# An Introduction to Social Choice Theory

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# What is Social Choice Theory?

- **Aim: study decision problems in which a *group* has to take a decision in a "democratic way"**
- **Abstract Theory**
  - **Nature of the decision**
  - **Size of the group**
  - **Nature of the group**
- **Many (deep) results**
  - **Economics, Political Science, Applied Mathematics, OR**
  - **Two Nobel Prizes (K.J. Arrow, A. Sen)**

# Applications

- **Political Elections**
- **Other Elections (Universities, Firms)**

- **Aggregation**
  - ⇨ **MCDA**
  - ⇨ **AI**

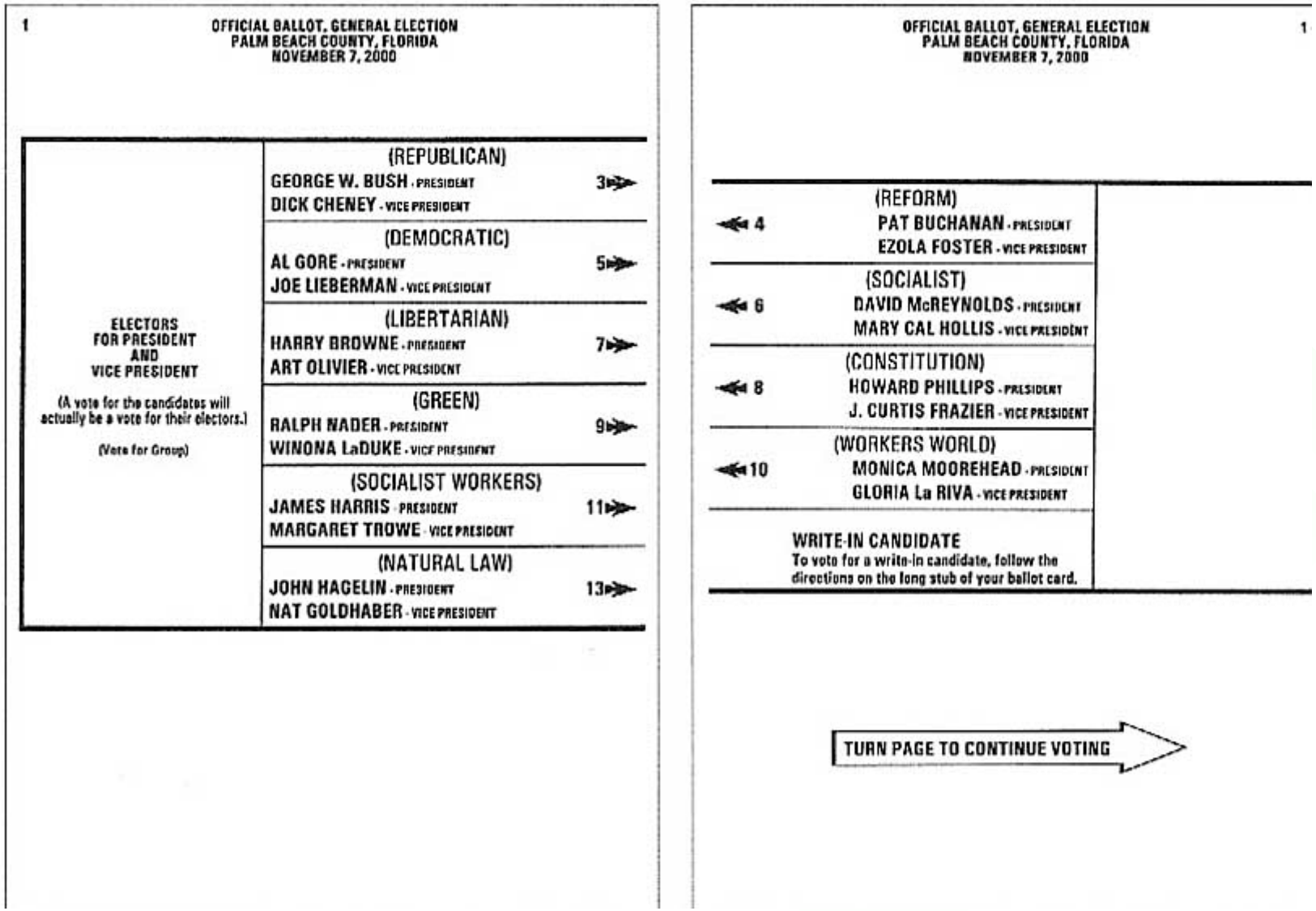
  - ⇨ **Several agents with different priorities**
  - ⇨ **Several decision rules indicating different actions**
  - ⇨ **Several states of nature with different consequences**
  - ⇨ **Several criteria**


UNIDADE POPULAR
RUMO AO SOCIALISMO


O OTELO A PRESIDENTE


VOTA PCP
Partido Comunista Português PCP
POR UMA MAIORIA DE ESQUERDA


CDS
PSD
UDP
AD
ALIANÇA DEMOCRÁTICA
OCML
MES
MRPP
VOTA PS
Partido Popular



1

OFFICIAL BALLOT, GENERAL ELECTION
PALM BEACH COUNTY, FLORIDA
NOVEMBER 7, 2000

ELECTORS FOR PRESIDENT AND VICE PRESIDENT

(A vote for the candidates will actually be a vote for their electors.)

(Vote for Group)

(REPUBLICAN)
GEORGE W. BUSH - PRESIDENT
DICK CHENEY - VICE PRESIDENT
3

(DEMOCRATIC)
AL GORE - PRESIDENT
JOE LIEBERMAN - VICE PRESIDENT
5

(LIBERTARIAN)
HARRY BROWNE - PRESIDENT
ART OLIVIER - VICE PRESIDENT
7

(GREEN)
RALPH NADER - PRESIDENT
WINONA LaDUKE - VICE PRESIDENT
9

(SOCIALIST WORKERS)
JAMES HARRIS - PRESIDENT
MARGARET TROWE - VICE PRESIDENT
11

(NATURAL LAW)
JOHN HAGELIN - PRESIDENT
NAT GOLDHABER - VICE PRESIDENT
13

OFFICIAL BALLOT, GENERAL ELECTION
PALM BEACH COUNTY, FLORIDA
NOVEMBER 7, 2000

1

(REFORM)
4
PAT BUCHANAN - PRESIDENT
EZOLA FOSTER - VICE PRESIDENT

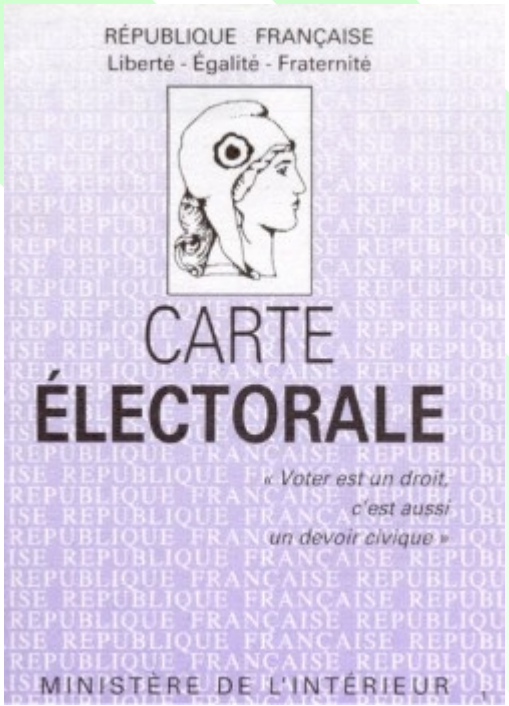
(SOCIALIST)
6
DAVID McREYNOLDS - PRESIDENT
MARY CAL HOLLIS - VICE PRESIDENT

(CONSTITUTION)
8
HOWARD PHILLIPS - PRESIDENT
J. CURTIS FRAZIER - VICE PRESIDENT

(WORKERS WORLD)
10
MONICA MOOREHEAD - PRESIDENT
GLORIA La RIVA - VICE PRESIDENT

WRITE-IN CANDIDATE
To vote for a write-in candidate, follow the directions on the long stub of your ballot card.

TURN PAGE TO CONTINUE VOTING


RÉPUBLIQUE FRANÇAISE
Liberté - Égalité - Fraternité
CARTE
ÉLECTORALE
« Voter est un droit,
c'est aussi
un devoir civique »
MINISTÈRE DE L'INTÉRIEUR

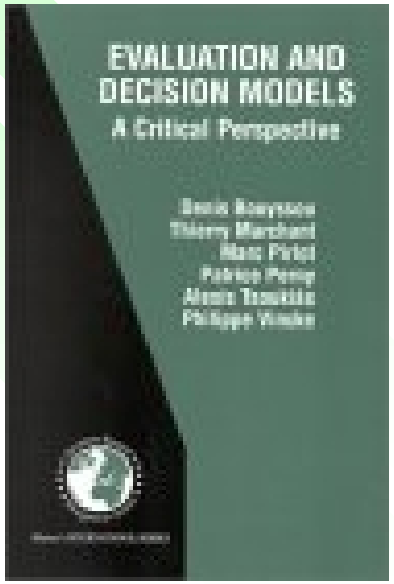

Santa Maria da Feira
CONSEIL EUROPEEN

# Outline

- Introduction
- Examples
  - ⇨ What can go wrong?
- Some results
  - ⇨ What can be expected?
- Extensions

# References

- **D. Bouyssou, Th. Marchant, M. Pirlot, P. Perny, A. Tsoukiàs and Ph. Vincke "Evaluation and Decision models : acritical Perspective", 2000, Kluwer, Ch. 2**



