Analyzing the correspondence between non-strict and strict outranking relations

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Outranking relations

Concordance / non-discordance

- alternative x is preferable to alternative y if

concordance condition the coalition of attributes supporting this assertion is "more important" than the coalition of attributes opposing it

non-discordance condition there is no attribute that strongly opposes this assertion

ELECTRE "preferable" means "at least as good as"

TACTIC "preferable" means "strictly better than"

Framework

Classical conjoint measurement setting

- $N = \{1, 2, \ldots, n\}$: set of attributes
- $X = \prod_{i=1}^{n} X_i$ with $n \geq 2$: set of alternatives
- $x = (x_1, x_2, \ldots, x_n) \in X$

- $\mathcal{S}$ reflexive binary relation on X interpreted as an "at least as good as" relation between alternatives

Definition of reflexive concordance relations

Reflexive Concordance Relations (RCR)

$$x \mathcal{S} y \Leftrightarrow S(x,y) \trianglerighteq S(y,x)$$

with $S(x,y) = \{i \in N : x_i \; S_i \; y_i\}$ and

- S_i: complete binary relation X_i
- $\trianglerighteq$: binary relation between subsets of attributes having N for union that is increasing w.r.t. inclusion

$$A \trianglerighteq B, C \supseteq A \text{ and } B \supseteq D \Rightarrow C \trianglerighteq D$$

  and such that $N \trianglerighteq N$

- S_i: preference relation on attribute $i \in N$
  - asymmetric part of S_i: S_i^a
  - symmetric part of S_i: S_i^s
- $\trianglerighteq$: importance relation between coalitions of attributes
  - because S_i is complete, $S(x,y) \cup S(y,x) = N$, for all $x, y \in X$

Definition of reflexive outranking relations

Reflexive Outranking Relations (ROR)

$$x \; \mathcal{S} \; y \Leftrightarrow [S(x,y) \trianglerighteq S(y,x) \text{ and } V(y,x) = \varnothing]$$

with $S(x,y) = \{i \in N : x_i \; S_i \; y_i\}$ and $V(y,x) = \{i \in N : y_i \; V_i \; x_i\}$

- S_i: complete binary relation X_i
- V_i: binary relation on X_i such that $V_i \subseteq S_i^a$
- $\trianglerighteq$: binary relation between subsets of attributes having N for union that is increasing w.r.t. inclusion

$$A \trianglerighteq B, C \supseteq A \text{ and } B \supseteq D \Rightarrow C \trianglerighteq D$$

- V_i: "far better than" relation on attribute $i \in N$

ELECTRE I

ELECTRE I

$$x \; \mathcal{S} \; y \Leftrightarrow \begin{cases} \dfrac{\sum_{i \in S(x,y)} w_i}{\sum_{j \in N} w_j} \geq s \\ \text{and} \\ V(y,x) = \varnothing \end{cases}$$

with:

- $s \in [0.5, 1]$: concordance threshold
- S_i: semi order (complete, Ferrers and semitransitive)
- $V_i \subseteq S_i^a$: strict semiorder (asymmetric, Ferrers and semitransitive)

Conjoint measurement framework

Model (M)

$$x \mathrel{\mathcal{S}} y \Leftrightarrow F(p_1(x_1, y_1), p_2(x_2, y_2), \ldots, p_n(x_n, y_n)) \geq 0 \qquad \text{(M)}$$

with

- p_i *skew symmetric* ($p_i(x_i, y_i) = -p_i(y_i, x_i)$)
- F *nondecreasing* in all its arguments
- $F(\mathbf{0}) \geq 0$

Interpretation

- p_i measures *preference differences* between levels on attribute $i \in N$
- F synthesizes these preference differences

Axiomatic analysis

Model (M)

- two conditions guaranteeing that preference differences on each attribute are well behaved

RCR

- axioms for model (M)
- two additional axioms guaranteeing that each p_i takes at most three distinct values: $+k_i, 0, -k_i$
- upper coarseness, lower coarseness

ROR

- axioms for model (M)
- two additional axioms guaranteeing that each p_i takes at most five distinct values: $+v_i, +k_i, 0, -k_i, -v_i$
- upper coarseness, weak lower coarseness

Asymmetric outranking relations

- same principles, except that "preferable" means "strictly preferred" instead of "at least as good as"

TACTIC, Vansnick, 1986

$$x \, \mathcal{P} \, y \Leftrightarrow \begin{cases} \sum_{i \in P(x,y)} w_i > \sum_{j \in P(y,x)} w_j + \varepsilon \\ \text{and} \\ W(y,x) = \varnothing \end{cases}$$

with:

- $P(x,y) = \{i \in N : x_i \; P_i \; y_i\}$
- $\varepsilon \geq 0$: threshold
- P_i: strict semi order (asymmetric, Ferrers and semitransitive)
- $W_i \subseteq P_i$: strict semiorder (asymmetric, Ferrers and semitransitive)

Definition of asymmetric concordance relations

Asymmetric Concordance relations (ACR)

$$x \mathcal{P} y \Leftrightarrow P(x,y) \Supset P(y,x)$$

with $P(x,y) = \{i \in N : x_i \; P_i \; y_i\}$ and

- P_i: asymmetric binary relation X_i
- $\Supset$: binary relation between disjoint subsets of attributes that is increasing w.r.t. inclusion

$$A \Supset B, C \supseteq A \text{ and } B \supseteq D \Rightarrow C \Supset D$$

and such that $Not[\varnothing \Supset \varnothing]$

Coduality

Coduality

- $\mathcal{T}$ is a binary relation on A

$$a \; \mathcal{T}^{cd} \; b \Leftrightarrow Not[\, b \; \mathcal{T} \; a \,]$$

RCR and ACR

- complete RCR and ACR correspond through coduality

Coduality

$$x\ \mathcal{P}\ y \Leftrightarrow P(x,y) \supseteq P(y,x)$$

- $P_i{}^{cd} = S_i$ (S_i is complete)
- $A \trianglerighteq B \Leftrightarrow Not[\,N \setminus B \supseteq N \setminus A\,]$

$$\begin{aligned} Not[\,y\ \mathcal{P}\ x\,] &\Leftrightarrow Not[\,P(x,y) \supseteq P(y,x)\,] \\ &\Leftrightarrow Not[\,N \setminus S(y,x) \supseteq N \setminus S(x,y)\,] \\ &\Leftrightarrow S(x,y) \trianglerighteq S(y,x) \\ &\Leftrightarrow x\ \mathcal{S}\ y \end{aligned}$$

- $\mathcal{S}$ codual of $\mathcal{P}$ is complete
- conversely the codual of the completion of a RCR is an ACR

Definition of asymmetric outranking relations

Asymmetric Outranking Relations (AOR)

$$x \mathcal{P} y \Leftrightarrow [P(x,y) \Supset P(y,x) \text{ and } W(y,x) = \varnothing]$$

with $P(x,y) = \{i \in N : x_i \ P_i \ y_i\}$ and and $W(y,x) = \{i \in N : y_i \ W_i \ x_i\}$

- P_i: asymmetric binary relation X_i
- W_i: binary relation on X_i such that $W_i \subseteq P_i$
- $\Supset$: binary relation between disjoint subsets of attributes that is increasing w.r.t. inclusion

$$A \Supset B, C \supseteq A \text{ and } B \supseteq D \Rightarrow C \Supset D$$

and such that $Not[\varnothing \Supset \varnothing]$

Results

$$\begin{aligned} Not[\, y\ \mathcal{P}\ x\,] &\Leftrightarrow Not[\, P(x,y) \supseteq P(y,x) \text{ and } W(x,y) = \varnothing\,] \\ &\Leftrightarrow Not[\, N \setminus S(y,x) \supseteq N \setminus S(x,y)\,] \text{ or } W(x,y) \neq \varnothing \\ &\Leftrightarrow S(x,y) \trianglerighteq S(y,x) \text{ or } W(x,y) \neq \varnothing \end{aligned}$$

Outranking relations with bonus

$$x\ \mathcal{S}\ y \Leftrightarrow S(x,y) \trianglerighteq S(y,x) \text{ or } W(x,y) \neq \varnothing$$

- we have a characterization of AOR
- we have a characterization of outranking relations with bonus through coduality

Asymmetric part of a ROR

$$x \; \mathcal{S}^a \; y \Leftrightarrow [x \; \mathcal{S} \; y \text{ and } Not[\, y \; \mathcal{S} \; x \,]$$

$$\Leftrightarrow \begin{cases} [S(x,y) \trianglerighteq S(y,x) \text{ and } V(y,x) = \varnothing] \\ \text{and} \\ [Not[\, S(y,x) \trianglerighteq S(x,y) \,] \text{ or } V(x,y) \neq \varnothing] \end{cases}$$

$$x \succsim^c y \text{ iff } S(x,y) \trianglerighteq S(y,x)$$

$$x \; \mathcal{S}^a \; y \Leftrightarrow \begin{cases} x \succ^c y \text{ and } V(y,x) = \varnothing \\ \text{or} \\ x \sim^c y,\, V(y,x) = \varnothing \text{ and } V(x,y) \neq \varnothing. \end{cases}$$

- double rôle of the veto relation: veto and bonus
- is used in some outranking methods (ELECTRE TRI)
- is not an AOR
- is not an ROR
- an axiomatic characterization is available

References

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