

A Manichean theory of Subjective Expected Utility

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Introduction

Decision making under uncertainty

- I have to make a **decision**
- the consequences of my decisions depend on **Nature's** decisions
- I have to decide **before** Nature decides

Three main Ingredients

- **States**: what Nature can decide, $N = \{1, 2, \dots, n\}$
- **Consequences**: what will ultimately happen to me, $\Gamma = \{\alpha, \beta, \gamma, \dots\}$
- **Acts**: mapping from states to consequences, $\mathcal{X} = \Gamma^N = \Gamma^n = \{a, b, \dots\}$

Example

Planning a reception

- states: {Sunny, Rainy, Cloudy }
- consequences: { ++, +, 0, -, -- }
- acts:

	Sunny	Rainy	Cloudy
Outdoor	++	--	0
Indoor	-	+	+
Tent	+	-	0

Subjective Expected Utility

Model

$$a \succsim b \Leftrightarrow SEU(a) = \sum_{i=1}^n p_i u(a_i) \geq SEU(b) = \sum_{i=1}^n p_i u(b_i)$$

- $SEU(a)$: sum over all states of the **utility** of the consequence in state i ($u(a_i)$) weighted by the **subjective probability** of the state i (p_i)
- \succsim : preference relation on the set of acts

SEU

- **simplicity**: separation of tastes (u) and beliefs (p_i)
- **analytical tractability**: linear model underlying **Decision Analysis**
- **many useful tools**: decision trees, influence diagrams, EVPI, etc.
- **normative appeal**: behavioral foundations, dynamic consistency
- **descriptive limitations**: Ellsberg & Allais

Behavioral foundations of SEU

Many approaches

- Shapiro (1979)
 - Savage (1954)
 - Wakker (1989)
 - Anscombe & Aumann (1963)
- all approaches use a **preference relation** on the set of all acts

Research Question

- deriving SEU from different and weaker premises

Manichean premises

- ordered partition: **attractive** and **unattractive** acts

Motivations

Theory

- a preference relation \succsim induces many ordered partitions
 - $\mathcal{A}^x = \{a \in \mathcal{X} : a \succ x\}$, $\mathcal{U}^x = \{a \in \mathcal{X} : x \succsim a\}$
- our premises are **weaker** than the classical ones

Practice

- the **status quo** plays an important rôle in comparing acts
- comparing to the status quo induces an ordered partition

Framework

Classical setting

- $N = \{1, 2, \dots, n\}$: set of **states**
- $\Gamma = \{\alpha, \beta, \gamma \dots\}$: set of **consequences**
- $\mathcal{X} = \Gamma^N = \Gamma^n = \{a, b, \dots\}$: set of **acts**
- notation: $(a_E, b_{-E}), (\alpha_E, b_{-E}), (a_i, b_{-i}), (\alpha_{ij}, b_{-ij})$

Primitives: ordered partition $\langle \mathcal{A}, \mathcal{U} \rangle$ of \mathcal{X}

- $\mathcal{A} \subseteq \mathcal{X}, \mathcal{U} \subseteq \mathcal{X}, \mathcal{A} \cup \mathcal{U} = \mathcal{X}, \mathcal{A} \cap \mathcal{U} = \emptyset$
- \mathcal{A} : set of acts that are “Atractive”
- \mathcal{U} : set of acts that are “Unattractive”

Useful interpretation

- position of acts vis-à-vis a **status quo**
- acts in \mathcal{A} are strictly better than the status quo
- all acts in \mathcal{A} (\mathcal{U}) are **not** equivalent

Definitions

Influence

- state $i \in N$ has **influence** if there are $\alpha, \beta \in \Gamma$ and $a \in \mathcal{X}$ such that $(\alpha_i, a_{-i}) \in \mathcal{A}$ and $(\beta_i, a_{-i}) \in \mathcal{U}$

Structural Assumption

There are at least **three** states
All states have **influence**

- the case of two states is quite different
- price to pay for using weak premises

Model

SEU

$$a \in \mathcal{A} \Leftrightarrow \sum_{i=1}^n p_i u(a_i) > 0$$

Interpretation

- $a_i \in \Gamma$ consequence of act $a \in \mathcal{X}$ if state $i \in N$ obtains
- u is a real-valued function on Γ
- p_i is the subjective probability of $i \in N$
 - $p_i \geq 0$, $\sum_{i=1}^n p_i = 1$
- the choice of the value “0” for the threshold is arbitrary
- influence of state i implies $p_i > 0$