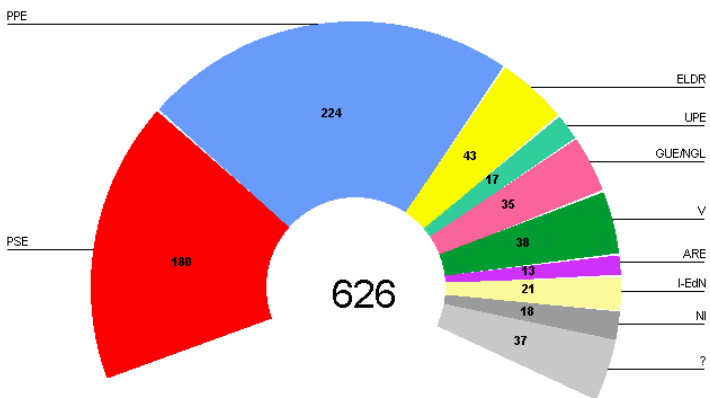

- **H. Nurmi, Comparing Voting Systems, D. Reidel, 1987**
- **H. Nurmi, Voting Paradox and how to deal with them?, Springer-Verlag, 1999**

# Introduction: Vocabulary

- **Group**
  - ⇨ ***Society***
- **Members of the Group**
  - ⇨ ***Voters***
- **Alternatives**
  - ⇨ ***Candidates***
- **Problem**
  - ⇨ **Choice of *one* among several *Candidates***

# Aside: Proportional representation

- **We'll study procedures selecting a *single* candidate**
- **Why not be interested in more refined procedures electing more than one candidate (Proportional Representation)?**
  - ⇨ **PR does not solve the decision problem in the Parliament!**
  - ⇨ **PR raises many difficult problems (What is a just PR? How to achieve it? PR and Power indices)**



# Portuguese Constitution - Art. 149

- **Constituencies**
  - **1.** Deputies shall be elected by electoral districts, the boundaries of which shall be laid down by law, which may also provide for the existence of plurinominal and uninominal electoral districts, as well as their respective kind and complementarity, in order to ensure the system of proportional representation and the Hondt highest average method when converting the votes into the number of mandates.
  - **2.** Except in the case where there is a national electoral district, the number of Deputies allocated to each plurinominal electoral district, shall be proportionate to the number of voters enrolled in the electoral register for that electoral district.

# Proportional representation

**Problem 1**

- **"Power" in Assembly ≠ "Number of seats"**
  - ⇨ **Assembly: 100 members (MP)**
  - ⇨ **Voting rule in assembly: 50% majority**
  - ⇨ **# of votes = # of seats**
  - ⇨ **Party A: 45%**
  - ⇨ **Party B: 15%**
  - ⇨ **Party C: 40%**

• alone each party is powerless
• *any* two party coalition will do

- **All parties have the same "power" (symmetry/50%)**

# Proportional representation

- **Problem 2: How to achieve a “just” representation**
- **# of voters >> # of MP**
- **# of MP is integer!**
  - ➪ **“Rounding off”**
  - ➪ **12 324 823 voters**
  - ➪ **3 987 345 votes for Party A**
  - ➪ **342 MP**
  - ➪ **Theoretical # of MP for party A:** $\frac{3\,987\,345}{12\,324\,823} \times 342 = 110{,}644$
  - ➪ **# of MP = 110? 111? Other?**

# Alabama Paradox (1881)

- 2 100 000 voters, 3 parties, 20 seats
- Results
  - A: 928 000
  - B: 635 000
  - C: 537 000
- Rule: Largest remainder (Hamilton's rule)
  - A: 8,84 — 8 seats + 1 seat = 9 seats
  - B: 6,05 — 6 seats
  - C: 5,11 — 5 seats

$$8{,}84 = \frac{928\,000}{2\,100\,000}$$

# Alabama Paradox

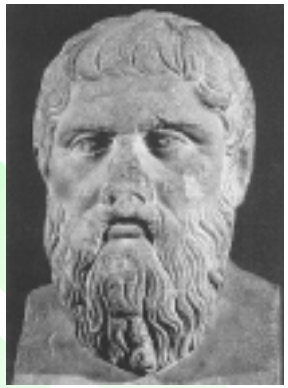
- **Increasing the # of MP**
- **21 seats**
  - **A: 9,28** — **9 seats (9)**
  - **B: 6,35** — **6 seats (6)**
  - **C: 5,37** — **5 seats + 1 seat = 6 seats (5)**
- **22 seats**
  - **A: 9,72** — **9 seats + 1 seat = 10 seats (9)**
  - **B: 6,65** — **6 seats + 1 seat = 7 seats (6)**
  - **C: 5,63** — **5 seats (6)**
  - **C is loosing one seat whereas the # of seats is increasing!**

# Introduction

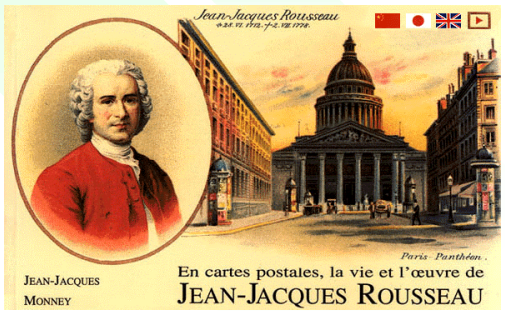
- **The choice of the candidate will affect all members of the society**
- **The choice of the candidate should take into account the *opinion* of the members of the society**

Democracy $\Rightarrow$ Elections $\Rightarrow$ Majority

# Elections



- “Philosophical problems”
  - ➪ “General will” and elections
  - ➪ Minorities vs. Majority
- “Political problems”
  - ➪ Direct vs. indirect democracy
  - ➪ Role of political parties
  - ➪ Who should vote? How often should we vote?
  - ➪ Who can be a candidate?
  - ➪ What kind of mandate?



# Technical problems

- **Majority decisions**
  - ➪ **Candidate *a* should beat candidate *b* if more voters prefer *a* to *b***
- **Two candidates ⇒ No problem: elect the candidate with more votes!**
- **How to extend the idea with more than 2 candidates?**
  - ➪ **Many ways to do so!**

# Types of Elections

- **Type of ballot that the voters can cast**
  - ➪ **Indicate the name of a candidate**
  - ➪ **Rank order the set of candidates**
  - ➪ **Other (acceptable or unacceptable candidates, grades, veto, etc.)**
- **Aggregation method**
  - ➪ **Technique used to tabulate the ballots and to designate the winner**

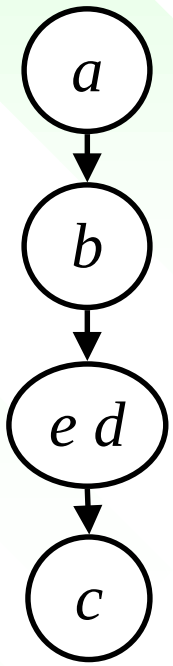
# Hypothesis

- **Each voter is able to *rank order* the set of candidates in terms of preference**

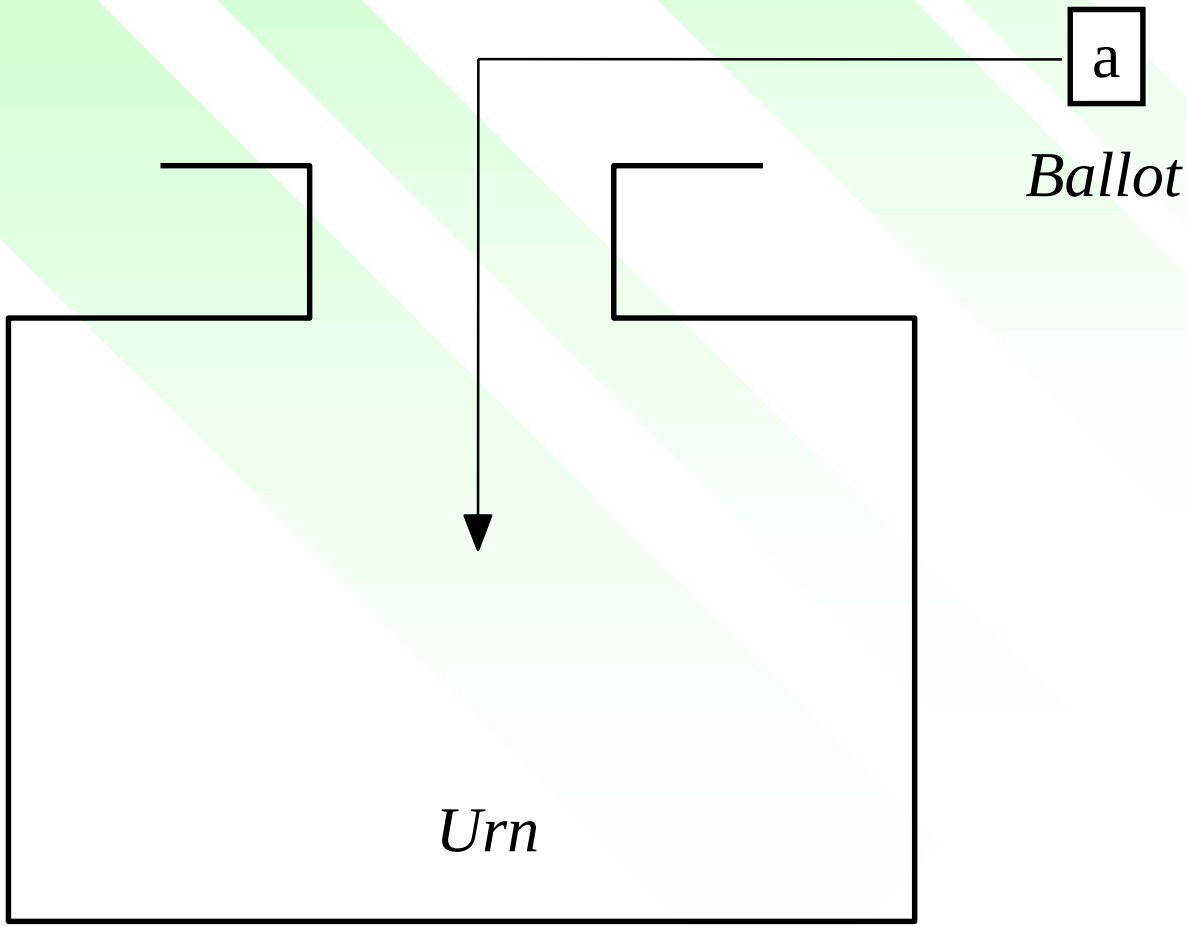
  ***a P b P* [*e I d*] *P c***

- **Voters are *sincere***

  **(If I if have to vote for *one* candidate, I vote for *a*)**



# Simple ballots



# Plurality voting (UK)

- **Ballots with a single name**
- **One round of voting**
- **The candidate with most votes is elected**

ties (not likely) are neglected

Give some special tie-breaking power to one of the voter

Give some special special statute to one of the candidate

3 candidates : {*a*, *b*, *c*}

21 voters (or 21 000 000 or 42 000 000)

*a* : Tories

*b* : Labour

*c* : LibDem

**Preferences of the voters**

10 : *a P b P c*

6 : *b P c P a*

5 : *c P b P a*

**Result**

*a* : 10 *b* : 6 *c* : 5

*a* is elected

Is the UK system that democratic?

