An axiomatic approach to ELECTRE TRI

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If you do not know Thierry…

Introduction

Context

- preference modelling for MCDA

Two main traditions

- **Axiomatic**: conjoint measurement and additive value functions
  - firm theoretical background (Krantz et al., 1971)
  - implementation often delicate: requires a detailed analysis of preferences
- **Pragmatic**: dominance relation and refinements
  - outranking relations based on a concordance-discordance principle
  - simple and intuitive... but difficult to compare to other methods (lack of axiomatic foundations)

Conjoint measurement

Ingredients

- $X \subseteq X_1 \times X_2 \times \cdots \times X_n$: set of objects evaluated on n attributes
- $\succsim$: *binary relation* on X

Aim: Study under what conditions $\succsim$ can be represented in a given measurement model and the uniqueness of this representation

$$x \succsim y \Leftrightarrow \sum_{i=1}^{n} v_i(x_i) \geq \sum_{i=1}^{n} v_i(y_i)$$

Bouyssou & Pirlot (2002, 2004 a & b, 2005)

- general conjoint measurement framework tolerating intransitivity and incompleteness
- analyze pragmatic methods within this framework

Conjoint measurement

Why be interested in conjoint measurement?

- axiomatic analysis allows to better understand models and to compare them
- for Psychologists: these results exhibit empirically testable conditions
- for Decision Analysts: these results give hints on how to build the representation and, hence, to assess preferences

Limitation

- restricted to *relative* evaluation models (models comparing alternatives between themselves)

Sorting methods

Absolute evaluation models

- compare alternative to norms
    - prototypes
    - limiting profiles

ELECTRE TRI (Wei, 1992, Roy & Bouyssou, 1993)

- compare alternatives to limiting profiles
- using a concordance / discordance approach

ELECTRE TRI

ELECTRE TRI vs other outranking methods

- keeps the idea of concordance and non-discordance
- seems to avoid "exploitation" problems

Active research

- interactive assessment of parameters (Mousseau et al.)
- many applications

Objectives

Propose a general framework for conjoint measurement

- adapted to sorting methods
- simple and intuitive but nontrivial
- having a numerical representation

Put this framework to work

- to characterize ELECTRE TRI
- not any characterization
  - having ad hoc axioms for a method is an easy (and rather futile) exercise

Outline

1. Background
   - Definitions and notation
   - Measurement framework
   - GMS
   - Electre Tri
2. The noncompensatory sorting model
   - Definitions
   - Axioms and results
   - Sugeno integral
   - Extensions
3. The noncompensatory sorting model with veto
   - Definitions
   - Axioms and results
   - Extensions
4. Conclusion

Setting

Classical conjoint measurement setting

- $N = \{1, 2, \dots, n\}$: set of attributes
- $X = \prod_{i=1}^{n} X_i$ with $n \geq 2$: set of alternatives
  - X_i are not supposed to have a special structure
- $\succsim$: binary relation on X
- notation: $(x_i, y_{-i}) \in X$

Our setting

- replace the binary relation $\succsim$ on X by a partition $\langle C^1, C^2, \dots, C^r \rangle$ of X

Setting

Variants

partition $\langle C^1, C^2, \ldots, C^r \rangle$ of X

- ordered vs unordered categories
- $r = 2$ vs $r > 2$
- presence or absence of a “frontier” between categories
- objects belonging to more than one category
- measurement models

Scope of today's talk: 2 categories

Primitives

- twofold partition $\langle \mathscr{A}, \mathscr{U} \rangle$ of the set X

Interpretation

- $\mathscr{A}$ contains sAtisfactory objects
- $\mathscr{U}$ contains Unsatisfactory objects

Remark

- with only two categories the distinction between ordered and unordered categories tends to be blurred
- the ordering of categories is not part of our primitives

Influence and degeneracy

Influence

Attribute $i \in N$ is *influent* if there are $x_i, y_i \in X_i$ and $a_{-i} \in X_{-i}$ such that

- $(x_i, a_{-i}) \in \mathscr{A}$
- $(y_i, a_{-i}) \notin \mathscr{A}$

An attribute that is not influent will be *degenerate*

We do not suppose that all attributes are influent

Measurement models

Decomposable threshold model (Goldstein, 1992)

$$x \in \mathscr{A} \Leftrightarrow F\left[u_1(x_1), u_2(x_2), \ldots, u_n(x_n)\right] > 0 \qquad (D)$$

