Multiattribute preference models with reference points

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Workshop OSGD
November 2012

Introduction

	cost	# of days	travel time	category of hotel	distance to beach	Wifi	cultural interest
A	200 €	15	12 h	***	45 km	Y	++
B	425 €	18	15 h	****	0 km	N	−−
C	150 €	4	7 h	**	250 km	N	+
D	300 €	5	10 h	***	5 km	Y	−

Decision making with multiple attributes

Two main trends

axiomatic trend: multiattribute value theory (and extensions)

pragmatic trend: outranking methods (and other ordinal approaches)

Multiattribute value theory

Multiattribute value theory

- additive value functions

$$x \succsim y \Leftrightarrow \sum_{i=1}^{n} v_i(x_i) \geq \sum_{i=1}^{n} v_i(y_i)$$

x, y : alternatives

x_i : evaluation of alternative x on attribute i

$v_i(x_i)$: real number

Remarks

- axiomatic foundations
- elicitation procedures

- precise analysis of tradeoffs
- independence is required

Ordinal approaches

Ordinal approaches

- aggregate ordinal information on each attribute

Example: outranking methods

$$x \succsim y \Leftrightarrow S(x, y) \trianglerighteq' S(y, x)$$

x is at least as good as y iff the attributes supporting this proposition are "more important" than the attributes supporting the reverse conclusion

Remarks

- simple and ordinal

- transitivity: Arrow's theorem, need for exploitation techniques
- axiomatic foundations: remain weak

Reference points in multiattribute decision making

Using reference points

- use special levels reference points
- compare alternatives wrt to these reference points only
  - remaining ordinal
  - alleviating transitivity problems (as for sorting problems)

Antoine Rolland & Patrice Perny

Objectives for today

- axiomatic analysis of models with reference points
- using a classical conjoint measurement framework
  - only primitive: $\succsim$ on X
  - axioms entirely phrased in terms of $\succsim$
- compare these models to existing ones

Related Literature

Doctoral dissertation of A. Rolland and several of his papers

- he studies many particular cases of models with reference points
- he gives many examples showing their interest
- he analyzes them using a different perspective
  - axiomatic analysis often supposing that the reference points are known beforehand
  - axioms not always phrased in terms of $\succsim$

Conjoint measurement framework (B. & Pirlot)

Additive value functions

$$x \succsim y \Leftrightarrow \sum_{i=1}^{n} v_i(x_i) \geq \sum_{i=1}^{n} v_i(y_i)$$

Models using traces on levels (B. & Pirlot, 2004)

$$x \succsim y \Leftrightarrow F(v_1(x_1), v_2(x_2), \ldots, v_n(x_n), v_1(y_1), v_2(y_2), \ldots, v_n(y_n)) \geq 0$$

F increasing (nondecreasing) in its first n arguments and decreasing (nonincreasing) in its last n arguments

Models using traces on differences (B. & Pirlot, 2002)

$$x \succsim y \Leftrightarrow G(p_1(x_1, y_1), p_2(x_2, y_2), \ldots, p_n(x_n, y_n)) \geq 0$$

G increasing (nondecreasing) in all its arguments, possibly odd
p_i possibly skew symmetric

Rough models

Outranking relations

- models using traces on differences

$$x \succsim y \Leftrightarrow G(p_1(x_1, y_1), p_2(x_2, y_2), \ldots, p_n(x_n, y_n)) \geq 0$$

- with rough traces on differences (B. & Pirlot, 2005, 2007)
  - the functions p_i take a limited number of values

Models using reference points

- models using traces on levels

$$x \succsim y \Leftrightarrow F(v_1(x_1), v_2(x_2), \ldots, v_n(x_n), v_1(y_1), v_2(y_2), \ldots, v_n(y_n)) \geq 0$$

- with rough traces on levels (B. & Marchant 2012)
  - the functions v_i take a limited number of values

Conjoint measurement

Classical setting

- $N = \{1, 2, \dots, n\}$ set of attributes
- X_i: set of possible levels on the ith attribute
- $X = \prod_{i=1}^{n} X_i$: set of all conceivable alternatives
  - X include the alternatives under study… and many others