Can we expect the voters to be sincere?

Extra-democratic choice of only two candidates

BUT…

An *absolute majority* of voters (11/21) prefer *all* other candidates to the candidate elected!

# Plurality voting with runoff (France – Presidential elections)

- Ballots with a single name
- 1st round of voting
  - ⇨ The candidate with most votes is elected if he receives more than 50% of the votes
  - ⇨ Otherwise go to a 2nd round of voting (runoff) with the two candidates having received most votes in the first round (again neglect ties)
- 2nd round of voting
  - ⇨ The candidate with most votes is elected

**Preferences of the voters**

10 : *a P b P* ~~*c*~~

6 : *b P* ~~*c*~~ *P a*

5 : ~~*c*~~ *P b P a*

Apparently much better than the UK system

With little added complexity

**1st round** (absolute majority = 11)

*a* : 10 *b* : 6 *c* : 5

**2nd round**

*a* : 10 *b* : 11

*b* is elected (11/21)

AND

no candidate is preferred to *b* by a majority of voters

(a : 11/21, c : 16/21)

4 candidates: {*a*, *b*, *c*, *d*}

21 voters

10 : *b P ~~a~~ P c P ~~d~~*

6 : *c P ~~d~~ P ~~d~~ P b*

5 : *~~d~~ P ~~d~~ P b P c*

**1st Round** (absolute majority = 11)

*a* : 5 *b* : 10 *c* : 6 *d* : 0

**2nd Round**

*b* : 15 *c* : 6

**Result**: *b* is (very well) elected (15/21)

BUT...

an absolute majority of voters (11/21) prefer candidates *a* and *d* to the candidate elected *b*!

The French system does only a little better than the UK system

Preferences used in the example are NOT bizarre

Sincerity?

Wasted votes

4 candidates : {*a*, *b*, *c*, *d*}

21 voters

10 : *b P a P c P d*

6 : *c P a P d P b*

5 : *a P d P b P c*

**Result** : *b* is elected

Manipulable methods ⇒
elections might not reveal
the true opinion of the voters

Advantage to clever voters
(knowing how to manipulate)

**Non sincere voting**

The 6 voters with *c P a P d P b*

*decide to vote* vote as if their preference was

*a P c P d P b*

(Do not waste your vote!)

**Result** : *a* is elected in the 1st round (11/21)

Voting non sincerely may be profitable

Method susceptible to **manipulation**

3 candidates: $\{a, b, c\}$

17 voters

**Opinion poll**

6 : $a\ P\ b\ P\ c$

5 : $c\ P\ a\ P\ b$

4 : $b\ P\ c\ P\ a$

2 : $b\ P\ a\ P\ c$

**1st Round** (absolute majority = 9)

$a$ : 6  $b$ : 6  $c$ : 5

**2nd Round**

$a$ : 11  $b$ : 6

Nothing to worry about

up to now on this example

$a$ starts a campaign against $b$

It works

2 voters: $b\ P\ a\ P\ c$

become

$a\ P\ b\ P\ c$

This change is favorable to $a$

which is the favorite

**New preferences (after campaign)**

6 : *a P b P c*

5 : *c P a P b*

4 : *b P c P a*

2 : *a P b P c*

**Non monotonic method**

**Sincerity of voters?**

**1st Round** (absolute majority = 9)

*a* : 8 *b* : 4 *c* : 5

**2nd Round**

*a* : 8 *c* : 9

*c* is elected!

The result of his successful campaign is fatal to *a*

3 candidates: {*a*, *b*, *c*}

11 voters

4 : *a P b P c*

4 : *c P b P a*

3 : *b P c P a*

What if some voters abstain?

Abstention should NOT be profitable

(otherwise why vote?!)

**1st round** (absolute majority = 6)

*a* : 4  *b* : 3  *c* : 4

**2nd round**

*a* : 4  *c* : 7

**Result**: *c* elected (7/11)

3 candidates: {*a*, *b*, *c*}

11 voters

2 voters among the 4 : *a P b P c* abstain

Abstaining was VERY rational for our two voters (they prefer *b* to *c*)

**Not participation incentive!**

**1st round** (majority = 5)

*a* : 2  *b* : 3  *c* : 4

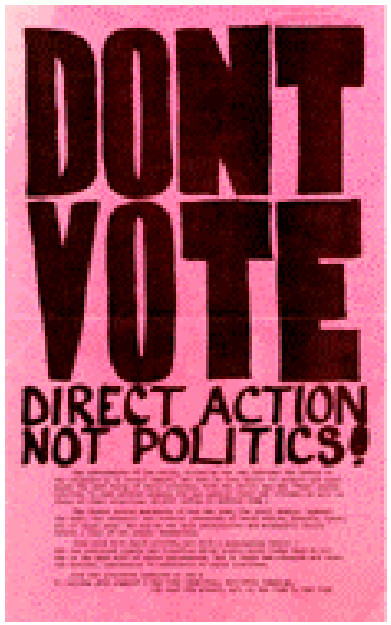
**2nd round**

*b* : 5  *c* : 4

**Result**: *b* elected (5/9)


NE PAS VOTER, C'EST LAISSER LES AUTRES DÉCIDER À VOTRE PLACE.
INSCRIVEZ-VOUS SUR LES LISTES ÉLECTORALES AVANT LE 31 DÉCEMBRE
cidem


voter c'est abdiquer
fédération anarchiste
s'abstenir c'est lutter


DONT VOTE
DIRECT ACTION NOT POLITICS!

3 candidates: {*a*, *b*, *c*}

26 voters: 13 in district 1, 13 in district 2

**District 1**

13 voters

4 : *a P b P c*

3 : *b P a P c*

3 : *c P a P b*

3 : *c P b P a*

**1st round** (majority = 7)

*a* : 4  *b* : 3  *c* : 6

**2nd round**

*a* : 7  *c* : 6

**Result**: *a* elected (7/13) in district 1

**District 2**

13 voters

4 : *a P b P c*

3 : *c P a P b*

3 : *b P c P a*

3 : *b P a P c*

**1st round** (majority = 7)

*a* : 4  *b* : 6  *c* : 3

**2nd round**

a : 7  b : 6

**Result**: *a* elected (7/13) in district 2

*a* is elected in both district...

AND THUS should be elected

26 voters

4 : *a P b P c*

3 : *b P a P c*

3 : *c P a P b*

3 : *c P b P a*

4 : *a P b P c*

3 : *c P a P b*

3 : *b P c P a*

3 : *b P a P c*

Entire Society

*a* is elected in both districts

but looses when grouped

**Non separable method**

Decentralized decisions?

**1st Round** (majority = 14)

*a* : 8 *b* : 9 *c* : 9 ***a* looses in the first round!**

**2nd Round**

*b* : 17 *c* : 9

**Result**: *b* elected (17/26)

# Summary

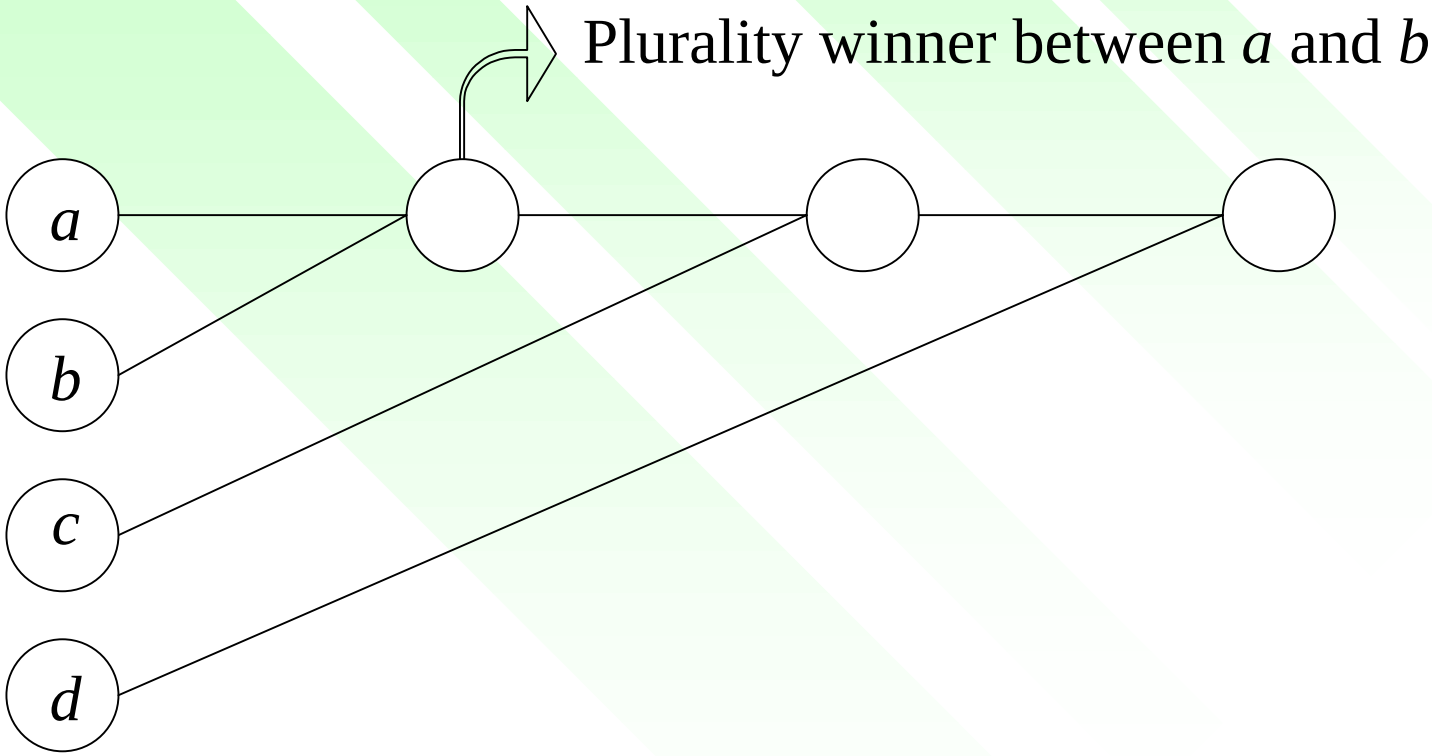
- **The French system does only a little better better than the UK one on the “democratic side”**
- **It has many other problems**
  - **not monotonic**
  - **no incentive to participate**
  - **manipulable**
  - **non separable**
- **Other (better!) systems?**

