A Manichean theory of Subjective Expected Utility

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Introduction

Decision making under uncertainty

- I have to make a decision
- the consequences of my decisions depend on Nature's decisions
- I have to decide before Nature decides

Three main Ingredients

- States: what Nature can decide, $N = \{1, 2, \ldots, n\}$
- Consequences: what will ultimately happen to me, $\Gamma = \{\alpha, \beta, \gamma \ldots\}$
- Acts: mapping from states to consequences, $\mathcal{X} = \Gamma^N = \Gamma^n = \{a, b, \ldots\}$

Example

Planning a reception

- states: {Sunny, Rainy, Cloudy }
- consequences: $\{ ++, +, 0, -, -- \}$
- acts:

	Sunny	Rainy	Cloudy
Outdoor	$++$	$--$	0
Indoor	$-$	$+$	$+$
Tent	$+$	$-$	0

Subjective Expected Utility

Model

$$a \succsim b \Leftrightarrow SEU(a) = \sum_{i=1}^{n} p_i u(a_i) \geq SEU(b) = \sum_{i=1}^{n} p_i u(b_i)$$

- $SEU(a)$: sum over all states of the utility of the consequence in state i ($u(a_i)$) weighted by the subjective probability of the state i (p_i)
- $\succsim$: preference relation on the set of acts

SEU

- simplicity: separation of tastes (u) and beliefs (p_i)
- analytical tractability: linear model underlying Decision Analysis
- many useful tools: decision trees, influence diagrams, EVPI, etc.
- normative appeal: behavioral foundations, dynamic consistency
- descriptive limitations: Ellsberg & Allais

Behavioral foundations of SEU

Many approaches

- Shapiro (1979)
- Savage (1954)
- Wakker (1989)
- Anscombe & Aumann (1963)

- all approaches use a preference relation on the set of all acts

Research Question

- deriving SEU from different and weaker premises

Manichean premises

- ordered partition: attractive and unattractive acts

Motivations

Theory

- a preference relation $\succsim$ induces many ordered partitions
  - $\mathscr{A}^x = \{a \in \mathcal{X} : a \succ x\}$, $\mathscr{U}^x = \{a \in \mathcal{X} : x \succsim a\}$
- our premises are weaker than the classical ones

Practice

- the status quo plays an important rôle in comparing acts
- comparing to the status quo induces an ordered partition

Framework

Classical setting

- $N = \{1, 2, \ldots, n\}$: set of states
- $\Gamma = \{\alpha, \beta, \gamma \ldots\}$: set of consequences
- $\mathcal{X} = \Gamma^N = \Gamma^n = \{a, b, \ldots\}$: set of acts
- notation: (a_E, b_{-E}), (α_E, b_{-E}), (a_i, b_{-i}), (α_{ij}, b_{-ij})

Primitives: ordered partition $\langle \mathscr{A}, \mathscr{U} \rangle$ of $\mathcal{X}$

- $\mathscr{A} \subseteq \mathcal{X}$, $\mathscr{U} \subseteq \mathcal{X}$, $\mathscr{A} \cup \mathscr{U} = \mathcal{X}$, $\mathscr{A} \cap \mathscr{U} = \varnothing$
- $\mathscr{A}$: set of acts that are "Attractive"
- $\mathscr{U}$: set of acts that are "Unattractive"

Useful interpretation

- position of acts vis-à-vis a status quo
- acts in $\mathscr{A}$ are strictly better than the status quo
- all acts in $\mathscr{A}$ ($\mathscr{U}$) are not equivalent

Definitions

Influence

- state $i \in N$ has influence if there are $\alpha, \beta \in \Gamma$ and $a \in \mathcal{X}$ such that $(\alpha_i, a_{-i}) \in \mathscr{A}$ and $(\beta_i, a_{-i}) \in \mathscr{U}$

Structural Assumption

There are at least three states
All states have influence

- the case of two states is quite different
- price to pay for using weak premises

Model

SEU

$$a \in \mathscr{A} \Leftrightarrow \sum_{i=1}^{n} p_i u(a_i) > 0$$

Interpretation

- $a_i \in \Gamma$ consequence of act $a \in \mathcal{X}$ if state $i \in N$ obtains
- u is a real-valued function on Γ
- p_i is the subjective probability of $i \in N$
  - $p_i \geq 0$, $\sum_{i=1}^{n} p_i = 1$
- the choice of the value “0” for the threshold is arbitrary
- influence of state i implies $p_i > 0$

Tradeoff Consistency

Tradeoff Consistency

$$\left.\begin{array}{rl} (\alpha_j, \lambda_k, a_{-jk}) \in \mathscr{A} & \text{and} \\ (\gamma_j, \mu_k, b_{-jk}) \in \mathscr{A} & \text{and} \\ (\delta_i, \tau_k, c_{-ik}) \in \mathscr{A} & \text{and} \\ (\beta_i, \xi_k, d_{-ik}) \in \mathscr{A} & \end{array}\right\} \Rightarrow \left\{\begin{array}{rl} (\beta_j, \mu_k, a_{-jk}) \in \mathscr{A} & \text{or} \\ (\delta_j, \lambda_k, b_{-jk}) \in \mathscr{A} & \text{or} \\ (\gamma_i, \xi_k, c_{-ik}) \in \mathscr{A} & \text{or} \\ (\alpha_i, \tau_k, d_{-ik}) \in \mathscr{A} & \end{array}\right.$$

