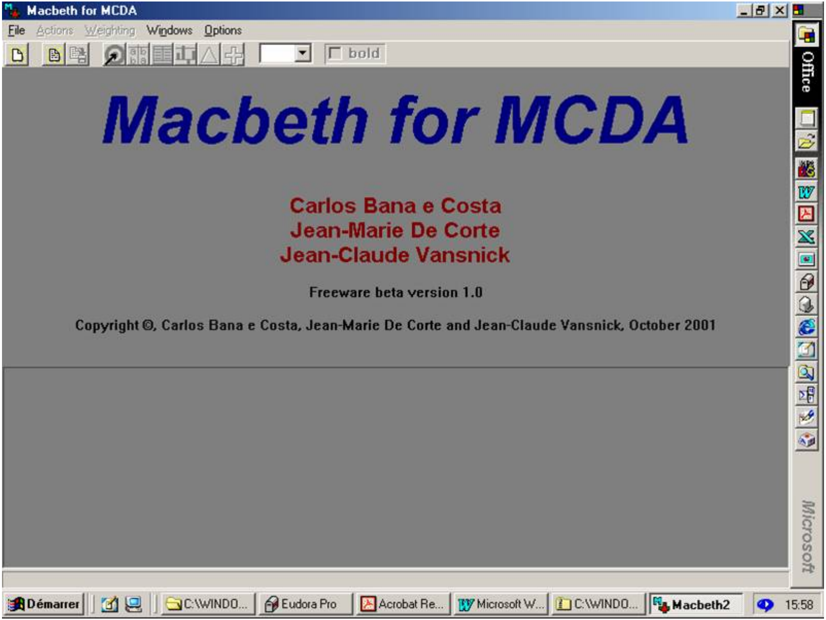
Categories	Description
C_0	null
C_1	
C_2	weak
C_3	
C_4	strong
C_5	
C_6	extreme

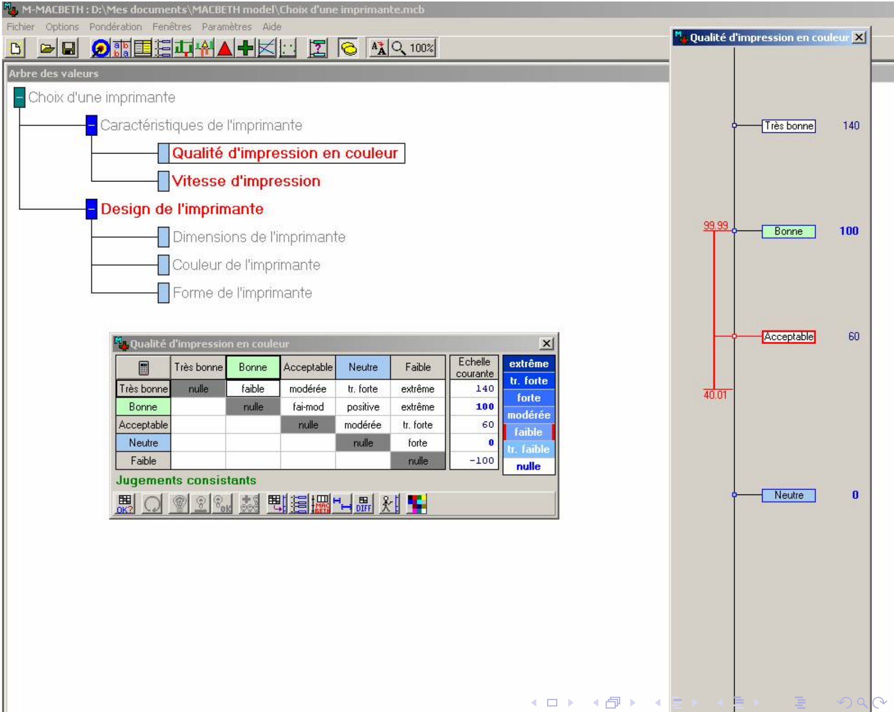
MACBETH

$$\left.\begin{array}{c}(a_i, b_i) \in C_k \\ (c_i, d_i) \in C_\ell \\ \ell > k\end{array}\right\} \Rightarrow u_i(a_i) - u_i(b_i) < u_i(c_i) - u_i(d_i)$$

Solution

- add normalization constraints $u_i(x_{i*}) = 0$, $u_i(x_i^*) = 1$
- add deviation variables
- use LP