# Amendment procedure

- **The majority method works well with two candidates**
- **When there are more than two candidates, organize a series of confrontations between two candidates according to an *agenda***
- **Method used in most parliaments**
  - ⇨ **amendments to a bill**
  - ⇨ **bill amended *vs.* status quo**

4 candidates {*a*, *b*, *c*, *d*}

Agenda: *a*, *b*, *c*, *d*




Example: *c* is a bill, *a* and *b* are amendments, *d* is the status quo

3 candidates: {*a*, *b*, *c*}

3 voters

1 voter: *a P b P c*

1 voter: *b P c P a*

1 voter: *c P a P b*

Agenda: *a*, *b*, *c* **Result**: *c*

Agenda: *b*, *c*, *a* **Result**: *a*

Agenda: *c*, *a*, *b* **Result**: *b*

Results depending on the arbitrary choice of an agenda

(power given to the agenda-setter)

Candidates are not treated equally (the later the better)

4 candidates: {*a*, *b*, *c*, *d*}

3 voters

1 voter: *b P a P d P c*

1 voter: *c P b P a P d*

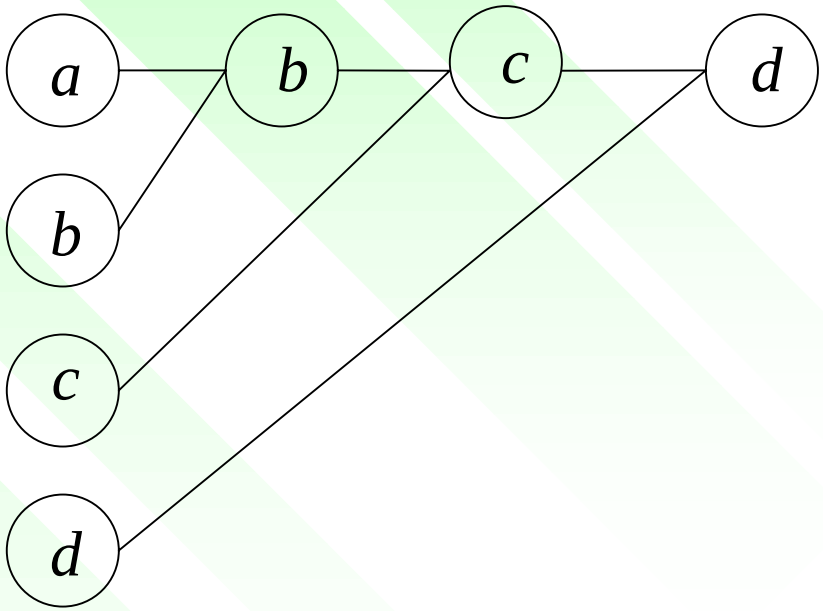
1 voter: *a P d P c P b*

Agenda: *a*, *b*, *c*, *d*

**Result**: *d* elected

BUT...

100% of voters prefer *a* to *d* !




Non *unanimous* method

26 candidates: {*a*, *b*, *c*, ..., *z*}

100 voters

51 voters: *a P b P c P ... P y P z*

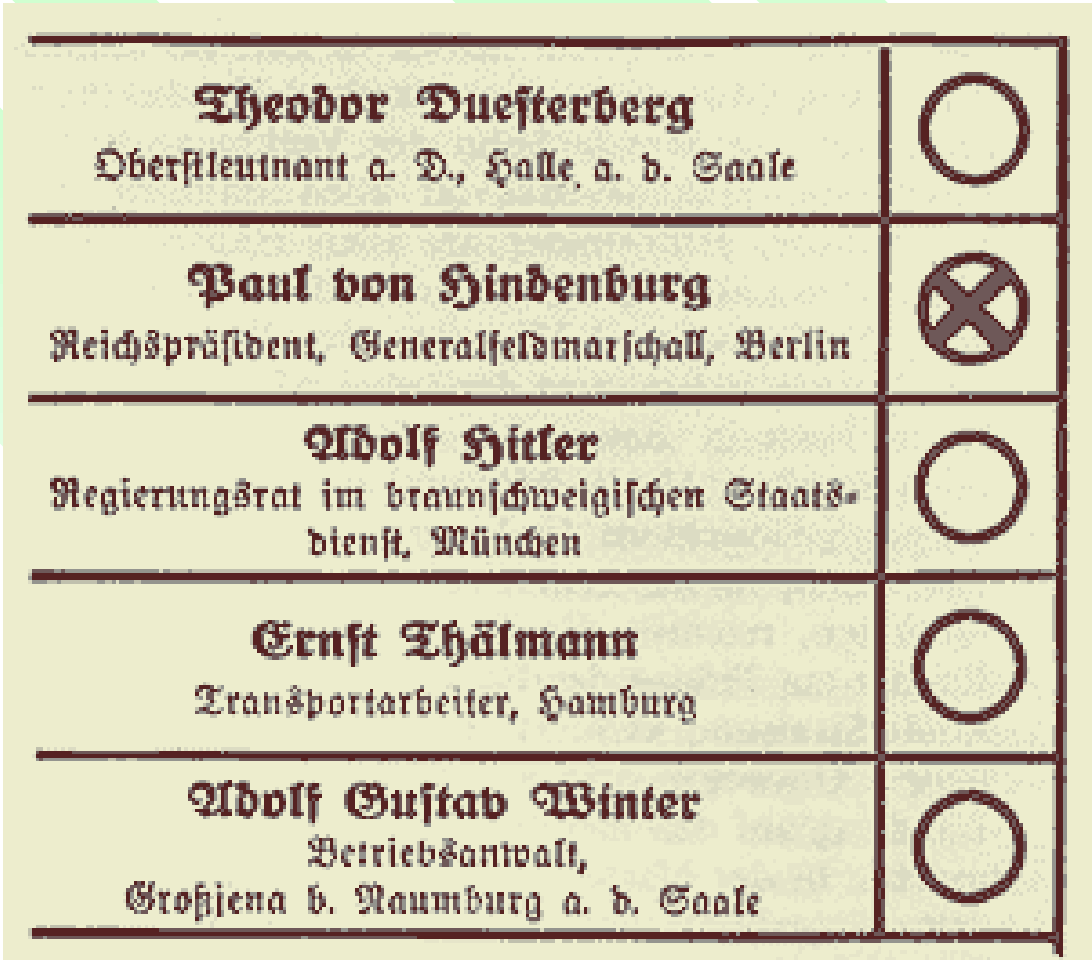
49 voters: *z P b P c P ... P y P a*

With sincere voters and with all majority-based systems with only one name per ballot, *a* is elected and the “compromise” candidate *b* is rejected

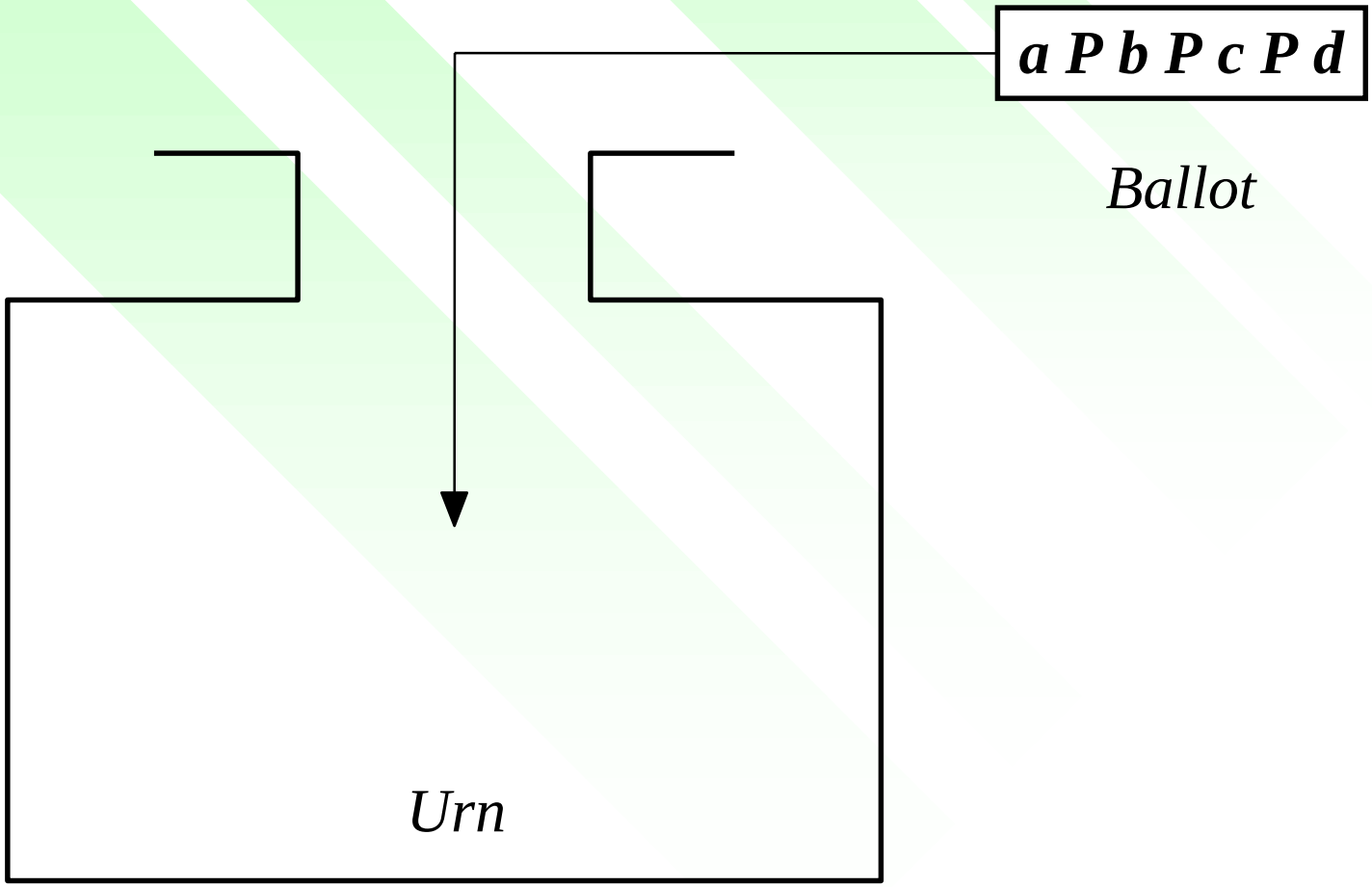
Dictature of the majority

(recent European history?)