- u_i is a real-valued function on X_i
- F is a real-valued function on $\prod_{i=1}^{n} u_i(X_i)$

Variants

- model $(D \nearrow)$: (D) with F nondecreasing in each variable
- model $(D \nearrow\!\!\nearrow)$: (D) with F increasing in each variable
- model (Add): (D) with $F = \sum$

$$(Add) \subseteq (D \nearrow\!\!\nearrow) \subseteq (D \nearrow) \subseteq (D)$$

Results

Analysis of model (D)

the relation $\sim_i$ on X_i such that:

$$x_i \sim_i y_i \Leftrightarrow [\text{for all } a_{-i} \in X_{-i}, (y_i, a_{-i}) \in \mathscr{A} \Leftrightarrow (x_i, a_{-i}) \in \mathscr{A}]$$

is an equivalence

Theorem (Goldstein)

$\langle \mathscr{A}, \mathscr{U} \rangle$ has a representation in model (D) if and only if, for all $i \in N$, there is a one-to-one correspondence between $X_i/\sim_i$ and some subset of $\mathbb{R}$

Axioms

Analysis of variants

the relation $\succsim_i$ on X_i such that:

$$x_i \succsim_i y_i \Leftrightarrow [\text{for all } a_{-i} \in X_{-i}, (y_i, a_{-i}) \in \mathscr{A} \Rightarrow (x_i, a_{-i}) \in \mathscr{A}]$$

is always reflexive and transitive (symmetric part is $\sim_i$)

- in all variants ($D \nearrow$, $D \nearrow\!\!\nearrow$, *Add*) introduced above, $\succsim_i$ is complete

Axiom: linearity on attribute i

$$\left.\begin{array}{c}(x_i, a_{-i}) \in \mathscr{A} \\ \text{and} \\ (y_i, b_{-i}) \in \mathscr{A}\end{array}\right\} \Rightarrow \left\{\begin{array}{c}(y_i, a_{-i}) \in \mathscr{A} \\ \text{or} \\ (x_i, b_{-i}) \in \mathscr{A}\end{array}\right. \qquad (\mathit{linear}_i)$$

for all $x_i, y_i \in X_i$ and all $a_{-i}, b_{-i} \in X_{-i}$
$\langle \mathscr{A}, \mathscr{U} \rangle$ is linear if it satisfies linear_i on all $i \in N$

Results

Lemma (Linearity)

1. *condition* linear_i *holds iff* $\succsim_i$ *is complete*
2. *if* $\langle \mathscr{A}, \mathscr{U} \rangle$ *has a representation in model* $(D \nearrow)$ *then it is linear*

Theorem (Model $(D \nearrow\nearrow)$)

There is a representation of $\langle \mathscr{A}, \mathscr{U} \rangle$ *in model* $(D \nearrow\nearrow)$ *if and only if it is linear and, for all* $i \in N$*, there is a finite or countably infinite set* $X'_i \subseteq X_i$ *that is dense in* X_i *for* $\succsim_i$*.*
Furthermore:

- *if* $\langle \mathscr{A}, \mathscr{U} \rangle$ *has a representation in model* $(D \nearrow\nearrow)$*, it has a representation in which, for all* $i \in N$*,* u_i *is a numerical representation of* $\succsim_i$*,*
- *models* $(D \nearrow\nearrow)$ *and* $(D \nearrow)$ *are equivalent.*

Model $(D \nearrow\nearrow)$

Remarks

- uniqueness is quite weak
- the role of $\mathscr{A}$ and $\mathscr{U}$ is entirely symmetric

Using model $(D \nearrow\nearrow)$

- model $(D \nearrow\nearrow)$ contains many sorting models as particular cases
  - UTADIS (model (Add))
  - conjunctive and disjunctive models
  - models using distances to prototypes
- we will show later that it contains (our version of) ELECTRE TRI as a particular case

Decision rules à la GMS (2001, 2002)

