An axiomatic approach to ELECTRE TRI [1]

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[1] This text is a much abridged version of Bouyssou and Marchant (2004a) to which we refer the readers for proofs.

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Abstract

In the literature on MCDM, many methods have been proposed in order to sort alternatives evaluated on several attributes into ordered categories. Most of them were proposed on an ad hoc basis. Using tools from conjoint measurement, this paper takes a more theoretical approach to these sorting methods. We provide an axiomatic analysis of the partitions of alternatives into two categories that can be obtained using what we call "noncompensatory sorting models". These models have strong links with the pessimistic version of ELECTRE TRI and our analysis allows to pinpoint what appears to be the main distinctive features of ELECTRE TRI when compared to other sorting methods.

Keywords: Decision with multiple attributes, Sorting models, Twofold partitions, Noncompensation, Conjoint measurement, ELECTRE TRI.

1 Introduction and motivation

MCDM has traditionally been concerned with decision situations in which the objective is either to select an alternative from a set of alternatives evaluated on several attributes or to rank order this set. In such situations, the usual practice is to build a recommendation on the basis of a binary relation comparing alternatives in terms of preference. The recommendation is therefore based on a *relative evaluation model* of the alternatives as given by the preference relation.

The use of relative evaluation models is not always appropriate to build meaningful recommendations. Indeed the "best" alternatives, while being superior to all others, may well not be desirable. When such elements appear important, one may turn to evaluation models having a more absolute character. This leads to comparing alternatives not between them but to pre-defined norms. The result of such an analysis is a partition of the set of alternatives into several categories defined with respect to these norms. This is what Roy (1996) called the sorting problem formulation. This is the subject of this paper.

Sorting problems come in two rather different guises depending on whether the categories used to sort the alternatives are ordered in terms of desirability. The situation in which they are not, called "classification" in Zopounidis and Doumpos (2002), is common in pattern recognition or medical diagnosis. Such problems have been studied rather intensively in Statistics, Operations Research and Artificial Intelligence. The situation in which the categories are ordered is also quite common. It occurs, for instance, when a credit analyst rates credit applications or when an academic programme is enrolling students. It has recently attracted much attention in the literature on MCDM (see Greco et al., 2002a; Zopounidis and Doumpos, 2000b, 2002, for reviews). Several methods have been designed to tackle such problems such as UTADIS (see Jacquet-Lagrèze, 1995; Zopounidis and Doumpos, 2000c) or ELECTRE TRI (see Mousseau et al., 2000; Roy and Bouyssou, 1993; Wei, 1992). Most of these sorting techniques were proposed on a more or less ad hoc basis. The main aim of this paper is to suggest a more theoretical approach to this problem. More precisely, we propose an axiomatic analysis of what we shall call "noncompensatory sorting models". The main characteristic of these models is that they rely on rather poor information on each attribute. These models have close connections with ELECTRE TRI and our analysis may be considered as an attempt to provide a firm axiomatic basis to this technique.

This paper concentrates on sorting problems with *two* categories. Besides being much simpler than the general case, this situation deserves special attention since it tends to blur the distinction between the case of ordered and unordered categories.

Technically, our strategy will be to use conjoint measurement techniques to deal with partitions, instead of binary relations, defined on Cartesian products. This strategy was first proposed by Goldstein (1991) and later developed in Greco et al. (2001b).

The rest of this paper is organized as follows. We introduce our setting and some background material in section 2. Section 3 deals with the case of noncompensatory sorting models. Section 4 extends this analysis to include the possibility of veto effects as in ELECTRE TRI. A final section discusses our findings and presents directions for future research.

2 Background

2.1 The setting

Let $n \geq 2$ be an integer and $X = X_1 \times X_2 \times \cdots \times X_n$ be a set of objects. Elements $x, y, z, \ldots$ of X will be interpreted as alternatives evaluated on a set $N = \{1, 2, \ldots, n\}$ of attributes. For any nonempty subset J of the set of attributes N, we denote by X_J (resp. X_{-J}) the set $\prod_{i \in J} X_i$ (resp. $\prod_{i \notin J} X_i$). With customary abuse of notation, (x_J, y_{-J}) will denote the element $w \in X$ such that $w_i = x_i$ if $i \in J$ and $w_i = y_i$ otherwise. When $J = \{i\}$ we shall simply write X_{-i} and (x_i, y_{-i}).