$\Rightarrow$ look for more refined ballots


Theodor Duesterberg
Oberstleutnant a. D., Halle a. d. Saale
Paul von Hindenburg
Reichspräsident, Generalfeldmarschall, Berlin
Adolf Hitler
Regierungsrat im braunschweigischen Staatsdienst, München
Ernst Thälmann
Transportarbeiter, Hamburg
Adolf Gustav Winter
Betriebsanwalt,
Großjena b. Naumburg a. d. Saale

# Ballots: Ordered lists



# Remarks

- **Much richer information**
  - ⇨ **practice?**
- **Ballots with one name are a particular case**
  - ⇨ **voting for *a***
  - ⇨ **voting like *a P [all others]***

| | |
|---|---|
| voter 1: | *a b c d e* |
| voter 2: | *b c e d a* |
| voter 3: | *e a b c d* |
| voter 4: | *a b d e c* |
| voter 5: | *b d c a e* |

# J.A.M.N.C. marquis de Condorcet

- **Compare all candidates by pair**
- **Declare that *a* is "socially preferred" to *b* if (strictly) more voters prefer *a* to *b***

  **(social indifference in case of a tie)**
- **Condorcet's principle: if one candidate is preferred to *all other* candidates, it should be elected.**

  ***Condorcet Winner* (must be unique)**

# Remarks

- **UK and French systems violate Condorcet's principle**
- **The UK system may elect a Condorcet looser**

3 candidates: {*a*, *b*, *c*}

21 voters

**Preferences of the voters**

10 : *a P b P c*

6 : *b P c P a*

5 : *c P b P a*

*a* is the plurality winner

*b* is the Condorcet Winner (11/21 over *a*, 16/21 over *c*)

*a* is the Condorcet Looser (10/21 over *b*, 10/21 over *c*)

4 candidates: {*a*, *b*, *c*, *d*}

21 voters

10 : *b P a P c P d*

6 : *c P a P d P b*

5 : *a P d P b P c*

*b* is the plurality with runoff winner

*a* is the Condorcet Winner

(11/21 over *b*, 15/21 over *c*, 21/21 over *d*)

# Remarks

- **Condorcet's principle does not solve the "dictature of the majority" difficulty**

26 candidates: {*a*, *b*, *c*, ..., *z*}

100 voters

51 voters: *a P b P c P ... P y P z*

49 voters: *z P b P c P ... P y P a*

*a* is the Condorcet winner

- **A Condorcet Winner is not necessarily “ranked high” by voters**

5 candidates: {*a*, *b*, *c*, *d*, *e*}

5 voters

1 voter: *a P b P c P d P e*

1 voter: *b P c P e P d P a*

1 voter: *e P a P b P c P d*

1 voter: *a P b P d P e P c*

1 voter: *b P d P c P a P e*

| Ranks | 1 | 2 | 3 | 4 | 5 |
|---|---|---|---|---|---|
| a | 2 | 1 | 0 | 1 | 1 |
| b | 2 | 2 | 1 | 0 | 0 |

*a* is the Condorcet winner

(3:2 win on all other candidates)

# Remarks

- **May be an attractive concept however BUT it is impossible to rely exclusively on it**

3 candidates: {*a*, *b*, *c*}

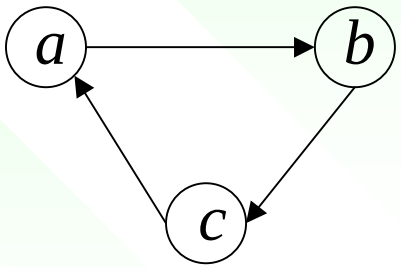
3 voters

1 : *a P b P c*

1 : *b P c P a*

1 : *c P a P b*

## Condorcet's Paradox



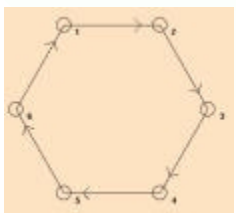

*a* is socially preferred to *b*

*b* is socially preferred to *c*

*c* is socially preferred to *a*

As the social preference relation may have cycles, a Condorcet winner does not always exist (probability 40% with 7 candidates and a large number of voters)

McGarvey's Theorem



# Condorcet

- **Weaken the principle so as to elect candidates that are not strictly beaten**

  **(Weak CW)**

  - ⇨ **they may not exist**
  - ⇨ **there may be more than one**
- **Find what to do when there is no (weak) Condorcet winner**

# Schwartz

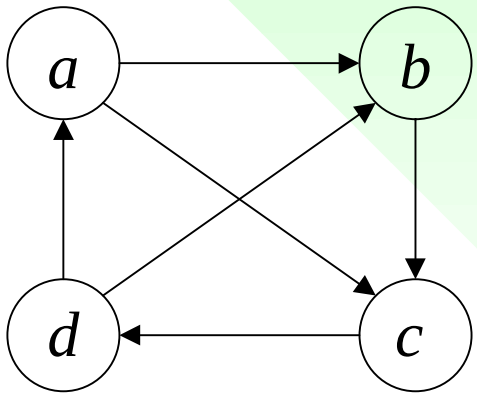
- **The strict social preference may not be transitive**
  - ⇨ **Take its transitive closure**
    - **smallest transitive relation containing the original relation**
  - ⇨ **Take the maximal elements of the resulting weak order**

4 candidates: {*a*, *b*, *c*, *d*}, 3 voters

1 : *a P b P c P d*

1 : *d P a P b P c*

1 : *c P d P a P b*



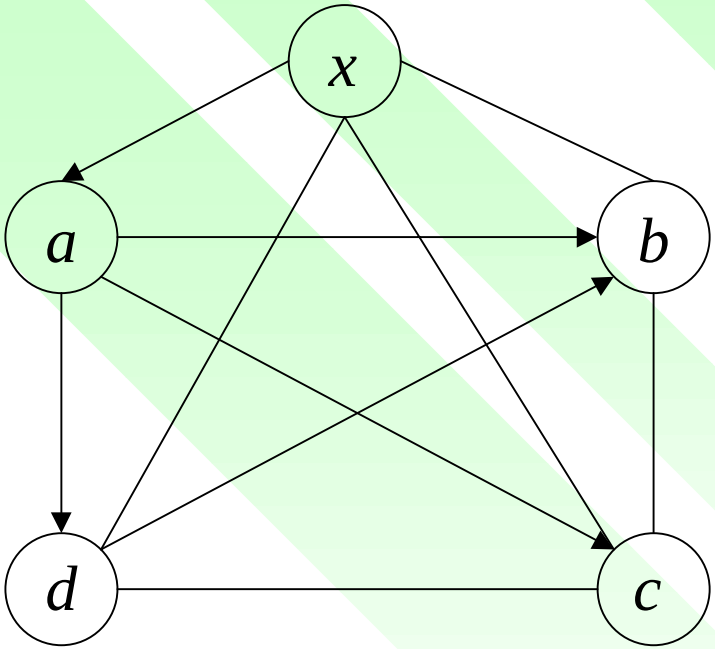

Taking the transitive closure,
all alternatives are indifferent

BUT....

100% of the voters prefer *a* to *b*

# Copeland

- **Count the number of candidates that are beaten by one candidate minus the number of candidates that beat him (Copeland score)**
- **Elect the candidate with the highest score**
- **Sports leagues**
  - **+2 for a victory, +1 for a tie**
  - **equivalent to Copeland's rule (round robin tournaments)**




## Copeland scores

| | |
|---|---|
| $x$ | **1** |
| $a$ | **2** |
| $b$ | **-2** |
| $c$ | **-1** |
| $d$ | **0** |

*x* is the only unbeaten candidate but is not elected

# Jean-Charles de Borda

- Each ballot is an ordered list of candidates (exclude ties for simplicity)
- On each ballot compute the rank of the candidates in the list
- Rank order the candidates according to the decreasing sum of their ranks

4 candidates: {*a*, *b*, *c*, *d*}

3 voters

2 : *b P a P c P d*

1 : *a P c P d P b*

| | 1st | 2nd | 3rd | 4th |
|---|---|---|---|---|
| a | 1 | 2 | 0 | 0 |
| b | 2 | 0 | 0 | 1 |
| c | 0 | 1 | 2 | 0 |
| d | 0 | 0 | 1 | 2 |

Borda Scores

*a* : 2×2 + 1×1 = 5  *b* : 6  *c* : 8  *d* : 11

Result: *a* elected

Remark: *b* is the (obvious) Condorcet winner

# Borda

- **Simple**
- **Efficient: always lead to a result**
- **Separable, monotonic, participation incentive**

**BUT...**

- **Violates Condorcet's Principle**
- **Has other problems**
  - ⇨ **consistency of choice in case of withdrawals**