Macbeth for MCDA
File Actions Weighting Windows Options
bold
Macbeth for MCDA
Carlos Bana e Costa
Jean-Marie De Corte
Jean-Claude Vansnick
Freeware beta version 1.0
Copyright ©, Carlos Bana e Costa, Jean-Marie De Corte and Jean-Claude Vansnick, October 2001
Office
Microsoft
Démarrer
C:\WINDO...
Eudora Pro
Acrobat Re...
Microsoft W...
C:\WINDO...
Macbeth2
15:58


M-MACBETH : D:\Mes documents\MACBETH model\Choix d'une imprimante.mcb
Fichier Options Pondération Fenêtres Paramètres Aide
Arbre des valeurs
Choix d'une imprimante
Caractéristiques de l'imprimante
Qualité d'impression en couleur
Vitesse d'impression
Design de l'imprimante
Dimensions de l'imprimante
Couleur de l'imprimante
Forme de l'imprimante
Qualité d'impression en couleur
Très bonne Bonne Acceptable Neutre Faible Echelle courante
Très bonne nulle faible modérée tr. forte extrême 140
Bonne nulle fai-mod positive extrême 100
Acceptable nulle modérée tr. forte 60
Neutre nulle forte 0
Faible nulle -100
extrême
tr. forte
forte
modérée
faible
tr. faible
nulle
Jugements consistants
Qualité d'impression en couleur
Très bonne 140
99.99
Bonne 100
Acceptable 60
40.01
Neutre 0

Scaling constants

$$x \succsim y \Leftrightarrow \sum_{i=1}^{n} \lambda_i u_i(x_i) \geq \sum_{i=1}^{n} \lambda_i u_i(y_i)$$

$$u_1(x_{1*}) = u_2(x_{2*}) = \ldots = u_n(x_{n*}) = 0$$

$$u_1(x_1^*) = u_2(x_2^*) = \ldots = u_n(x_n^*) = 1$$

Scaling constants

- once the u_i are known...
- comparing alternatives leads to a constraint on the λ_i

MACBETH

Repeat the procedure with the alternatives:

$$(x_1^*, x_{2*}, \ldots, x_{n*}), (x_{1*}, x_2^*, \ldots, x_{n*}) \ldots (x_{1*}, x_{2*}, \ldots, x_n^*)$$

Summary

Conjoint measurement

- highly consistent theory
- together with practical assessment techniques

Why consider extensions?

- hypotheses may be violated
- assessment is demanding
  - time
  - cognitive effort

Part II

A glimpse at possible extensions

Summary

Additive value function model

- requires independence
- requires a finely grained analysis of preferences

Two main types of extensions

1. models with interactions
2. more ordinal models

Interactions

Two extreme models

- additive value function model
  - independence
- decomposable model
  - only weak independence

$$x \succsim y \Leftrightarrow \sum_{i=1}^{n} v_i(x_i) \geq \sum_{i=1}^{n} v_i(y_i)$$

$$x \succsim y \Leftrightarrow F[v_1(x_1), \ldots v_n(x_n)] \geq F[v_1(y_1), \ldots v_n(y_n)]$$

Decomposable models

$$x \succsim y \Leftrightarrow F[v_1(x_1), \dots v_n(x_n)] \geq F[v_1(y_1), \dots v_n(y_n)]$$

F increasing in all arguments

Result

Under mild conditions, any weakly independent weak order may be represented in the decomposable model

Problem

- all possible types of interactions are admitted
- assessment is a very challenging task

Two main directions

Extensions

1. work with the decomposable model
   - rough sets
2. find models "in between additive" and decomposable
   - CP-nets, GAI
   - fuzzy integrals

Rough sets

Basic ideas

- work within the general decomposable model
- use the same principle as in UTA
- replacing the numerical model by a symbolic one
- infer decision rules

IF

$x_1 \geq a_1, \ldots, x_i \geq a_i, \ldots, x_n \geq a_n$ and

$y_1 \leq b_1, \ldots, y_i \leq b_i, \ldots, y_n \leq b_n$

THEN

$x \succsim y$

- many possible variants
- Greco, Matarazzo, Słowiński

GAI: Example

Choice of a meal: 3 attributes

$X_1 = \{\text{Steak}, \text{Fish}\}$
$X_2 = \{\text{Red}, \text{White}\}$
$X_3 = \{\text{Cake}, \text{sherBet}\}$