Tradeoff Consistency

Tradeoff Consistency

$$\left. \begin{array}{l}
 (\alpha_j, \lambda_k, a_{-jk}) \in \mathcal{A} \quad \text{and} \\
 (\gamma_j, \mu_k, b_{-jk}) \in \mathcal{A} \quad \text{and} \\
 (\delta_i, \tau_k, c_{-ik}) \in \mathcal{A} \quad \text{and} \\
 (\beta_i, \xi_k, d_{-ik}) \in \mathcal{A}
 \end{array} \right\} \Rightarrow \left\{ \begin{array}{l}
 (\beta_j, \mu_k, a_{-jk}) \in \mathcal{A} \quad \text{or} \\
 (\delta_j, \lambda_k, b_{-jk}) \in \mathcal{A} \quad \text{or} \\
 (\gamma_i, \xi_k, c_{-ik}) \in \mathcal{A} \quad \text{or} \\
 (\alpha_i, \tau_k, d_{-ik}) \in \mathcal{A}
 \end{array} \right.$$

Interpretation

Consistent tradeoff between consequences that are independent from states:

$$p_j u(\alpha) - p_j u(\beta) > p_k u(\mu) - p_k u(\lambda)$$

$$p_k u(\mu) - p_k u(\lambda) > p_j u(\delta) - p_j u(\gamma)$$

$$p_i u(\delta) - p_i u(\gamma) > p_k u(\xi) - p_k u(\tau)$$

$$p_k u(\xi) - p_k u(\tau) > p_i u(\alpha) - p_i u(\beta)$$

Necessary for SEU

Inspired from [Wakker](#) (1989)

r-1-Linearity

r-1-Linearity

$$\left. \begin{array}{l} (\alpha_i, a_{-i}) \in \mathcal{A} \\ \text{and} \\ (\beta_j, b_{-j}) \in \mathcal{A} \end{array} \right\} \Rightarrow \left\{ \begin{array}{l} (\beta_i, a_{-i}) \in \mathcal{A} \\ \text{or} \\ (\alpha_j, b_{-j}) \in \mathcal{A} \end{array} \right.$$

Interpretation

$\alpha \succsim^{\mathcal{A}} \beta \Leftrightarrow [(\beta_i, a_{-i}) \in \mathcal{A} \Rightarrow (\alpha_i, a_{-i}) \in \mathcal{A}, \text{ for all } i \in N \text{ and all } a \in \mathcal{X}]$

- $\succsim^{\mathcal{A}}$ is always reflexive and transitive
- the above axiom implies that it is complete: consistent ordering of consequences across states
- **necessary** for SEU
- **implied** by Tradeoff Consistency

Connectedness

Remarks

- under r-1-Linearity, the set Γ is weakly ordered by $\succsim^{\mathcal{A}}$
- we use the order topology induced by $\succsim^{\mathcal{A}}$ on Γ
- we use the product topology on $\mathcal{X} = \Gamma^n$

Connectedness

When $\succsim^{\mathcal{A}}$ is a weak order, the set Γ is **connected** in the order topology generated by $\succsim^{\mathcal{A}}$

Interpretation

- the set Γ is “rich”
- $\alpha \succ^{\mathcal{A}} \beta \Rightarrow \alpha \succ^{\mathcal{A}} \gamma \succ^{\mathcal{A}} \beta$, for some $\gamma \in \Gamma$
- **not** necessary for SEU

Openness

Openness

The set \mathcal{A} is **open** in the product topology on \mathcal{X}

Interpretation

- **necessary** for SEU whenever u is continuous
 - if $a \in \mathcal{A}$, there is a neighborhood of a included in \mathcal{A}
- implies that \mathcal{U} is closed in the product topology on \mathcal{X}

Unboundedness*

Unboundedness*

For all $i \in N$ and all $a \in \mathcal{X}$, $(\alpha_i, a_{-i}) \in \mathcal{A}$ and $(\beta_i, a_{-i}) \in \mathcal{U}$, for some $\alpha, \beta \in \Gamma$

Interpretation

- strong axiom
- **not** necessary for SEU
- implies that the image of Γ by u is \mathbb{R}
- only introduced to keep things simple

New behavioral foundations for SEU

Theorem, B & Marchant, 2010

Suppose that $\langle \mathcal{A}, \mathcal{U} \rangle$ is an ordered partition on \mathcal{X} such that the Structural Assumption holds.

Suppose that $\langle \mathcal{A}, \mathcal{U} \rangle$ satisfies Tradeoff Consistency, Connectedness, Openness, and Unboundedness*.

Then there are:

- a continuous real-valued function u on Γ such that $u(\Gamma) = \mathbb{R}$
- n strictly positive numbers p_1, p_2, \dots, p_n adding up to 1

such that SEU holds.

The numbers p_1, p_2, \dots, p_n are unique. The function u is unique up to a multiplication by a strictly positive constant.

Remarks

- full characterization of SEU when $n \geq 3$
- tight uniqueness properties

Summary

- SEU with tight uniqueness properties derived from Manichean premises
- Manichean premises are observable
- reasonably simple conditions that can be tested in experiments

Further work

Theoretical side

- get rid of Unboundedness*
 - technical but important
- use similar analysis for NEU models (CEU, CPT)
 - likely to be difficult

Experimental side

- test Tradeoff Consistency
- are SEU violations less severe with weaker premises?
 - unlikely!
 - the paper gives variants of Allais' problem and Ellsberg's problem adapted to our setting

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