4 candidates: $\{a, b, c, d\}$

3 voters

2 : $b \; P \; a \; P \; c \; P \; d$

1 : $a \; P \; c \; P \; d \; P \; b$

Borda Scores

$a$ : $2\times2 + 1\times1 = 5$ $\quad$ $b$ : 6 $\quad$ $c$ : 8 $\quad$ $d$ : 11

Result: *a* elected

Suppose that *c* and *d* withdraw from the competition

Borda Scores

$a$ : $2\times2 + 1\times1 = 5$ $\quad$ $b$ : 4

Result: *b* elected

# Is the choice of a method important?

4 candidates: {*a*, *b*, *c*, *d*}, 27 voters

5 : *a P b P c P d*

4 : *a P c P b P d*

2 : *d P b P a P c*

6 : *d P b P c P a*

8 : *c P b P a P d*

2 : *d P c P b P a*

*d* is the plurality winner

*a* is the plurality with runoff winner

*b* is the Borda winner

*c* is the Condorcet winner

# Many other proposals

- **Dodgson (Lewis Carroll)**
- **Nansson**
- **etc.**

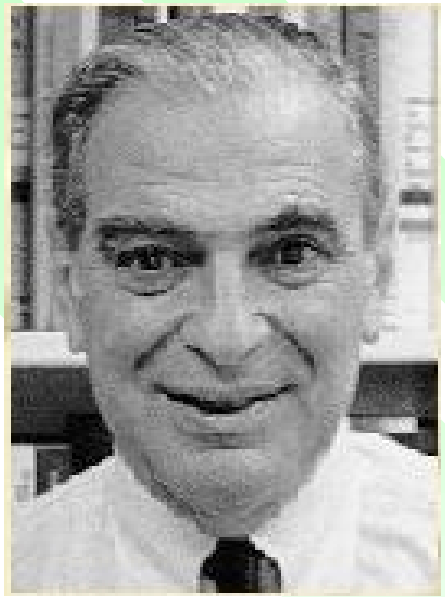
# What are we looking for?

- **“Democratic method”**
  - **always giving a result like Borda**
  - **always electing the Condorcet winner**
  - **consistent wrt withdrawals**
  - **monotonic, separable, incentive to participate, not manipulable, etc.**

# K.J. Arrow

- $n \geq 3$ candidates (otherwise use plurality)
- $m$ voters ($m \geq 2$ and *finite*)
- ballots = “ordered list” of candidates

- Problem: find all “methods” respecting a small number of “desirable” principles



**Kenneth J. Arrow**: Nobel Prize in Economics (1972)
"for his pioneering contributions to general economic equilibrium theory and welfare theory"
(together with John R. Hicks)

- ***Universality*: the method should be able to deal with any configuration of ordered lists**
- ***Transitivity*: the result of the method should be an ordered list of candidates**
- ***Unanimity*: the method should respect a unanimous preference of the voters**
- ***Absence of dictator*: the method should not allow for dictators**
- ***Independence*: the comparison of two candidates should be based only on their respective standings in the ordered lists of the voters**

# Arrow's Theorem (1951)

- **Theorem: There is no method respecting the five principles**
  - ⇨ **Borda is**
    - **universal, transitive, unanimous with no dictator**
  - ⇒ **it cannot be independent**
  - ⇨ **Condorcet is**
    - **universal, unanimous, independent with no dictator**
  - ⇒ **it cannot be transitive**

# Sketch of the proof

- $V \subseteq N$ **is *decisive* for** $(a,b)$ **if whenever**
  $a \; P_i \; b$ **for all** $i \in V$ **then** $a \; P \; b$
- $V \subseteq N$ **is *almost decisive* for** $(a,b)$ **if whenever**
  $a \; P_i \; b$ **for all** $i \in V$ **and**
  $b \; P_j \; a$ **for all** $j \notin V$
  **then** $a \; P \; b$

# Lemma 1

- **If *V* is almost decisive over some ordered pair (*a*,*b*), it is decisive over all ordered pairs.**

  $\{a, b, x, y\}$ and use *universality* to obtain:

  $V : x\ P\ a\ P\ b\ P\ y$

  $N\backslash V : x\ P\ a,\ b\ P\ y,\ b\ P\ a$ (position of $x$ and $y$ unspecified)

  *Unanimity* $\Rightarrow x\ P\ a$ and $b\ P\ y$

  $V$ is almost decisive for $(a,b) \Rightarrow a\ P\ b$

  $\Rightarrow x\ P\ y$ (*transitivity*)

  *Independence* $\Rightarrow$ the ordering of $a$ and $b$ is irrelevant

# Lemma 2

- **If $V$ is decisive and card($V$) > 1, then some proper subset of $V$ is decisive**

  $\{x, y, z\}$ use *universality* to obtain:

  $V1 : x\ P\ y\ P\ z$

  $V2 : y\ P\ z\ P\ x$

  $N \backslash V : z\ P\ x\ P\ y$

  $V$ decisive $\Rightarrow y\ P\ z$

  If $x\ P\ z$ then $V1$ is almost decisive for ($x$, $z$) and thus decisive (lemma 1)

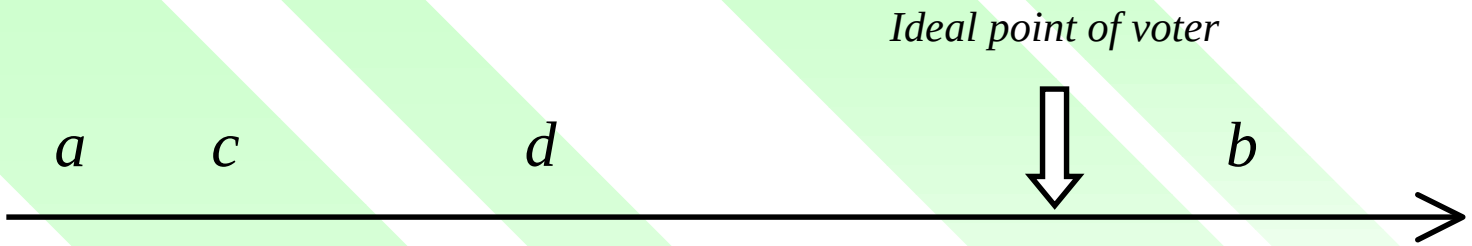
  If $z\ R\ x$ then $y\ P\ x$ (*transitivity*) and $V2$ is almost decisive for ($y$, $x$) and thus decisive (lemma 1)

# Proof

- ***Unanimity*** $\Rightarrow$ ***N*** **is decisive**
- **Since *N* is finite the iterated use of lemma 2 leads to the existence of a dictator**

# Principles

- ***Unanimity*: no apparent problem**
- ***Absence of dictator*: minimal requirement of democracy!**
- ***Universality*: a group adopting functioning rules that would not function in "difficult situations" could be in big trouble!**
  - ⇨ **Black: unimodal preferences (no weird voters)**