The "at least" decision rule model of GMS

- $\mathcal{S}_i$: complete and transitive relation on each X_i
- decision rule d
  - a subset $N^d \subseteq N$ of attributes
  - for each $i \in N^d$, in an element $\delta_i^d \in X_i$
- "at least" decision rule d

$$[x_i \; \mathcal{S}_i \; \delta_i^d, \forall i \in N^d] \Rightarrow x \in \mathscr{A}$$

Representation in the decision rule model

A set of "at least" decision rules $\mathcal{D}$ is said to represent $\langle \mathscr{A}, \mathscr{U} \rangle$ if

- for each $x \in \mathscr{A}$, there is one decision rule in $d \in \mathcal{D}$ that matches x, $(x_i \; \mathcal{S}_i \; \delta_i^d, \forall i \in N^d)$
- for each $y \in \mathscr{U}$, there is no decision rule in $\mathcal{D}$ that matches y

Result

Theorem (GMS, 2001, 2002)

A partition $\langle \mathscr{A}, \mathscr{U} \rangle$ can be represented in the "at least" decision rule model iff it is linear

skip relational model

Relational model à la GMS

Relational model of GMS

- $\mathcal{S}_i$: complete and transitive relation on each X_i
- binary relation $\mathcal{S}$ on X compatible with $\mathcal{S}_i$

$$[x \,\mathcal{S}\, y, z_i \,\mathcal{S}_i\, x_i, y_j \,\mathcal{S}_j\, w_j] \Rightarrow (z_i, x_{-i}) \,\mathcal{S}\, (w_j, y_{-j})$$

- an element $\tau \in X$

such that

$$x \in \mathscr{A} \Leftrightarrow x \,\mathcal{S}\, \tau$$

Theorem (GMS, 2001, 2002)

A partition $\langle \mathscr{A}, \mathscr{U} \rangle$ can be represented in the relational model iff it is linear

(our version of) ELECTRE TRI

Ingredients

- profile: $\pi = (\pi_1, \pi_2, \dots, \pi_n)$
- $\widehat{X}_i = X_i \cup \{\pi_i\}$ and $\widehat{X} = \prod_{i=1}^n \widehat{X}_i$
- semiorder: S_i on $\widehat{X}_i$
- strict semiorder: V_i on $\widehat{X}_i$ such that $V_i \subseteq P_i$
- normalized weights: w_i $(\sum_{i=1}^n w_i = 1)$
- majority threshold: $\lambda \in [0.5, 1]$

Outranking relation

- S on $\widehat{X}$

$$x \; S \; y \Leftrightarrow \sum_{i \in S(x,y)} w_i \geq \lambda \text{ and } \left[Not[y_i \; V_i \; x_i], \text{ for all } i \in N \right]$$

Electre Tri

Pessimistic version

$$x \in \mathscr{A} \Leftrightarrow x\ S\ \pi$$

Optimistic version

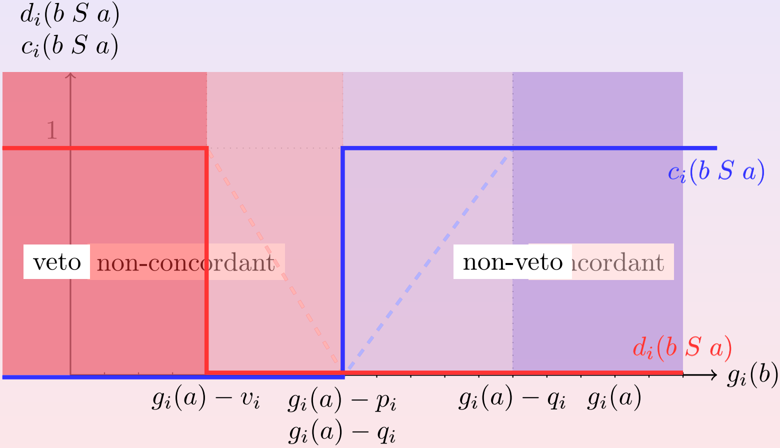
$$x \in \mathscr{A} \Leftrightarrow Not[\pi\ P\ x]$$

Remarks

- no weak preference zone
- discordance occur in an “all or nothing” way

Close to the original when X_i are discrete

Electre Tri


$d_i(b\ S\ a)$
$c_i(b\ S\ a)$
1
$c_i(b\ S\ a)$
veto
non-concordant
non-veto
concordant
$d_i(b\ S\ a)$
$g_i(b)$
$g_i(a) - v_i$
$g_i(a) - p_i$
$g_i(a) - q_i$
$g_i(a) - q_i$
$g_i(a)$