- $J \subseteq N$: subset of attributes
- $X_J = \prod_{j \in J} X_j$, $X_{-J} = \prod_{j \notin J} X_j$
- $(x_J, y_{-J}) \in X$
- $(x_i, y_{-i}) \in X$

Primitives

- $\succsim$: binary relation on X: "at least as good as"
  - $x \succ y \Leftrightarrow x \succsim y$ and $Not[y \succsim x]$
  - $x \sim y \Leftrightarrow x \succsim y$ and $y \succsim x$

Marginal preference and independence

Independence

- $J \subseteq N$ is independent if

$$(x_J, z_{-J}) \succsim (y_J, z_{-J}), \text{ for some } z_{-J} \in X_{-J} \Rightarrow$$
$$(x_J, w_{-J}) \succsim (y_J, w_{-J}), \text{ for all } w_{-J} \in X_{-J}$$

- common levels on attributes other than J do not affect preference

- $\succsim$ is independent if, for all $J \subseteq N$, J is independent

Influence

Influential attribute

- attribute $i \in N$ is influential for $\succsim$ if there are $x_i, y_i, z_i, w_i \in X_i$ and $a_{-i}, b_{-i} \in X_{-i}$ such that:

$$(x_i, a_{-i}) \succsim (y_i, b_{-i})$$
$$(z_i, a_{-i}) \not\succsim (w_i, b_{-i})$$

- attribute $i \in N$ is degenerate if it is not influential

Hypothesis

- a degenerate attribute has no impact on $\succsim$ and may be suppressed from N
- we suppose throughout that all attributes are influential

Preference models with a single reference point

Ingredients

- S_i: semiorder on X_i (complete, Ferrers and semitransitive relation)
- $\pi \in X$: reference point
- $\trianglerighteq$: importance relation on 2^N monotonic w.r.t. inclusion

$$[C \supseteq A, B \supseteq D, A \trianglerighteq B] \Rightarrow C \trianglerighteq D$$

- compare x and y to π only using ordinal considerations

$$\mathcal{S}(x) = \{i \in N : x_i \ S_i \ \pi_i\} \quad \mathcal{S}(y) = \{i \in N : y_i \ S_i \ \pi_i\}$$

$$x \succsim y \Leftrightarrow \mathcal{S}(x) \trianglerighteq \mathcal{S}(y) \qquad \text{(RSRP)}$$

$$\mathcal{A}_i = \{x_i \in X_i : x_i \ S_i \ \pi_i\}$$

- $x \succsim y$ if the set of attributes on which x is "acceptable" is "more important" than the set of attributes for which y is acceptable

Elementary properties

Lemma

- a RSRP has a unique representation in terms of the sets $\mathcal{A}_i$ and the relation $\trianglerighteq$
- for all $i \in N$, $\varnothing \subsetneq \mathcal{A}_i \subsetneq X_i$

Lemma

1. $\succsim$ is reflexive iff $\trianglerighteq$ is reflexive
2. $\succsim$ is complete iff $\trianglerighteq$ is complete
3. $\succsim$ is transitive iff $\trianglerighteq$ is transitive
4. $\succsim$ is independent iff, for all $i \in N$ and all $A, B \subseteq N$ such that $i \notin A$ and $i \notin B$, $A \trianglerighteq B$ iff $A \cup \{i\} \trianglerighteq B \cup \{i\}$