*Preference of the voter : b P d P c P a*

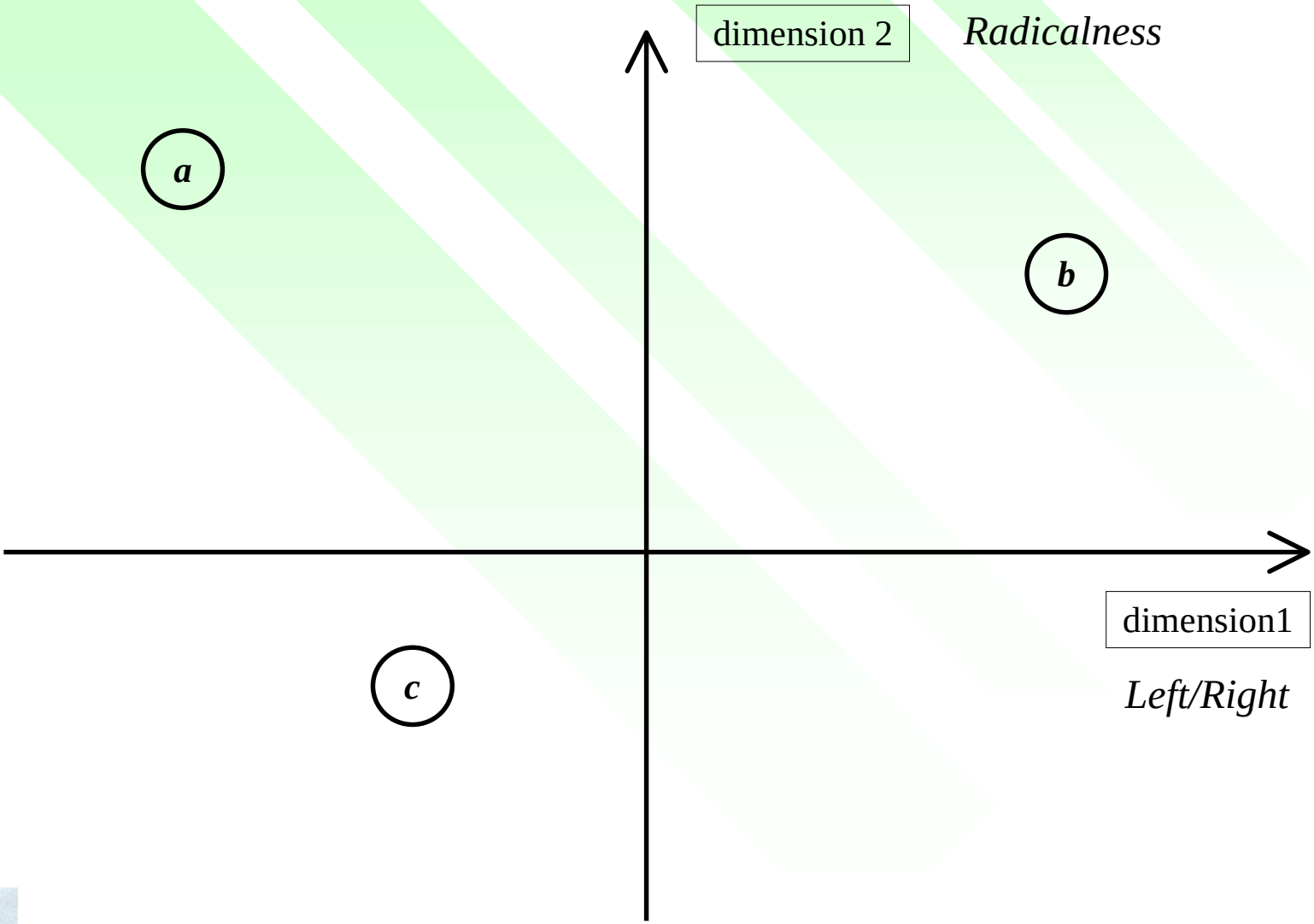
*"Impossible" preferences :*

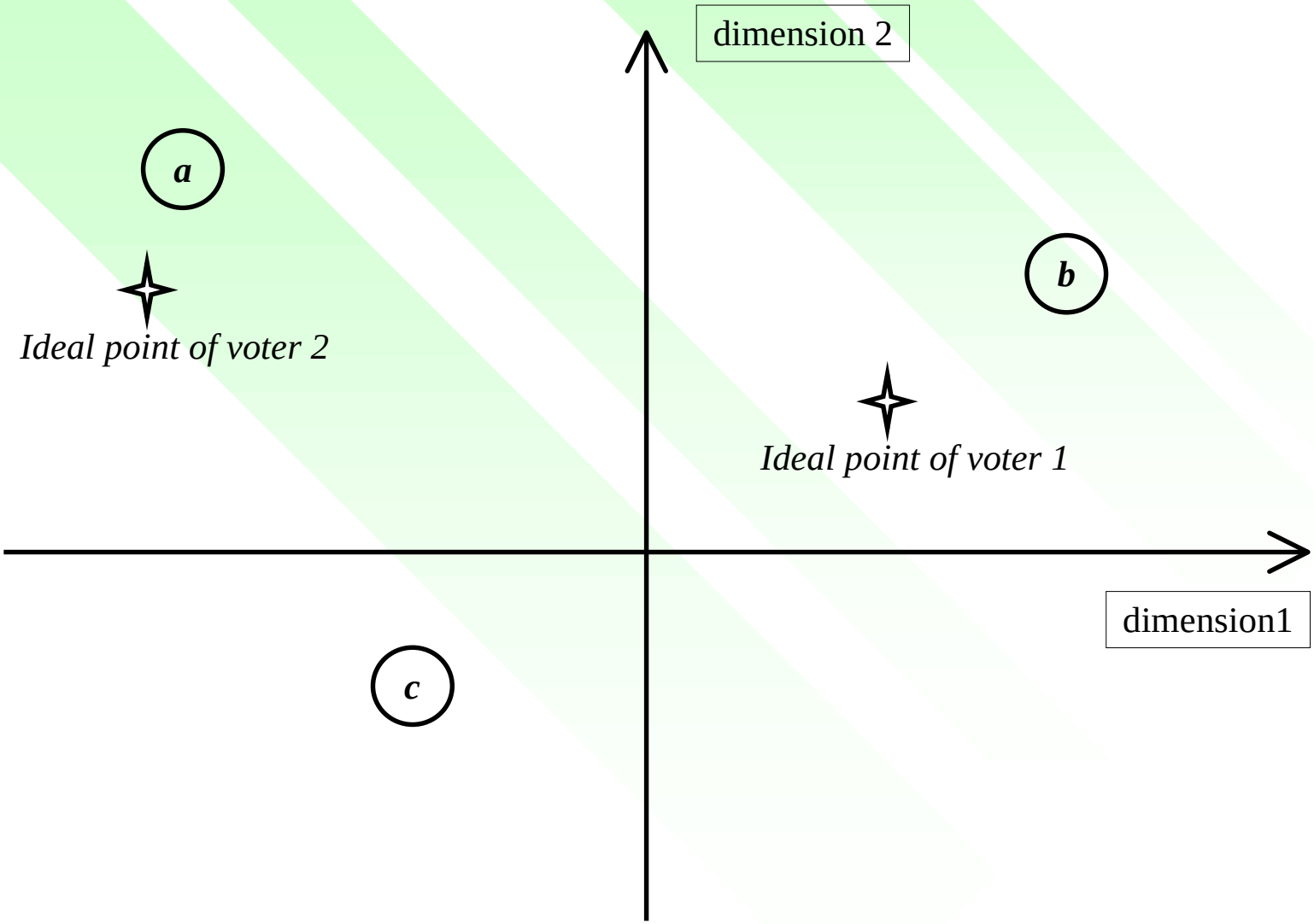
*a P b P c P d*

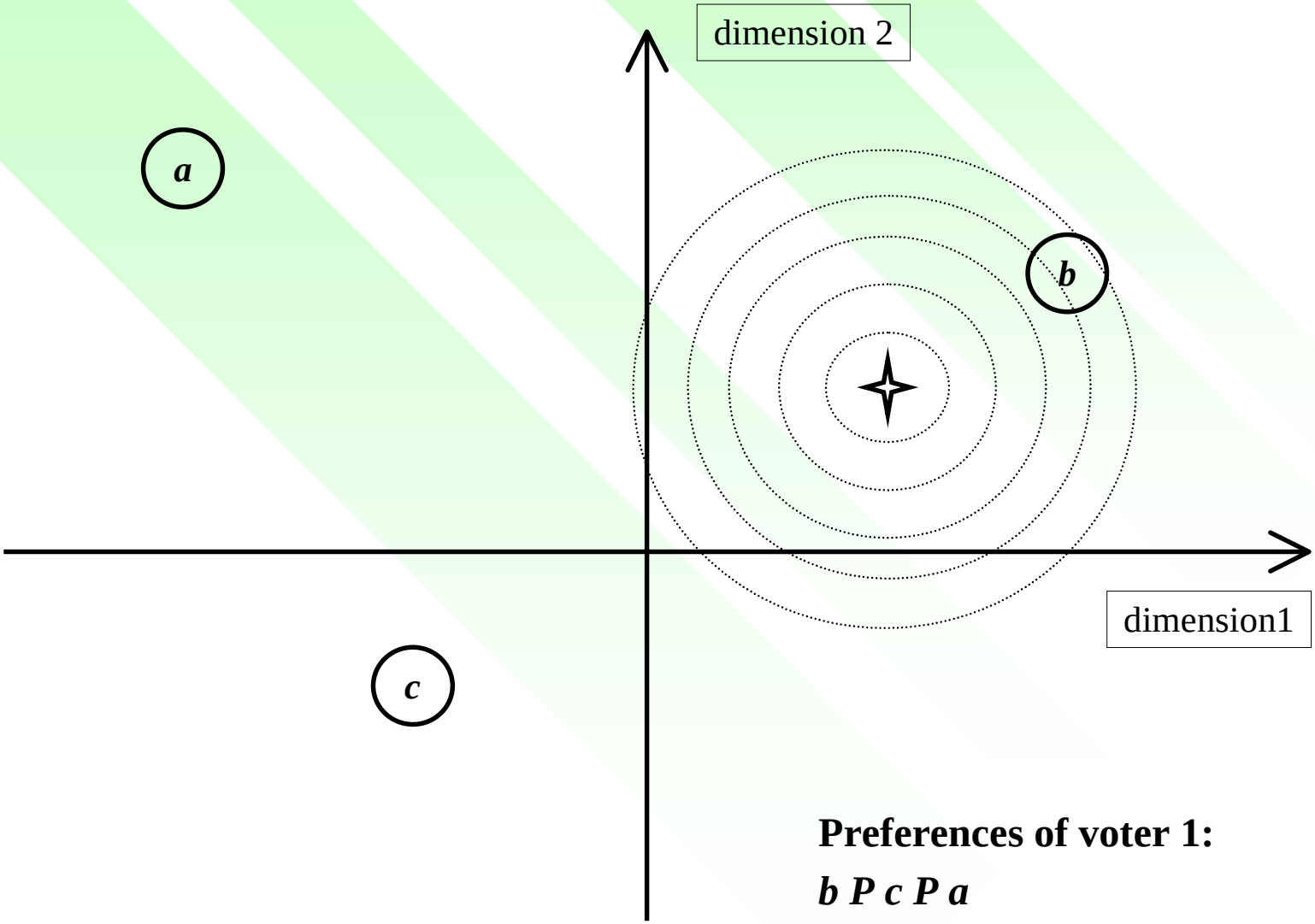
*d P a P b P c*

- **If this property can be accepted Universality can be abandoned**
- **Only work with *one* dimension**

# Spatial models of voting




dimension 2
a
b
Ideal point of voter 2
Ideal point of voter 1
dimension1
c




**Preferences of voter 1:**

***b P c P a***

# *Independence*

- **no *intensity of preference* considerations**
  - ⇨ **I "intensely" or "barely" prefer *a* to *b***
    - **practice, manipulation, interpersonal comparisons?**
- **no consideration of a *third alternative* to rank order *a* and *b***

# Borda and Independence

4 candidates: $\{a, b, c, d\}$, 3 voters

2 voters: $c\ P\ a\ P\ b\ P\ d$

1 voter: $a\ P\ b\ P\ c\ P\ d$

Borda: $a\ P\ c\ P\ b\ P\ d$ (scores : 5, 6, 7 and 11)

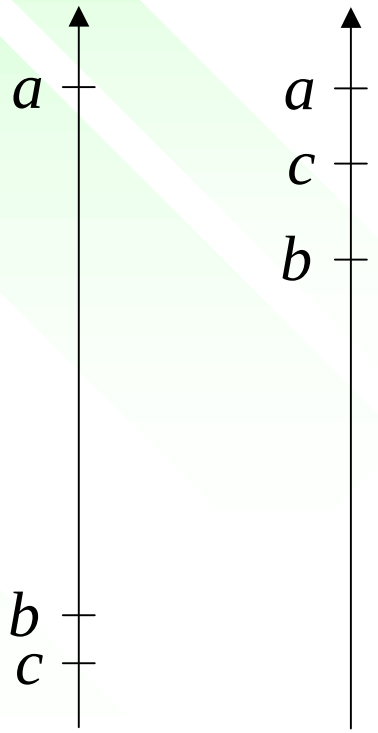
2 voters: $c\ P\ a\ P\ b\ P\ d$

1 voter: $a\ P\ c\ P\ b\ P\ d$

Borda: $c\ P\ a\ P\ b\ P\ d$ (scores : 4, 5, 9 and 12)