Our primitives consist in a twofold partition $\langle \mathscr{A}, \mathscr{U} \rangle$ of the set X; this means that the sets $\mathscr{A}$ and $\mathscr{U}$ are nonempty and disjoint and that their union is the entire set X. Our central aim is to study various models allowing to represent the information contained in $\langle \mathscr{A}, \mathscr{U} \rangle$. We interpret the partition $\langle \mathscr{A}, \mathscr{U} \rangle$ as the result of a sorting model applied to the alternatives in X. Although the ordering of the categories is not part of our primitives, it is useful to interpret the set $\mathscr{A}$ as containing sAtisfactory objects, while $\mathscr{U}$ contains Unsatisfactory ones.

2.2 Binary relations

We use a standard vocabulary for binary relations. An equivalence (resp. a weak order, a semiorder) is a reflexive, symmetric and transitive (resp. complete and transitive, complete, Ferrers and semi-transitive) relation.

Let R be a binary relation on A. Following e.g. Krantz et al. (1971, Chapter 2), we say that B is dense in A for R if, for all $a, b \in A$, $[a \; R \; b$ and $Not[b \; R \; a]] \Rightarrow [a \; R \; c$ and $c \; R \; b$, for some $c \in B]$. It is well-known (Fishburn, 1970; Krantz et al., 1971) that there is a real-valued function f on A such that, for all $a, b \in A$, $a \; R \; b \Leftrightarrow f(a) \geq f(b)$, if and only if R is a weak order and there is a finite or countably infinite set $B \subseteq A$ that is dense in A for R.

2.3 Goldstein's (1991) framework

Goldstein (1991) was the first to suggest the use of conjoint measurement techniques for the analysis of twofold and threefold partitions of a set of multi-attributed alternatives. We briefly recall here the main points of his analysis for the case of twofold partitions.

Consider a model in which, for all $x \in X$,

$$x \in \mathscr{A} \Leftrightarrow F(u_1(x_1), u_2(x_2), \ldots, u_n(x_n)) > 0, \qquad (D)$$

where u_i is a real-valued function on X_i and F is a real-valued function on $\prod_{i=1}^{n} u_i(X_i)$ that is increasing in each of its arguments. Model (D) contains as a particular case the additive model for sorting in which, for all $x \in X$,

$$x \in \mathscr{A} \Leftrightarrow \sum_{i=1}^{n} u_i(x_i) > 0, \tag{1}$$

that is at the heart of the UTADIS technique (Jacquet-Lagrèze, 1995; Zopounidis and Doumpos, 2000c) and its variants (Zopounidis and Doumpos, 2000a). We shall see below that model (D) has also close links with the ELECTRE TRI technique.

In order to analyze model (D), we define on each X_i the binary relation $\succsim_i$ letting, for all $x_i, y_i \in X_i$,

$$x_i \succsim_i y_i \Leftrightarrow [\text{for all } a_{-i} \in X_{-i}, (y_i, a_{-i}) \in \mathscr{A} \Rightarrow (x_i, a_{-i}) \in \mathscr{A}].$$

It is not difficult to see that, by construction, $\succsim_i$ is reflexive and transitive. We denote by $\succ_i$ (resp. $\sim_i$) the asymmetric (resp. symmetric) part of $\succsim_i$.

We say that the partition $\langle \mathscr{A}, \mathscr{U} \rangle$ is linear on attribute $i \in N$ (condition *linear*$_i$) if, for all $x_i, y_i \in X_i$ and all $a_{-i}, b_{-i} \in X_{-i}$,

$$\left.\begin{array}{c} (x_i, a_{-i}) \in \mathscr{A} \\ \text{and} \\ (y_i, b_{-i}) \in \mathscr{A} \end{array}\right\} \Rightarrow \left\{\begin{array}{c} (y_i, a_{-i}) \in \mathscr{A} \\ \text{or} \\ (x_i, b_{-i}) \in \mathscr{A} \end{array}\right. \tag{$linear_i$}$$

The partition is said to be linear if it is linear on all $i \in N$. The following lemma takes note of the consequences of condition *linear*$_i$ on relations $\succsim_i$ and shows that linearity is necessary for model (D).