Example

	1	2	3
x	α_1	α_2	α_3
y	β_1	β_2	α_3
z	β_1	β_2	β_3
w	α_1	α_2	β_3

$$x \succ y \succ z \succ w$$

$$\pi = (\alpha_1, \beta_2, \alpha_3)$$

$$\alpha_1 \; P_1 \; \beta_1 \quad \beta_2 \; P_2 \; \alpha_2 \quad \alpha_3 \; P_3 \; \beta_3$$

$$\mathcal{A}_1 = \{\alpha_1\} \quad \mathcal{A}_2 = \{\beta_2\} \quad \mathcal{A}_3 = \{\alpha_3\}$$

$$\Big[\{1,2,3\} \triangleq {\color{red}\{1,3\}}\Big] \rhd \Big[{\color{red}\{2,3\}} \triangleq \{3\}\Big] \rhd \Big[\{1,2\} \triangleq {\color{red}\{2\}}\Big] \rhd \Big[{\color{red}\{1\}} \triangleq \varnothing\Big]$$

$$\mathcal{S}(x) = \{1,3\} \quad \mathcal{S}(y) = \{2,3\} \quad \mathcal{S}(z) = \{2\} \quad \mathcal{S}(w) = \{1\}$$

Traces on levels

Definition

$$x_i \succsim_i^+ y_i \Leftrightarrow \forall a, b \in X, [(y_i, a_{-i}) \succsim b \Rightarrow (x_i, a_{-i}) \succsim b]$$
$$x_i \succsim_i^- y_i \Leftrightarrow \forall a, b \in X, [a \succsim (x_i, b_{-i}) \Rightarrow a \succsim (y_i, b_{-i})]$$
$$x_i \succsim_i^\pm y_i \Leftrightarrow [x_i \succsim_i^+ y_i \text{ and } x_i \succsim_i^- y_i]$$

Remarks

- $\succsim_i^+$, $\succsim_i^-$ and $\succsim_i^\pm$ are always reflexive and transitive
- they may be incomplete

Properties of Traces

Properties

$$[x \succsim y, z_i \succsim_i^+ x_i] \Rightarrow (z_i, x_{-i}) \succsim y$$
$$[x \succsim y, y_i \succsim_i^- w_i] \Rightarrow x \succsim (w_i, y_{-i})$$
$$[z_i \succsim_i^\pm x_i, y_i \succsim_i^\pm w_i] \Rightarrow \begin{cases} x \succsim y \Rightarrow (z_i, x_{-i}) \succsim (w_i, y_{-i}) \\ \text{and} \\ x \succ y \Rightarrow (z_i, x_{-i}) \succ (w_i, y_{-i}) \end{cases}$$
$$[x_i \sim_i^\pm z_i, y_i \sim_i^\pm w_i \text{ for all } i \in N] \Rightarrow \begin{cases} x \succsim y \Leftrightarrow z \succsim w \\ \text{and} \\ x \succ y \Leftrightarrow z \succ w \end{cases}$$

Axioms for complete traces

$AC1$, $AC2$, $AC3$

$$AC1_i \text{ if } \left.\begin{array}{c}(x_i, a_{-i}) \succsim c \\ \text{and} \\ (y_i, b_{-i}) \succsim d\end{array}\right\} \Rightarrow \left\{\begin{array}{c}(y_i, a_{-i}) \succsim c \\ \text{or} \\ (x_i, b_{-i}) \succsim d\end{array}\right.$$

$$AC2_i \text{ if } \left.\begin{array}{c}c \succsim (y_i, a_{-i}) \\ \text{and} \\ d \succsim (x_i, b_{-i})\end{array}\right\} \Rightarrow \left\{\begin{array}{c}c \succsim (x_i, a_{-i}) \\ \text{or} \\ d \succsim (y_i, b_{-i})\end{array}\right.$$

$$AC3_i \text{ if } \left.\begin{array}{c}(x_i, a_{-i}) \succsim c \\ \text{and} \\ d \succsim (x_i, b_{-i})\end{array}\right\} \Rightarrow \left\{\begin{array}{c}(y_i, a_{-i}) \succsim c \\ \text{or} \\ d \succsim (y_i, b_{-i})\end{array}\right.$$

Properties

Lemma

1. $AC1_i \Leftrightarrow \succsim_i^+$ is complete
2. $AC2_i \Leftrightarrow \succsim_i^-$ is complete
3. $AC3_i \Leftrightarrow [Not[y_i \succsim_i^+ x_i] \Rightarrow x_i \succsim_i^- y_i]$
4. $[AC1_i,\ AC2_i$ and $AC3_i] \Leftrightarrow \succsim_i^\pm$ is complete
5. in the class of all semiorders on X, $AC1$, $AC2$ and $AC3$ are independent conditions