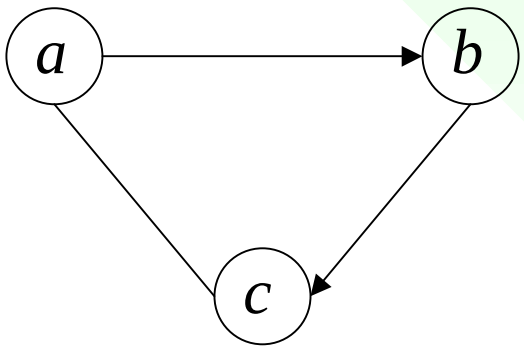
The ranking of $a$ and $c$ is reversed

BUT... the respective positions of $a$ and $c$ is unchanged in the individual lists



# *Transitivity*

- **maybe too demanding if the only problem is to elect a candidate**
- **BUT... guarantees consistency**




In $\{a, b, c\}$, $a$ is elected

In $\{a, c\}$, both $a$ and $c$ are elected

# Relaxing transitivity

- **Semi-orders and interval order**
  - ⇨ **no change (if more than 4 candidates)**
- **Transitivity of strict preference**
  - ⇨ **oligarchy: group *O* of voters st**

    $a \, P_i \, b \; \forall \; i \in O \Rightarrow a \, P \, b$

    $i \in O$ **and** $a \, P_i \, b \Rightarrow$ **Not[**$b \, P \, a$ **]**
- **Absence of cycles**
  - ⇨ **some voter has a veto power**

    $a \, P_i \, b \Rightarrow$ **Not[**$b \, P \, a$ **]**

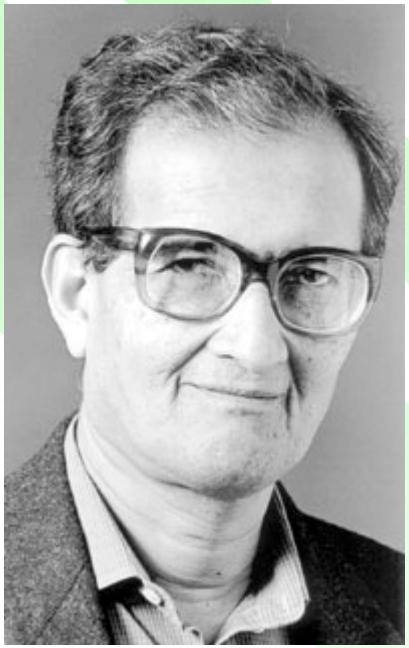
# Message

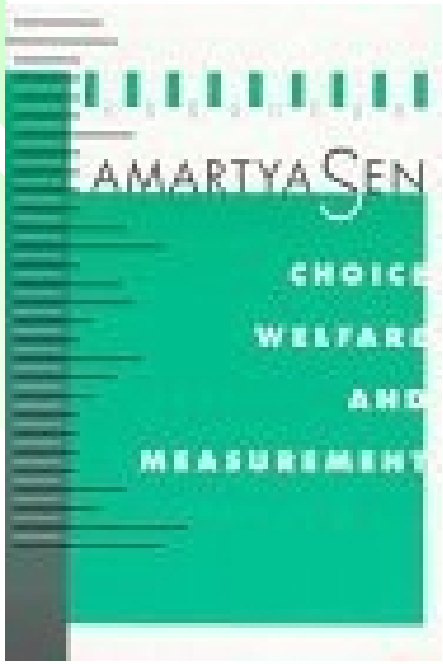
- **Despair?**
  - ⇨ **no “ideal” method**
  - ⇨ **this would be dull!**

- **A group is more complex than an individual**
- **Analyze the pros and cons of each method**
- **Beware of “method-sellers”**

# Extensions

- **Impossibility results**
  - ⇨ **Arrow**
  - ⇨ **Gibbard-Sattherthwaite**
    - **All “reasonable methods” may be manipulated (more or less easily or frequently)**
  - ⇨ **Moulin**
    - **No separable method can be Condorcet**
    - **No Condorcet method can give an incentive to participate**
  - ⇨ **Sen**
    - **Tensions between unanimity and individual freedom**






**Amartya Sen**: Nobel Prize in Economics (1998)

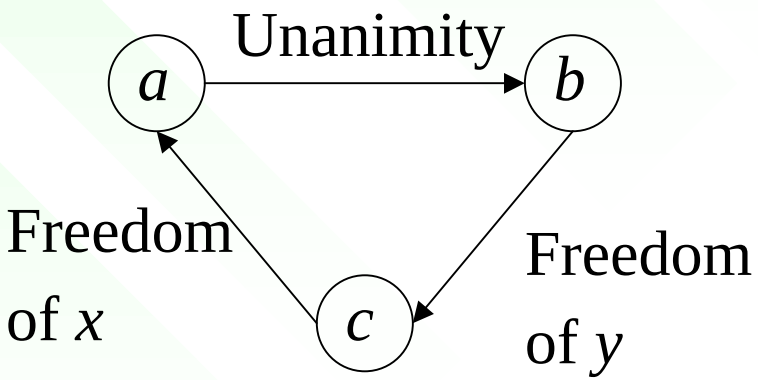
“for his contributions to welfare economics”

# Paretian Liberal Paradox

- **Fact: there are tensions between the majority principle and the respect of individual rights**
  - ⇨ **A majority wants me to do something I do not want to do!**
- ***Paradox*: there are tensions between the respect of individual rights and the unanimity principle**
- ***Theorem*: Unanimity + Universality + Respect of individual rights ⇒ Problems**

# Example

- **2 individuals (males) on a desert island**
  - **Mr. *x* the Puritan and Mr. *y* the Liberal**
- **A pornographic brochure**
  - **3 social states: *a, b, c***
    - ***a*: *x* reads**
    - ***b*: *y* reads**
    - ***c*: nobody reads**
  - **Preferences**
    - ***x*: *c P a P b***
    - ***y*: *a P b P c***

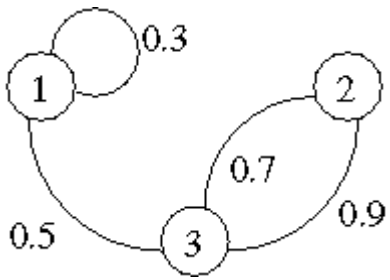


# Extensions

- **Characterization results**
  - ⇨ **find a list of properties that a method is the only one to satisfy simultaneously**
    - **Borda**
    - **Copeland**
    - **Plurality**
  - ⇨ **Neutral, anonymous and separable method are of Borda-type (H.P. Young 1975)**
- **Analysis results**
  - ⇨ **find a list of desirable properties**
  - ⇨ **fill up the methods $\times$ properties table**

# Conclusion

- **Little hope to find THE method**
- **Immense literature: DO NOT re-invent the wheel**
  - ⇨ **these problems and results generalize easily to other settings**
    - **fuzzy preference**
    - **states of nature**
    - **etc.**



# Other (important!) aspects

- **Institutional setting**
  - ⇨ **Control of political action, functioning of parties, financing campaigns, etc.**
- **Welfare judgments**
  - ⇨ **A majority can decide to put all taxes on a minority!**
- **Direct vs. indirect democracy**
  - ⇨ **Ostrogorski paradox**
  - ⇨ **Referendum paradox**
- **Electoral platforms**
  - ⇨ **bundle of issues, vote trading, logrolling**
- **Paradox of voting (why vote?)**

# Paradox of voting

- **Voting has a cost**
  - ⇨ **“I have to go to the polling station”**
  - ⇨ **“I had rather go fishing”**
- **The probability that my vote will change the results is nil**
- **Why should I bother?**

- **Economic explanations**
- **Sociological explanations**

# Ostrogorski's Paradox

- **Problem: *Representative democracy***
  - ⇨ **Referendum *vs.* Assembly**
- **You vote for a *party* that has a position on several *issues* (economic, social, international, etc.)**
- **No party can be expected to represent your opinion on every issue**
- **Why vote for *parties* instead of *issues*?**

# Ostrogorski's Paradox

- **5 *voters*, 2 *parties* (X and Y), 3 *issues***

| | issue 1 | issue 2 | issue 3 |
|---|---|---|---|
| voter 1 | X | X | Y |
| voter 2 | Y | Y | Y |
| voter 3 | Y | X | X |
| voter 4 | X | Y | X |
| voter 5 | X | Y | X |

- **On *issue* 1, *voter* 1 agrees with *party* X**

# Ostrogorski's Paradox

| | issue 1 | issue 2 | issue 3 | Party supported |
|---|---|---|---|---|
| voter 1 | X | Y | Y | Y |
| voter 2 | Y | X | Y | Y |
| voter 3 | Y | Y | X | Y |
| voter 4 | X | X | X | X |
| voter 5 | X | X | X | X |

- **If each voter vote for the party with which he/she agrees on a majority of issues, Y *wins***
- **yet, the *losing* party X represents the views of a *majority* of voters on *every* issue!**

# Anscombe's paradox

| | issue 1 | issue 2 | issue 3 | |
|---|---|---|---|---|
| voter 1 | X | X | Y | *minority* |
| voter 2 | Y | Y | Y | *minority* |
| voter 3 | Y | X | X | *minority* |
| voter 4 | X | Y | X | *majority* |
| voter 5 | X | Y | X | *majority* |
| result | *X* | *Y* | *X* | |

- **Voting on issues by simple majority**
- **A *majority* of voters may be frustrated on a *majority* of issues!**

# Referendum Paradox

- **Direct democracy (referendum) and indirect democracy (via MP) are indeed different**

  **… even when each MP votes according to the opinion of the majority of his/her electors**

| | MP 1 | … | MP 167 | MP 168 | … | MP 200 |
|---|---|---|---|---|---|---|
| Yes | 7 000 | … | 7 000 | 15 000 | … | 15 000 |
| No | 8 000 | … | 8 000 | 0 | … | 0 |

- **In the parliament *No* wins (167/200 = 83%)**
- **In a referendum *Yes* wins (55%)**

$$167 \times 7\,000 + 33 \times 15\,000 = 1\,664\,000$$

$$167 \times 8\,000 = 1\,336\,000$$