Definition

Noncompensatory sorting model

A partition $\langle \mathscr{A}, \mathscr{U} \rangle$ has a representation in the *noncompensatory sorting model* if

- for all $i \in N$ there is a set $\mathscr{A}_i \subseteq X_i$
- there is a subset $\mathscr{F}$ of 2^N such that, for all $I, J \in 2^N$

$$[I \in \mathscr{F} \text{ and } I \subseteq J] \Rightarrow J \in \mathscr{F}$$

such that, for all $x \in X$,

$$x \in \mathscr{A} \Leftrightarrow \{i \in N : x_i \in \mathscr{A}_i\} \in \mathscr{F}$$

Interpretation

- $x \in \mathscr{A}$ iff x is "satisfactory" ($x_i \in \mathscr{A}_i$) on a subset of attributes that is "sufficiently important" ($\in \mathscr{F}$)

Remarks

Consequences of influence

- if attribute $i \in N$ is influent then $\varnothing \subsetneq \mathscr{A}_i \subsetneq X_i$
- if $\langle \mathscr{A}, \mathscr{U} \rangle$ has a representation $\langle \mathscr{F}, \langle \mathscr{A}_i \rangle_{i \in N} \rangle$ in the noncompensatory sorting model, this representation is unique iff all attributes are influent

Particular cases of the noncompensatory sorting model

- $\mathscr{F} = \{N\}$: conjunctive model
- $\{i\} \in \mathscr{F}$, for all $i \in N$: disjunctive model

Noncompensatory sorting model and ELECTRE TRI

ELECTRE TRI (pessimistic)

- When $V_i = \varnothing$, for all $i \in N$, (our version of) ELECTRE TRI is a particular case of the noncompensatory sorting model

$$x \in \mathscr{A} \Leftrightarrow x \; S \; \pi \Leftrightarrow \sum_{i \in S(x,\pi)} w_i \geq \lambda$$

- $\mathscr{A}_i = \{x_i \in X_i : x_i \; S_i \; \pi_i\}$
- $I \in \mathscr{F}$ iff $\sum_{i \in I} w_i \geq \lambda$

ELECTRE TRI (optimistic)

- ELECTRE TRI (optimistic) is *not* a particular case of the noncompensatory sorting model

Observations and Result

Observations

Suppose that $\langle \mathscr{A}, \mathscr{U} \rangle$ has a representation in the noncompensatory sorting model. Then

- it is linear, so that $\succsim_i$ are weak orders
- all relations $\succsim_i$ have at most two distinct equivalence classes

Proposition

$\langle \mathscr{A}, \mathscr{U} \rangle$ has a representation in the noncompensatory sorting model iff it has a representation in model $(D \nearrow\nearrow)$ in which each u_i takes at most two distinct values

Axiom

2-gradedness

$\langle \mathscr{A}, \mathscr{U} \rangle$ satisfies condition 2-*graded*$_i$ if

$$\left.\begin{array}{c}(x_i, a_{-i}) \in \mathscr{A} \\ \text{and} \\ (y_i, a_{-i}) \in \mathscr{A} \\ \text{and} \\ (y_i, b_{-i}) \in \mathscr{A}\end{array}\right\} \Rightarrow \left\{\begin{array}{c}(x_i, b_{-i}) \in \mathscr{A} \\ \text{or} \\ (z_i, a_{-i}) \in \mathscr{A}\end{array}\right. \qquad (2\text{-}graded_i)$$

for all $x_i, y_i, z_i \in X_i$ and all $a_{-i}, b_{-i} \in X_{-i}$
We say that $\langle \mathscr{A}, \mathscr{U} \rangle$ is 2-graded if it is 2-*graded*$_i$ for all $i \in N$

$$\left.\begin{array}{l}(x_i, a_{-i}) \in \mathscr{A} \text{ and } (z_i, a_{-i}) \notin \mathscr{A} \Rightarrow Not[z_i \succsim_i x_i] \\ (y_i, a_{-i}) \in \mathscr{A} \text{ and } (z_i, a_{-i}) \notin \mathscr{A} \Rightarrow Not[z_i \succsim_i y_i] \\ (y_i, b_{-i}) \in \mathscr{A} \text{ and } (x_i, b_{-i}) \notin \mathscr{A} \Rightarrow Not[x_i \succsim_i y_i]\end{array}\right\} \Rightarrow 3 \text{ classes}$$