Numerical representation

Theorem (B. & Pirlot, 2004)

Suppose that, for all $i \in N$, the set $X_i/\sim_i^{\pm}$ is at most countably infinite. Then there are real-valued functions u_i on X_i and a real-valued function F on $[\prod_{i=1}^{n} u_i(X_i)]^2$ such that, for all $x, y \in X$:

$$x \succsim y \Leftrightarrow F([u_i(x_i)]; [u_i(y_i)]) \geq 0,$$

where F is increasing in its first n arguments and decreasing in its last n arguments
iff
$\succsim$ satisfies $AC1$, $AC2$ and $AC3$

Weak orders

Lemma

If $\succsim$ is a weak order then conditions $AC1_i$, $AC2_i$ and $AC3_i$ are equivalent

Theorem (B. & Pirlot, 2004)

Let $\succsim$ be a weak order on X such the set $X/\sim$ is at most countably infinite
Then there are real-valued functions u_i on X_i and a real-valued function U on $\prod_{i=1}^{n} u_i(X_i)$ such that, for all $x, y \in X$:

$$x \succsim y \Leftrightarrow U([u_i(x_i)]) \geq U([u_i(y_i)])$$

with U is nondecreasing in each of its arguments
iff
$\succsim$ satisfies $AC1$

Properties

Simple observation

If $\succsim$ is a RSRP, then, for all $i \in N$, the relation $\succsim_i^{\pm}$ is a weak order having two distinct equivalence classes

Consequence

A RSRP satisfies $AC1$, $AC2$ and $AC3$

Lemma

If $\succsim$ is such that, for all $i \in N$, the relation $\succsim_i^{\pm}$ is a weak order having two distinct equivalence classes, then $\succsim$ is a RSRP

Consequence

Devise axioms ensuring that $\succsim_i^{\pm}$ are weak orders having two equivalence classes

Axioms

$$AC1_i^* \text{ if } \left.\begin{array}{c}(x_i, a_{-i}) \succsim c \\ \text{and} \\ (y_i, b_{-i}) \succsim d\end{array}\right\} \Rightarrow \left\{\begin{array}{c}(y_i, a_{-i}) \succsim c \\ \text{or} \\ (z_i, b_{-i}) \succsim d\end{array}\right.$$

$$AC2_i^* \text{ if } \left.\begin{array}{c}c \succsim (y_i, a_{-i}) \\ \text{and} \\ d \succsim (z_i, b_{-i})\end{array}\right\} \Rightarrow \left\{\begin{array}{c}c \succsim (x_i, a_{-i}) \\ \text{or} \\ d \succsim (y_i, b_{-i})\end{array}\right.$$

$$AC3_i^* \text{ if } \left.\begin{array}{c}(x_i, a_{-i}) \succsim c \\ \text{and} \\ d \succsim (z_i, b_{-i})\end{array}\right\} \Rightarrow \left\{\begin{array}{c}(y_i, a_{-i}) \succsim c \\ \text{or} \\ d \succsim (y_i, b_{-i})\end{array}\right.$$

$$AC4_i^* \text{ if } \left.\begin{array}{c}c \succsim (y_i, a_{-i}) \\ \text{and} \\ (y_i, b_{-i}) \succsim d\end{array}\right\} \Rightarrow \left\{\begin{array}{c}c \succsim (x_i, a_{-i}) \\ \text{or} \\ (z_i, b_{-i}) \succsim d\end{array}\right.$$

Interpretation

$$AC1_i^* \qquad \left.\begin{array}{c}(x_i, a_{-i}) \succsim c \\ \text{and} \\ (y_i, b_{-i}) \succsim d\end{array}\right\} \Rightarrow \left\{\begin{array}{c}(y_i, a_{-i}) \succsim c \\ \text{or} \\ (z_i, b_{-i}) \succsim d\end{array}\right.$$