Lemma 1

1. *Condition linear$_i$ holds iff $\succsim_i$ is complete,*
2. *If $\langle \mathscr{A}, \mathscr{U} \rangle$ has a representation in model (D) then it is linear.*

This leads to:

Proposition 1 (Goldstein (1991))
Let $\langle \mathscr{A}, \mathscr{U} \rangle$ be a twofold partition of a set X. There is a representation of $\langle \mathscr{A}, \mathscr{U} \rangle$ in model (D) if and only if it is linear and, for all $i \in N$, there is a finite or countably infinite set $X_i' \subseteq X_i$ that is dense in X_i for $\succsim_i$.

2.4 ELECTRE TRI

For the ease of future reference, we briefly recall here the main points of the ELECTRE TRI sorting technique with two categories. For a more detailed description, we refer the reader to Mousseau et al. (2000), Roy and Bouyssou (1993, ch. 6) or Wei (1992). We suppose below that preference and indifference thresholds are equal and that discordance effects occur in an "all or nothing" way. This will allow to keep things simple while preserving what we believe to be the general spirit of the method.

The aim of ELECTRE TRI is to sort alternatives evaluated on several attributes between two ordered categories $\mathscr{A}$ and $\mathscr{U}$, with $\mathscr{A}$ containing the most desirable alternatives. This is done as follows. There is a profile p being the lower limit of category $\mathscr{A}$ and the upper limit of $\mathscr{U}$. This profile p is defined by its evaluations $(p_1, p_2, \dots, p_n)$ on the attributes in N. Define $\widehat{X}_i = X_i \cup \{p_i\}$ and $\widehat{X} = \prod_{i=1}^{N} \widehat{X}_i$.

On each $i \in N$, there is a semiorder S_i on $\widehat{X}_i$. This relation is interpreted as an "at least as good" relation on $\widehat{X}_i$. A strict semiorder (i.e. an irreflexive, Ferrers and semi-transitive relation) V_i is also defined on $\widehat{X}_i$. It is interpreted as a "far better than" relation on $\widehat{X}_i$. For consistency reasons, it is supposed that V_i is included in the asymmetric part of S_i.

A nonnegative weight w_i is assigned to each attribute $i \in N$. We suppose wlog that weights are normalized so that $\sum_{i=1}^{n} w_i = 1$. Let λ be a real number between 1/2 and 1.

In ELECTRE TRI, a binary relation S is built on $\widehat{X}$ letting, for all $x, y \in \widehat{X}$,

$$x \; S \; y \Leftrightarrow \sum_{i \in S(x,y)} w_i \geq \lambda \text{ and } [Not[y_i \; V_i \; x_i], \text{ for all } i \in N], \tag{2}$$

where $S(x, y) = \{i \in N : x_i \; S_i \; y_i\}$. Hence, we have $x \; S \; y$ when x is judged "at least as good as" y on a qualified weighted majority of attributes (concordance condition) and there is no attribute on which y is judged "far better" than x (non-discordance condition).

The sorting of an alternative $x \in X$ is based the comparison of x with the profile p using the relations S. In the pessimistic version of ELECTRE TRI, we have, for all $x \in X$, $x \in \mathscr{A} \Leftrightarrow x \; S \; p$. In the optimistic version of ELECTRE TRI, we have, for all $x \in X$, $x \in \mathscr{A} \Leftrightarrow Not[p \; P \; x]$, where P is the asymmetric part of S.

Note that if we have $x \in \mathscr{A}$ in the pessimistic version of ELECTRE TRI, we have $x \; S \; p$ so that $Not[p \; P \; x]$. Hence, we must have $x \in \mathscr{A}$ with the optimistic version of ELECTRE TRI. This explains the names of the two versions of the method.

We shall shortly see that if a partition $\langle \mathscr{A}, \mathscr{U} \rangle$ has been obtained using the pessimistic version of ELECTRE TRI it will always have a representation in model (D).