Preferences

$$x^1 = (S, R, C) \quad x^2 = (S, R, B) \quad x^3 = (S, W, C) \quad x^4 = (S, W, B)$$
$$x^5 = (F, R, C) \quad x^6 = (F, R, B) \quad x^7 = (F, W, C) \quad x^8 = (F, W, B)$$

$$x^2 \succ x^1 \succ x^7 \succ x^8 \succ x^4 \succ x^3 \succ x^5 \succ x^6$$

- the important is to match main course and wine
- I prefer Steak to Fish
- I prefer Cake to sherBet if Fish
- I prefer sherBet to Cake if Steak

Example

$$x^1 = (S, R, C) \quad x^2 = (S, R, B) \quad x^3 = (S, W, C) \quad x^4 = (S, W, B)$$
$$x^5 = (F, R, C) \quad x^6 = (F, R, B) \quad x^7 = (F, W, C) \quad x^8 = (F, W, B)$$

$$x^2 \succ x^1 \succ x^7 \succ x^8 \succ x^4 \succ x^3 \succ x^5 \succ x^6$$

Independence

$$x^1 \succ x^5 \Rightarrow v_1(S) > v_1(F)$$
$$x^7 \succ x^3 \Rightarrow v_1(F) > v_1(S)$$

Grouping main course and wine?

$$x^7 \succ x^8 \Rightarrow v_3(C) > v_3(B)$$
$$x^2 \succ x^1 \Rightarrow v_3(B) > v_3(C)$$

Example

$$x^1 = (S, R, C) \quad x^2 = (S, R, B) \quad x^3 = (S, W, C) \quad x^4 = (S, W, B)$$
$$x^5 = (F, R, C) \quad x^6 = (F, R, B) \quad x^7 = (F, W, C) \quad x^8 = (F, W, B)$$

$$x^2 \succ x^1 \succ x^7 \succ x^8 \succ x^4 \succ x^3 \succ x^5 \succ x^6$$

Model

$$x \succsim y \Leftrightarrow u_{12}(x_1, x_2) + u_{13}(x_1, x_3) \geq u_{12}(y_1, y_2) + u_{13}(y_1, y_3)$$

$$u_{12}(S, R) = 6 \quad u_{12}(F, W) = 4 \quad u_{12}(S, W) = 2 \quad u_{12}(F, R) = 0$$
$$u_{13}(S, C) = 0 \quad u_{13}(S, B) = 1 \quad u_{13}(F, C) = 1 \quad u_{13}(F, S) = 0$$

Generalized Additive Independence

GAI (Gonzales & Perny)

- axiomatic analysis
- if interdependences are known
  - assessment techniques
  - efficient algorithms (compactness of representation)

What R. L. Keeney would probably say

- the attribute “richness” of meal is missing

GAI

- interdependence within a framework that is quite similar to that of classical theory
- powerful generalization of recent models in Computer Science

Fuzzy integrals

Origins

- decision making under uncertainty
  - homogeneous Cartesian product
- mathematics
  - integrating w.r.t. a non-additive measure
- game theory
  - cooperative TU games
- multiattribute decisions
  - generalizing the weighted sum

Example

	Physics	Maths	Economics
a	18	12	6
b	18	7	11
c	5	17	8
d	5	12	13

$$a \succ b \quad d \succ c$$

Preferences

a is fine for Engineering d is fine for Economics

Interpretation: interaction

- having good grades in both
  - Math *and* Physics or
  - Maths *and* Economics
- better than having good grades in both
  - Physics *and* Economics

Weighted sum

	Physics	Maths	Economics
a	18	12	6
b	18	7	11
c	5	17	8
d	5	12	13

$$a \succ b \Rightarrow 18w_1 + 12w_2 + 6w_3 > 18w_1 + 7w_2 + 11w_3 \Rightarrow w_2 > w_3$$
$$d \succ c \Rightarrow 5w_1 + 17w_2 + 8w_3 > 5w_1 + 12w_2 + 13w_3 \Rightarrow w_3 > w_2$$

Choquet integral

Capacity

$$\mu : 2^N \longrightarrow [0, 1]$$
$$\mu(\varnothing) = 0, \mu(N) = 1$$
$$A \subseteq B \Rightarrow \mu(A) \leq \mu(B)$$