Implies $AC1_i$ so that $\succsim_i^+$ is complete

Suppose $(x_i, a_{-i}) \succsim c$ and $(y_i, a_{-i}) \not\succsim c$

It is not true that $y_i \succsim_i^+ x_i$

$AC1_i^*$ requires that $(y_i, b_{-i}) \succsim d$ implies $(z_i, b_{-i}) \succsim d$

y_i is below all other elements of X_i w.r.t. the relation $\succsim_i^+$

$\succsim_i^+$ can only have two distinct equivalence classes

$AC2_i^*$ does the same thing for $\succsim_i^-$

$AC3_i^*$ and $AC4_i^*$ ensure that $\succsim_i^+$ and $\succsim_i^-$ are compatible

Properties

Lemma

1. $AC1_i^* \Leftrightarrow [Not[y_i \succsim_i^+ x_i] \Rightarrow z_i \succsim_i^+ y_i]$
2. $AC2_i^* \Leftrightarrow [Not[y_i \succsim_i^- x_i] \Rightarrow z_i \succsim_i^- y_i]$
3. $AC3_i^* \Leftrightarrow [Not[y_i \succsim_i^+ x_i] \Rightarrow z_i \succsim_i^- y_i]$
4. $AC4_i^* \Leftrightarrow [Not[y_i \succsim_i^- x_i] \Rightarrow z_i \succsim_i^+ y_i]$

Lemma

If $\succsim$ is a RSRP then it satisfies $AC1^*$, $AC2^*$, $AC3^*$, and $AC4^*$

Lemma

The relation $\succsim$ satisfies $AC1_i^*$, $AC2_i^*$, $AC3_i^*$, and $AC4_i^*$ iff the binary relation $\succsim_i^\pm$ is a weak order having two distinct equivalence classes

Result

Theorem

$\succsim$ is a RSRP iff it satisfies $AC1^*$, $AC2^*$, $AC3^*$, and $AC4^*$

In the class of all semiorders on X, conditions $AC1^*$, $AC2^*$, $AC3^*$, and $AC4^*$ are independent

Remark

- does not use $AC1$, $AC2$, $AC3$
- need for a factorization of $AC1^*$, $AC2^*$, $AC3^*$, and $AC4^*$

Refined axioms

$$AC1_i^{**} \text{ if } \left.\begin{array}{c} (x_i, a_{-i}) \succsim c \\ \text{and} \\ (y_i, b_{-i}) \succsim d \\ \text{and} \\ (x_i, b_{-i}) \succsim d \end{array}\right\} \Rightarrow \left\{\begin{array}{c} (y_i, a_{-i}) \succsim c \\ \text{or} \\ (z_i, b_{-i}) \succsim d \end{array}\right.$$

$$AC2_i^{**} \text{ if } \left.\begin{array}{c} c \succsim (y_i, a_{-i}) \\ \text{and} \\ d \succsim (z_i, b_{-i}) \\ \text{and} \\ c \succsim (z_i, a_{-i}) \end{array}\right\} \Rightarrow \left\{\begin{array}{c} c \succsim (x_i, a_{-i}) \\ \text{or} \\ d \succsim (y_i, b_{-i}) \end{array}\right.$$

Refined axioms

$$AC3_i^{**} \text{ if } \left.\begin{array}{c}(x_i, a_{-i}) \succsim c \\ \text{and} \\ d \succsim (z_i, b_{-i})\end{array}\right\} \Rightarrow \left\{\begin{array}{c}(y_i, a_{-i}) \succsim c \\ \text{or} \\ d \succsim (y_i, b_{-i}) \\ \text{or} \\ d \succsim (x_i, b_{-i})\end{array}\right.$$

$$AC4_i^{**} \text{ if } \left.\begin{array}{c}c \succsim (y_i, a_{-i}) \\ \text{and} \\ (y_i, b_{-i}) \succsim d \\ \text{and} \\ (x_i, b_{-i}) \succsim d\end{array}\right\} \Rightarrow \left\{\begin{array}{c}c \succsim (x_i, a_{-i}) \\ \text{or} \\ (z_i, b_{-i}) \succsim d\end{array}\right.$$