3 The noncompensatory sorting model for twofold partitions

3.1 Definitions

We say that $\langle \mathscr{A}, \mathscr{U} \rangle$ has a representation in the *noncompensatory sorting model* if:

- for all $i \in N$ there is a set $\mathscr{A}_i \subseteq X_i$,
- there is a subset $\mathscr{F}$ of 2^N that is monotonic wrt inclusion (i.e. such that $[I \in \mathscr{F}$ and $I \subset J] \Rightarrow J \in \mathscr{F}$),

such that, for all $x \in X$,

$$x \in \mathscr{A} \Leftrightarrow \{i \in N : x_i \in \mathscr{A}_i\} \in \mathscr{F}. \tag{3}$$

In this case, we say, that $\langle \mathscr{F}, \mathscr{A}_1, \mathscr{A}_2, \ldots, \mathscr{A}_n \rangle$ or, for short, $\langle \mathscr{F}, \langle \mathscr{A}_i \rangle_{i \in N} \rangle$ is a representation of $\langle \mathscr{A}, \mathscr{U} \rangle$ in the noncompensatory sorting model. When there is no risk of confusion on the underlying sets $\mathscr{A}_i$, we write $A(x)$ instead of $\{i \in N : x_i \in \mathscr{A}_i\}$. In this section, we write $\mathscr{U}_i = X_i \setminus \mathscr{A}_i$.

We may interpret the noncompensatory sorting model as follows. On each $i \in N$, we isolate, within the set X_i a set $\mathscr{A}_i$ of "satisfactory" levels. In order for an alternative $x \in X$ to be globally satisfactory, i.e. that $x \in \mathscr{A}$, it is necessary and sufficient that x is judged satisfactory on a subset of attributes that is judged "sufficiently important", as indicated by the set $\mathscr{F}$. The fact that $\mathscr{F}$ is supposed to be monotonic wrt inclusion means that replacing an evaluation in $\mathscr{U}_i = X_i \setminus \mathscr{A}_i$ by an evaluation in $\mathscr{A}_i$ cannot turn a satisfactory alternative into an unsatisfactory one.

The rationale for the name "noncompensatory" comes from the fact that these sorting models do not allow to distinguish more than two types of elements in X_i: those in $\mathscr{A}_i$ and those in $\mathscr{U}_i$. Suppose that x is not in $\mathscr{A}$ because $A(x)$ does not belong to $\mathscr{F}$. In a compensatory model, it would be possible to improve the affectation of x by sufficiently improving its evaluation on any attribute. In our models, altering the evaluation of x on any attribute in $A(x)$ will never lead to modify the affectation of x in $\mathscr{U}$.

The pessimistic version of ELECTRE, when preference and indifference thresholds are equal and when discordance is not involved (i.e. $V_i = \varnothing$, for all $i \in N$), is a particular case of the noncompensatory sorting model. Using the notation of section 2.4, we have, for all $x \in X$, $x \in \mathscr{A} \Leftrightarrow x \; S \; p \Leftrightarrow \sum_{i \in S(x,p)} w_i \geq \lambda$. Defining $\mathscr{A}_i = \{x_i \in X_i : x_i \; S_i \; p_i\}$ and letting $I \in \mathscr{F}$ when $\sum_{i \in I} w_i \geq \lambda$ shows that such a model is a particular case of the noncompensatory sorting model.

In the optimistic version of ELECTRE TRI method, when preference and indifference thresholds are equal and when discordance is not involved, we have, for all $x \in X$, $x \in \mathscr{A} \Leftrightarrow Not[p \; P \; x] \Leftrightarrow \left[\sum_{i \in S(p,x)} w_i < \lambda \text{ or } \sum_{i \in S(x,p)} w_i \geq \lambda\right]$. Beyond surface, the two versions of the ELECTRE TRI method are rather different. Indeed, as shown by the following example, due to its use of P, the optimistic version of ELECTRE TRI does not fit into the framework of noncompensatory sorting model.

Example 1
Suppose that $n = 5$, $X_1 = X_2 = \ldots = X_5 = \{9, 10, 11\}$ and $p = (10, 10, 10, 10, 10)$. For all $i \in \{1, 2, \ldots, 5\}$, let $w_i = 1/5$ and $S_i = \geq$. Let $\lambda = 4/5$. Using the optimistic version of ELECTRE TRI, we obtain $x = (11, 11, 9, 9, 9) \in \mathscr{A}$ (we have $Not[p \; S \; x]$ and $Not[x \; S \; p]$), $y = (10, 11, 9, 9, 9) \in \mathscr{U}$ (we have $p \; S \; y$ and $Not[y \; S \; p]$) and $z = (10, 10, 10, 10, 10) \in \mathscr{A}$ (we have $p \; S \; z$ and $z \; S \; p$).