Choquet integral

$$0 = x_{(0)} \leq x_{(1)} \leq \cdots \leq x_{(n)}$$

$$\begin{array}{rr} x_{(1)} - x_{(0)} & \mu(\{(1),(2),(3),(4)\ldots,(n)\}) \\ x_{(2)} - x_{(1)} & \mu(\{(2),(3),(4)\ldots,(n)\}) \\ x_{(3)} - x_{(2)} & \mu(\{(3),(4)\ldots,(n)\}) \\ \ldots & \ldots \\ x_{(n)} - x_{(n-1)} & \mu(\{(n)\}) \end{array}$$

$$\mathcal{C}_\mu(x) = \sum_{i=1}^{n} \left[x_{(i)} - x_{(i-1)}\right] \mu(A_{(i)})$$

$$A_{(i)} = \{(i),(i+1),\ldots,(n)\}$$

Application

	Physics	Maths	Economics
a	18	12	6
b	18	7	11
c	5	17	8
d	5	12	13

$$\mu(M) = 0.1, \mu(P) = 0.5, \mu(E) = 0.5$$
$$\mu(M, P) = 1 > \mu(M) + \mu(P)$$
$$\mu(M, E) = 1 > \mu(M) + \mu(E)$$
$$\mu(P, E) = 0.6 < \mu(P) + \mu(E)$$

$$\mathcal{C}_\mu(a) = 6 \times 1 + (12 - 6) \times 1 + (18 - 12) \times 0.5 = 15.0$$
$$\mathcal{C}_\mu(b) = 7 + (11 - 7) \times 0.6 + (18 - 11) \times 0.5 = 12.9$$
$$\mathcal{C}_\mu(c) = 5 + (8 - 5) \times 1 + (17 - 8) \times 0.1 = 8.9$$
$$\mathcal{C}_\mu(d) = 5 + (12 - 5) \times 1 + (13 - 12) \times 0.5 = 12.5$$

Choquet integral in MCDM

Properties

- monotone, idempotent, continuous
- preserves weak separability
- tolerates violation of independence
- contains many other aggregation functions as particular cases

Capacities

Fascinating mathematical object:

- Möbius transform
- Shapley value
- interaction indices

Questions

Hypotheses

- I can compare x_i with x_j
  - attributes are (level) commensurable

Classical model

- I can indirectly compare $[x_i, y_i]$ with $[x_j, y_j]$

Central research question

- how to assess $u : \bigcup_{i=1}^{n} X_i \rightarrow \mathbb{R}$ so that the levels are commensurate?

Choquet integral

Assessment

- variety of mathematical programming based approaches

Extensions

- Choquet integral with a reference point (statu quo)
- Sugeno integral (median)
- axiomatization as aggregation functions
- k-additive capacities

Observations

Classical model

- deep analysis of preference that may not be possible
  - preference are not well structured
  - several or no DM
  - prudence

Idea

- it is not very restrictive to suppose that levels on each X_i can be ordered
- aggregate these orders
- possibly taking importance into account

Social choice

- aggregate the preference orders of the voters to build a collective preference

Outranking methods

ELECTRE

$x \succsim y$ if

Concordance a "majority" of attributes support the assertion

Discordance the opposition of the minority is not "too strong"

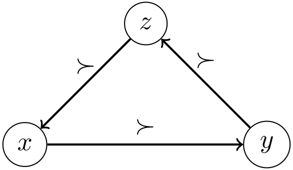
$$x \succsim y \Leftrightarrow \begin{cases} \sum_{i: x_i \succsim_i y_i} w_i \geq s \\ Not[y_i \; V_i \; x_i], \forall i \in N \end{cases}$$

Condorcet's paradox

$$x \succsim y \Leftrightarrow |\{i \in N : x_i \succsim_i y_i\}| \geq |\{i \in N : y_i \succsim_i x_i\}|$$

$$1 : x_1 \succ_1 y_1 \succ_1 z_1$$
$$2 : z_2 \succ_2 x_2 \succ_2 y_2$$
$$3 : y_3 \succ_3 z_3 \succ_3 x_3$$

$$x = (x_1, x_2, x_3)$$
$$y = (y_1, y_2, y_3)$$
$$z = (z_1, z_2, z_3)$$



Arrow's theorem

Theorem

The only ways to aggregate weak orders while remaining ordinal are not very attractive…

- dictator (weak order)
- oligarchy (transitive $\succ$)
- veto (acyclic $\succ$)

Ways out

Accepting intransitivity

- find way to extract information in spite of intransitivity
    - ELECTRE I, II, III, IS
    - PROMETHEE I, II

Do not use paired comparisons

- only compare x with carefully selected alternatives
    - ELECTRE TRI
    - methods using reference points

Conclusion

Fascinating field

- theoretical point of view
  - measurement theory
  - decision under uncertainty
  - social choice theory
- practical point of view
  - rating firms from a social point of view
  - evaluating H_2-propelled cars