Properties

Lemma

1. $\succsim$ satisfies $AC1_i^*$ iff it satisfies $AC1_i$ and $AC1_i^{**}$
2. $\succsim$ satisfies $AC2_i^*$ iff it satisfies $AC2_i$ and $AC2_i^{**}$
3. $\succsim$ satisfies $AC3_i^*$ iff it satisfies $AC3_i$ and $AC3_i^{**}$
4. $\succsim$ satisfies $AC4_i^*$ iff it satisfies $AC3_i$ and $AC4_i^{**}$

Lemma

In the class of all semiorders on X, conditions $AC1_i$, $AC2_i$, $AC3_i$, $AC1^{**}$, $AC2^{**}$, $AC3^{**}$, and $AC4^{**}$ are independent

Result

Theorem

$\succsim$ is a RSRP
iff
it satisfies $AC1$, $AC2$, $AC3$, $AC1^{**}$, $AC2^{**}$, $AC3^{**}$, and $AC4^{**}$

In the class of all semiorders on X, conditions $AC1_i$, $AC2_i$, $AC3_i$, $AC1^{**}$, $AC2^{**}$, $AC3^{**}$, and $AC4^{**}$ are independent

Interpretation

- a RSRP is a model having complete traces on levels with rough traces
- $AC1^{**}$, $AC2^{**}$, $AC3^{**}$, and $AC4^{**}$ characterize RSRP within the class of models having complete traces on levels

Relation to concordance relations

Remarks

$$S_i\text{: semiorder on } X_i$$
$$S(x, y) = \{i \in N : x_i \; S_i \; y_i\}$$
$$x \succsim y \Leftrightarrow S(x, y) \trianglerighteq' S(y, x)$$

- concordance relations do not require that the relations $\succsim_i^{\pm}$ are rough
- concordance relations require independence

Consequences

- there are concordance relations that are not RSRP and vice versa

Remark

- an independent RSRP is a concordance relation
  - an independent RSRP is a doubly rough concordance relation

Weak orders

Lemma

Let $\succsim$ be a weak order on a set X
Then conditions $AC1_i^*$, $AC2_i^*$, $AC3_i^*$ and $AC4_i^*$ are equivalent

Lemma

In the class of all weak orders on X, conditions $AC1_i$ and $AC1_i^{**}$ are independent

Theorem

A weak order is a RSRP iff it satisfies $AC1^*$ iff it satisfies $AC1$ and $AC1^{**}$

Sugeno integral model

Background

- P a finite set
- ν a normalized capacity on 2^P
- $\beta \in \mathbb{R}^p$

$$\text{Sug}_\nu[\beta] = \bigvee_{T \subseteq P} \left[\nu(T) \wedge \left(\bigwedge_{i \in T} \beta_i \right) \right]$$

Definition

- u_i real valued function on X_i
- μ normalized capacity on 2^N

$$x \succsim y \Leftrightarrow$$
$$\text{Sug}_\mu[(u_1(x_1), u_2(x_2), \ldots, u_n(x_n))] \geq \text{Sug}_\mu[(u_1(y_1), u_2(y_2), \ldots, u_n(y_n))]$$

Axiom

2^*-gradedness

$$\left.\begin{array}{c}(x_i, a_{-i}) \succsim c \\ \text{and} \\ (y_i, b_{-i}) \succsim d \\ \text{and} \\ d \succsim c\end{array}\right\} \Rightarrow \left\{\begin{array}{c}(y_i, a_{-i}) \succsim c \\ \text{or} \\ (z_i, b_{-i}) \succsim d\end{array}\right.$$

Theorem (GMS, 2004, B. & Marchant & Pirlot, 2009)

Let $\succsim$ be a weak order on X such that $X/\!\sim$ is at most countably infinite
Then $\succsim$ has a representation in the discrete Sugeno integral model iff it satisfies condition 2^*-graded