Suppose that this partition can be represented in the noncompensatory sorting model. Since $x \in \mathscr{A}$ and $y \in \mathscr{U}$, we must have $10 \in \mathscr{U}_1$ and $11 \in \mathscr{A}_1$. The problem being entirely symmetric, we easily obtain that, for all $i \in N$, $10 \in \mathscr{U}_i$ and $11 \in \mathscr{A}_i$. Because $z \in \mathscr{A}$, we must have $\varnothing \in \mathscr{F}$, so that all alternatives in X should belong to $\mathscr{A}$. Hence, this partition cannot be represented in the noncompensatory sorting model. $\diamond$

3.2 Axioms and results

Let us first observe that if $\langle \mathscr{A}, \mathscr{U} \rangle$ has a representation in the noncompensatory sorting model then it must be linear and, for all $i \in N$, the weak order $\succsim_i$ can have at most two distinct equivalence classes.

Lemma 2
Suppose that $\langle \mathscr{A}, \mathscr{U} \rangle$ has a representation in the noncompensatory sorting model. Then, for all $i \in N$,

1. *condition* linear$_i$ *holds, so that $\succsim_i$ is a weak order,*

2. *the weak order $\succsim_i$ can have at most two distinct equivalence classes.*

In view of lemma 2, a characterization of the noncompensatory sorting model will be at hand if we add to linearity a condition implying that all relations $\succsim_i$ have at most two equivalence classes. We say that $\langle \mathscr{A}, \mathscr{U} \rangle$ is 2-graded on attribute $i \in N$ (condition 2-graded$_i$) if

$$\left.\begin{array}{c} (x_i, a_{-i}) \in \mathscr{A} \\ \text{and} \\ (y_i, a_{-i}) \in \mathscr{A} \\ \text{and} \\ (y_i, b_{-i}) \in \mathscr{A} \end{array}\right\} \Rightarrow \left\{\begin{array}{c} (x_i, b_{-i}) \in \mathscr{A} \\ \text{or} \\ (z_i, a_{-i}) \in \mathscr{A} \end{array}\right. \qquad (\text{2-graded}_i)$$

for all $x_i, y_i, z_i \in X_i$ and all $a_{-i}, b_{-i} \in X_{-i}$. We say that $\langle \mathscr{A}, \mathscr{U} \rangle$ is 2-graded if it is 2-graded on all attributes $i \in N$. Condition 2-graded is inspired by related works in the context of binary relations by Bouyssou and Pirlot (2002, 2004) and Greco et al. (2001a). We have:

Lemma 3

1. *Conditions* linear$_i$ *and* 2-graded$_i$ *hold iff $\succsim_i$ is a weak order having at most two distinct equivalence classes.*

2. *Conditions* linear$_i$ *and* 2-graded$_i$ *are independent.*

Our main result in this section says that linearity and 2-gradedness characterize the noncompensatory sorting model for twofold partitions.

Theorem 1
A partition $\langle \mathscr{A}, \mathscr{U} \rangle$ has a representation in the noncompensatory sorting model iff it is linear and 2-graded.

4 The noncompensatory sorting model with veto for twofold partitions

4.1 Definitions

Let $\langle \mathscr{A}, \mathscr{U} \rangle$ be a twofold partition of X. We say that $\langle \mathscr{A}, \mathscr{U} \rangle$ has a representation in the *noncompensatory sorting model with veto* if:

- for all $i \in N$ there are disjoint sets $\mathscr{A}_i, \mathscr{V}_i \subseteq X_i$,
- there is a subset $\mathscr{F}$ of 2^N that is monotonic wrt inclusion (i.e. such that $[I \in \mathscr{F}$ and $I \subset J] \Rightarrow J \in \mathscr{F})$,

such that, for all $x \in X$,

$$x \in \mathscr{A} \Leftrightarrow [\{i \in N : x_i \in \mathscr{A}_i\} \in \mathscr{F} \text{ and } \{i \in N : x_i \in \mathscr{V}_i\} = \varnothing]. \tag{4}$$