Observations and result

Lemma

1. *Conditions* linear_i *and* 2-graded_i *hold iff* $\succsim_i$ *is a weak order having at most two distinct equivalence classes*
2. *Conditions* linear_i *and* 2-graded_i *are independent*

Theorem

A partition $\langle \mathscr{A}, \mathscr{U} \rangle$ *has a representation in the noncompensatory sorting model iff it is linear and 2-graded*

(Discrete) Sugeno integral model

Ingredients

- a non-negative real valued function f_i on X_i, for all $i \in N$
- a real valued function μ (a capacity) on 2^N that is nondecreasing w.r.t. inclusion (i.e., such that $A \subseteq B$ implies $\mu(A) \leq \mu(B)$) and such that $\mu(\varnothing) = 0$

such that, for all $x \in X$,

$$x \in \mathscr{A} \Leftrightarrow \bigvee_{I \subseteq N} \left[\mu(I) \wedge \left(\bigwedge_{i \in I} [f_i(x_i)] \right) \right] > 0 \qquad (Su)$$

GMS (2002)

Model (Su) has been characterized (without proof!) by GMS (2002)

Result

Theorem

A partition $\langle \mathscr{A}, \mathscr{U} \rangle$ of a set X has a representation in the noncompensatory sorting model iff it has a representation in the Sugeno integral model (Su)

Remarks

- gives a new and simple interpretation of the Sugeno integral
- shows the power and interest of axiomatic analysis
- hint: define f_i and μ letting
  - $f_i(x_i) = 1$ if $x_i \in \mathscr{A}_i$ and 0 otherwise
  - $\mu(I) = 1$ if $I \in \mathscr{F}$ and 0 otherwise

skip other extensions

Models without linearity

2-graded$_i$

$$\left.\begin{array}{c}(x_i, a_{-i}) \in \mathscr{A} \\ \text{and} \\ (y_i, a_{-i}) \in \mathscr{A} \\ \text{and} \\ (y_i, b_{-i}) \in \mathscr{A}\end{array}\right\} \Rightarrow$$

$$\left\{\begin{array}{c}(x_i, b_{-i}) \in \mathscr{A} \\ \text{or} \\ (z_i, a_{-i}) \in \mathscr{A}\end{array}\right.$$

2-graded$_i^*$

$$\left.\begin{array}{c}(x_i, a_{-i}) \in \mathscr{U} \\ \text{and} \\ (y_i, a_{-i}) \in \mathscr{U} \\ \text{and} \\ (y_i, b_{-i}) \in \mathscr{U}\end{array}\right\} \Rightarrow$$

$$\left\{\begin{array}{c}(x_i, b_{-i}) \in \mathscr{U} \\ \text{or} \\ (z_i, a_{-i}) \in \mathscr{U}\end{array}\right.$$

Results

Lemma

1. *Conditions* 2-graded$_i$ *and* 2-graded$_i^*$ *are independent*
2. *In presence of* linear$_i$, *conditions* 2-graded$_i$ *and* 2-graded$_i^*$ *are equivalent*
3. [2-graded$_i$ *and* 2-graded$_i^*$] *do not imply* linear$_i$

Lemma

$\langle \mathscr{A}, \mathscr{U} \rangle$ *satisfies* 2-graded$_i$ *and* 2-graded$_i^*$ *iff* $\sim_i$ *has at most two equivalence classes*

The generalized noncompensatory sorting model

Definition

- for all $i \in N$, there is a set $\mathscr{A}_i \subseteq X_i$
- there is a subset $\mathscr{F}$ of 2^N

such that, for all $x \in X$,

$$x \in \mathscr{A} \Leftrightarrow \{i \in N : x_i \in \mathscr{A}_i\} \in \mathscr{F}$$

Interpretation

- identical to the noncompensatory sorting model except that $\mathscr{F}$ is not supposed to be compatible with set inclusion
- combinations of levels in $\mathscr{A}_i$ are typical of $\mathscr{A}$