Interpretation

Consistent tradeoff between consequences that are independent from states:

$$p_j u(\alpha) - p_j u(\beta) > p_k u(\mu) - p_k u(\lambda)$$
$$p_k u(\mu) - p_k u(\lambda) > p_j u(\delta) - p_j u(\gamma)$$
$$p_i u(\delta) - p_i u(\gamma) > p_k u(\xi) - p_k u(\tau)$$
$$p_k u(\xi) - p_k u(\tau) > p_i u(\alpha) - p_i u(\beta)$$

Necessary for SEU

Inspired from Wakker (1989)

r-1-Linearity

r-1-Linearity

$$\left.\begin{array}{c}(\alpha_i, a_{-i}) \in \mathscr{A} \\ \text{and} \\ (\beta_j, b_{-j}) \in \mathscr{A}\end{array}\right\} \Rightarrow \left\{\begin{array}{c}(\beta_i, a_{-i}) \in \mathscr{A} \\ \text{or} \\ (\alpha_j, b_{-j}) \in \mathscr{A}\end{array}\right.$$

Interpretation

$$\alpha \succsim^{\mathscr{A}} \beta \Leftrightarrow [(\beta_i, a_{-i}) \in \mathscr{A} \Rightarrow (\alpha_i, a_{-i}) \in \mathscr{A}, \text{ for all } i \in N \text{ and all } a \in \mathcal{X}]$$

- $\succsim^{\mathscr{A}}$ is always reflexive and transitive
- the above axiom implies that it is complete: consistent ordering of consequences across states
- necessary for SEU
- implied by Tradeoff Consistency

Connectedness

Remarks

- under r-1-Linearity, the set Γ is weakly ordered by $\succsim^{\mathscr{A}}$
- we use the order topology induced by $\succsim^{\mathscr{A}}$ on Γ
- we use the product topology on $\mathcal{X} = \Gamma^n$

Connectedness

When $\succsim^{\mathscr{A}}$ is a weak order, the set Γ is connected in the order topology generated by $\succsim^{\mathscr{A}}$

Interpretation

- the set Γ is "rich"
- $\alpha \succ^{\mathscr{A}} \beta \Rightarrow \alpha \succ^{\mathscr{A}} \gamma \succ^{\mathscr{A}} \beta$, for some $\gamma \in \Gamma$
- not necessary for SEU

Openness

Openness

The set $\mathscr{A}$ is open in the product topology on $\mathcal{X}$

Interpretation

- necessary for SEU whenever u is continuous
  - if $a \in \mathscr{A}$, there is a neighborhood of a included in $\mathscr{A}$
- implies that $\mathscr{U}$ is closed in the product topology on $\mathcal{X}$

Unboundedness*

Unboundedness*

For all $i \in N$ and all $a \in \mathcal{X}$, $(\alpha_i, a_{-i}) \in \mathscr{A}$ and $(\beta_i, a_{-i}) \in \mathscr{U}$, for some $\alpha, \beta \in \Gamma$

Interpretation

- strong axiom
- not necessary for SEU
- implies that the image of Γ by u is $\mathbb{R}$
- only introduced to keep things simple

New behavioral foundations for SEU

Theorem, B & Marchant, 2010

Suppose that $\langle \mathscr{A}, \mathscr{U} \rangle$ is an ordered partition on $\mathcal{X}$ such that the Structural Assumption holds.

Suppose that $\langle \mathscr{A}, \mathscr{U} \rangle$ satisfies Tradeoff Consistency, Connectedness, Openness, and Unboundedness*.

Then there are:

- a continuous real-valued function u on Γ such that $u(\Gamma) = \mathbb{R}$
- n strictly positive numbers $p_1, p_2, \ldots p_n$ adding up to 1

such that SEU holds.

The numbers $p_1, p_2, \ldots, p_n$ are unique. The function u is unique up to a multiplication by a strictly positive constant.

Remarks

- full characterization of SEU when $n \geq 3$
- tight uniqueness properties

Summary

- SEU with tight uniqueness properties derived from Manichean premises
- Manichean premises are observable
- reasonably simple conditions that can be tested in experiments

Further work

Theoretical side

- get rid of Unboundedness*
  - technical but important
- use similar analysis for NEU models (CEU, CPT)
  - likely to be difficult

Experimental side

- test Tradeoff Consistency
- are SEU violations less severe with weaker premises?
  - unlikely!
  - the paper gives variants of Allais' problem and Ellsberg's problem adapted to our setting

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