In this case, we say, that $\langle \mathscr{F}, \langle \mathscr{A}_i, \mathscr{V}_i \rangle_{i \in N} \rangle$ is a representation of $\langle \mathscr{A}, \mathscr{U} \rangle$ in the noncompensatory sorting model with veto. We write $A(x)$ and $V(x)$ instead of $\{i \in N : x_i \in \mathscr{A}_i\}$ and $\{i \in N : x_i \in \mathscr{V}_i\}$ when there is no risk of confusion on the underlying sets $\mathscr{A}_i$ and $\mathscr{V}_i$. We define, in this section, $\mathscr{U}_i$ as $X_i \setminus [\mathscr{A}_i \cup \mathscr{V}_i]$.

The interpretation of this model is similar to the one considered in the preceding section. The only difference here is that, there is a subset $\mathscr{V}_i$ of elements of X_i that are "repulsive" for $\mathscr{A}$ in that, as soon as one of the evaluations of $x \in X$ is repulsive, it is impossible to have $x \in \mathscr{A}$. Note that, with the presence of repulsive levels for $\mathscr{A}$, the roles of $\mathscr{A}$ and $\mathscr{U}$ are no more symmetric in the noncompensatory sorting model with veto.

The pessimistic version of ELECTRE, when preference and indifference thresholds are equal, is a particular case of the noncompensatory sorting model. Indeed, using the notation of section 2.4, we have, for all $x \in X$,

$$x \in \mathscr{A} \Leftrightarrow x \; S \; p \Leftrightarrow \left[\sum_{i \in S(x,p)} w_i \geq \lambda \text{ and } [Not[p_i \; V_i \; x_i], \text{ for all } i \in N] \right].$$

Defining $\mathscr{A}_i = \{x_i \in X_i : x_i \; S_i \; p_i\}$, $\mathscr{V}_i = \{x_i \in X_i : p_i \; V_i \; x_i\}$ and letting $I \in \mathscr{F}$ if and only if $\sum_{i \in I} w_i \geq \lambda$, shows that such a model is a particular case of the noncompensatory sorting model with veto. Note that the sets $\mathscr{A}_i$ and $\mathscr{V}_i$ are indeed disjoint because we have supposed that V_i is included in the asymmetric part of S_i: if $x_i \; S_i \; p_i$, we cannot have $p_i \; V_i \; x_i$.

4.2 Axioms and results

Let us first observe that if a partition $\langle \mathscr{A}, \mathscr{U} \rangle$ has a representation in the noncompensatory sorting model with veto then it must be linear.

Lemma 4
If $\langle \mathscr{A}, \mathscr{U} \rangle$ has a representation in the noncompensatory sorting model with veto then it is linear.

It remains to see what must be added to linearity in order to characterize the noncompensatory sorting model with veto. Again, this will require limiting the number of distinct equivalence classes of $\succsim_i$, this time taking into account the possible existence of an equivalence class corresponding to repulsive levels for $\mathscr{A}$.

We say that the partition $\langle \mathscr{A}, \mathscr{U} \rangle$ is 3-graded with veto on attribute $i \in N$ (condition *3v-graded$_i$*) if, for all $x_i, y_i, z_i \in X_i$ and all $a_{-i}, b_{-i}, c_i \in X_{-i}$,

$$\left.\begin{array}{c} (x_i, a_{-i}) \in \mathscr{A} \\ \text{and} \\ (y_i, a_{-i}) \in \mathscr{A} \\ \text{and} \\ (y_i, b_{-i}) \in \mathscr{A} \\ \text{and} \\ (z_i, c_{-i}) \in \mathscr{A} \end{array}\right\} \Rightarrow \left\{\begin{array}{c} (x_i, b_{-i}) \in \mathscr{A} \\ \text{or} \\ (z_i, a_{-i}) \in \mathscr{A} \end{array}\right. \tag{3v-graded$_i$}$$

$\langle \mathscr{A}, \mathscr{U} \rangle$ is said to be 3-graded with veto if it satisfies *3v-graded$_i$* for all $i \in N$. This condition is inspired by Greco et al. (2001a) who study veto effects in the context of binary relations. It is apparent that condition *2-graded$_i$* implies condition *3v-graded$_i$*. As shown below, the role of condition *3v-graded$_i$* is to limit the number of distinct equivalence classes of $\sim_i$, taking into account the possible existence of repulsive levels. We have:

Lemma 5

1. *If $\langle \mathscr{A}, \mathscr{U} \rangle$ has a representation in the noncompensatory sorting model with veto then it is 3-graded with veto.*

2. *Conditions* linear$_i$ *and* 3v-graded$_i$ *are independent.*

3. *Conditions* linear$_i$ *and* 3v-graded$_i$ *imply that $\succsim_i$ is a weak order having at most three equivalence classes.*

   Furthermore if $\succsim_i$ has exactly three distinct equivalence classes and if x_i belongs to the last equivalence class of $\succsim_i$ then $(x_i, a_{-i}) \in \mathscr{U}$, for all $a_{-i} \in X_{-i}$.

Our main result in this section says that linearity and 3-gradedness with veto characterize the noncompensatory sorting model with veto.

Theorem 2

A partition $\langle \mathscr{A}, \mathscr{U} \rangle$ is representable in the noncompensatory sorting model with veto iff it is linear and 3-graded with veto.

5 Discussion

This paper has analyzed a number of sorting models for multi-attributed alternatives into two categories. The common feature of these models is that they particularize the general decomposable models proposed by Goldstein (1991) in the direction of using poor information on each attribute. Indeed, when there is no veto effect involved, non-compensatory models only distinguish two types of elements on each attribute. The possibility of veto effects adds a possible third type of elements.

The conditions that we have exhibited are reasonably simple and could well be the subject of empirical tests. A psychologist may, for instance, want to use them in order to know whether a partition of alternatives given by a subject can be explained using a noncompensatory model. On a more conceptual level, our conditions allow to pinpoint what appears to be the main distinctive feature of noncompensatory models à la ELECTRE within the general framework of decomposable sorting models. This was already shown in a series of papers (see Bouyssou and Pirlot, 2002, 2004; Dubois et al., 2003; Greco et al., 2001a) for the case of models involving binary relations. Our analysis can be considered as an extension of these paper to the case of sorting models.

Our theoretical analysis also has practical implications. In particular, it shows that, beyond surface, the two versions of ELECTRE TRI are rather different: only the pessimistic version fits into the framework of noncompensatory sorting models. This is related with the fact that most works trying to infer the parameters of an ELECTRE TRI model from assignment examples (i.e. from a partition defined on a subset of X) using mathematical programming techniques (see Dias et al., 2002; Dias and Mousseau, 2002; Mousseau et al., 2001; Mousseau and Słowiński, 1998; Ngo The and Mousseau, 2002) have only considered the pessimistic version of the method. Indeed, our models seem to show that the optimistic version of ELECTRE TRI is at variance with the general principles underlying most of the other ELECTRE-like techniques. Furthermore, it can be shown that the conditions ensuring the uniqueness of a representation in the noncompensatory sorting model with veto are rather stringent. Such a non-uniqueness will all the more be an issue for methods designed to infer all the parameters of an ELECTRE TRI model from assignment examples (see Mousseau and Słowiński, 1998) since they work on the basis of even less information than we do here. This possible, and likely, non-uniqueness of the representation probably explains why this type of method, independently of its computational complexity involving the solution of nonlinear programmes, have been abandoned and replaced by techniques inferring only one type of parameter (e.g. weights, veto thresholds or category limits) at a time. Finally, it should be mentioned that, contrary to what happens in ELECTRE TRI, the models proposed in this paper do not assume numerical weights, let alone the possibility to add then in order to test if a coalition of attributes is judged "sufficiently" important. Such a generalization is well in line with the use of symbolic inference techniques derived from Artificial Intelligence as shown in Greco et al. (2001c, 2002b). Therefore, this absence of weights should not be considered as an impediment to the practical use of such models.

The analysis proposed in this paper can be extended in several directions. First it is clearly necessary to extend our results concerning noncompensatory models to more than two categories. This is done in Bouyssou and Marchant (2004b). Quite a different line of extension is linked with the study of additive models for sorting. Using standard techniques, such an analysis is relatively straightforward when the set of alternatives is finite; it nevertheless raises difficult questions in the general case. This is the subject of an ongoing research.

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