Result

Observation

Condition 2^*-graded$_i$ is a weakening of $AC1^*_i$

$$\left.\begin{array}{c}(x_i, a_{-i}) \succsim c \\ \text{and} \\ (y_i, b_{-i}) \succsim d \\ \text{and} \\ \color{red}{d \succsim c}\end{array}\right\} \Rightarrow \left\{\begin{array}{c}(y_i, a_{-i}) \succsim c \\ \text{or} \\ (z_i, b_{-i}) \succsim d\end{array}\right. \qquad \left.\begin{array}{c}(x_i, a_{-i}) \succsim c \\ \text{and} \\ (y_i, b_{-i}) \succsim d\end{array}\right\} \Rightarrow \left\{\begin{array}{c}(y_i, a_{-i}) \succsim c \\ \text{or} \\ (z_i, b_{-i}) \succsim d\end{array}\right.$$

Proposition

A weak order that is a RSRP always has a representation in the discrete Sugeno integral model

Elicitation

Problem

- elicit the sets $\mathcal{A}_i$ and the relation $\trianglerighteq$ based on preference judgments

Method

- when $\succsim$ is a weak order
- a MILP can test whether $\succsim$ is a RSRP with $\trianglerighteq$ being represented by a 2-additive capacity and elicit the sets $\mathcal{A}_i$ and $\trianglerighteq$
- not detailed here!
- a different approach has been proposed by Zheng, Rolland, and Mousseau (2012)

Definition

Ingredients

- R_i: a semiorder on X_i
- $L = \{1, 2, \ldots, \ell\}$
- $\pi^1, \pi^2, \ldots, \pi^\ell$: ℓ reference points in X
- $\trianglerighteq_L$: binary relation on $(2^N)^\ell$ monotonic w.r.t. inclusion

$$\left.\begin{array}{r} (A^1, \ldots, A^\ell) \trianglerighteq_L (B^1, \ldots, B^\ell) \\ C^k \supseteq A^k \\ B^k \supseteq D^k \end{array}\right\} \Rightarrow (C^1, \ldots, C^\ell) \trianglerighteq_L (D^1, \ldots, D^\ell)$$

Definition RMRP

$$\mathcal{R}^k(x) = \{i \in N : x_i \; R_i \; \pi_i^k\}$$

$$x \succsim y \Leftrightarrow (\mathcal{R}^1(x), \mathcal{R}^2(x), \ldots, \mathcal{R}^\ell(x)) \trianglerighteq_L (\mathcal{R}^1(y), \mathcal{R}^2(y), \ldots, \mathcal{R}^\ell(y))$$

Results

Lemma

If a binary relation on X is a RMRP with ℓ reference points, it satisfies $AC1$, $AC2$ and $AC3$, so that all relations $\succsim_i^{\pm}$ are weak orders

Furthermore, for all $i \in N$, the weak order $\succsim_i^{\pm}$ has at most $\ell + 1$ distinct equivalence classes

Lemma

If $\succsim$ satisfies $AC1$, $AC2$, and $AC3$ and, for all $i \in N$, the sets $X_i/{\sim_i^{\pm}}$ are finite then $\succsim$ is a RMRP

Result

Proposition

$\succsim$ is a RMRP
iff
it satisfies $AC1$, $AC2$, and $AC3$ and all relations $\succsim_i^{\pm}$ have a finite number of equivalence classes

Corollary

If X is finite
$\succsim$ is a RMRP
iff
it satisfies $AC1$, $AC2$, and $AC3$

Remarks

Remarks

- a RMRP always has a representation in which all relations R_i are weak orders
- a RMRP always has a representation in which the reference points dominates each other w.r.t. $\succsim_i^{\pm}$
- a RMRP that is a weak order does not necessarily have a representation in the Sugeno integral model
- a RMRP has close connections with the "decision rule" model by GMS

Discussion

Models with reference points

- are interesting
  - in practice: simple and intuitive
  - in theory: allow to explore another side of "rough" models
- fit quite well into the framework of models having complete traces on levels
- can easily be characterized in this framework
- efficient elicitation methods can be proposed

Future work

- include an idea of discordance in these models
- study particular cases of RMRP as proposed by A. Rolland
- refine elicitation techniques

References

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