Result

Theorem

*A partition $\langle \mathscr{A}, \mathscr{U} \rangle$ has a representation in the generalized noncompensatory sorting model iff it is 2-graded and 2-graded**

The noncompensatory sorting model with veto

Definition

A partition $\langle \mathscr{A}, \mathscr{U} \rangle$ has a representation in the noncompensatory sorting model noncompensatory sorting model with veto if

- for all $i \in N$ there is a set $\mathscr{A}_i \subseteq X_i$
- for all $i \in N$ there is a set $\mathscr{V}_i \subseteq X_i$ such that $\mathscr{A}_i \cap \mathscr{V}_i = \varnothing$
- there is a subset $\mathscr{F}$ of 2^N such that, for all $I, J \in 2^N$

$$[I \in \mathscr{F} \text{ and } I \subseteq J] \Rightarrow J \in \mathscr{F}$$

such that, for all $x \in X$,

$$x \in \mathscr{A} \Leftrightarrow \{i \in N : x_i \in \mathscr{A}_i\} \in \mathscr{F} \text{ and } \{i \in N : x_i \in \mathscr{V}_i\} = \varnothing$$

Remarks

Interpretation

- $x \in \mathscr{A}$ iff x is satisfactory ($x_i \in \mathscr{A}_i$) on a subset of attributes that is "sufficiently important" ($\in \mathscr{F}$) and x has no repulsive level for $\mathscr{A}$ ($x_i \in \mathscr{V}_i$)

Lemma

If $\langle \mathscr{A}, \mathscr{U} \rangle$ has a representation in the noncompensatory sorting model with veto then it is linear

Consequence

- the noncompensatory sorting model with veto is a particular case of model $(D \nearrow\nearrow)$

ELECTRE TRI

Remark

- (our version of) ELECTRE TRI (pessimistic) is a particular case of the noncompensatory sorting model with veto

$$x \in \mathscr{A} \Leftrightarrow x \; S \; \pi \Leftrightarrow$$

$$\sum_{i \in S(x,\pi)} w_i \geq \lambda \text{ and } [Not[\pi_i \; V_i \; x_i], \text{ for all } i \in N]$$

- $\mathscr{A}_i = \{x_i \in X_i : x_i \; S_i \; \pi_i\}$
- $\mathscr{V}_i = \{x_i \in X_i : \pi_i \; V_i \; x_i\}$
- $I \in \mathscr{F}$ iff $\sum_{i \in I} w_i \geq \lambda$

Axiom

3-gradedness with veto

$\langle \mathscr{A}, \mathscr{U} \rangle$ satisfies condition *3v-graded*$_i$ if

$$\left.\begin{array}{c} (x_i, a_{-i}) \in \mathscr{A} \\ \text{and} \\ (y_i, a_{-i}) \in \mathscr{A} \\ \text{and} \\ (y_i, b_{-i}) \in \mathscr{A} \\ \text{and} \\ (z_i, c_{-i}) \in \mathscr{A} \end{array}\right\} \Rightarrow \left\{\begin{array}{c} (x_i, b_{-i}) \in \mathscr{A} \\ \text{or} \\ (z_i, a_{-i}) \in \mathscr{A} \end{array}\right. \qquad (\textit{3v-graded}_i)$$

We say that $\langle \mathscr{A}, \mathscr{U} \rangle$ is 3-graded with veto if it is *3v-graded*$_i$ for all $i \in N$

Observations

Lemma

1. *If* $\langle \mathscr{A}, \mathscr{U} \rangle$ *has a representation in the noncompensatory sorting model with veto then it is 3-graded with veto*
2. *Conditions* linear$_i$ *and* 3v-graded$_i$ *are independent*
3. *Conditions* linear$_i$ *and* 3v-graded$_i$ *imply that* $\succsim_i$ *is a weak order having at most three equivalence classes*
   Furthermore if $\succsim_i$ *has exactly three distinct equivalence classes and if* x_i *belongs to the last equivalence class of* $\succsim_i$ *then* $(x_i, a_{-i}) \in \mathscr{U}$*, for all* $a_{-i} \in X_{-i}$

Main result

Theorem

A partition $\langle \mathscr{A}, \mathscr{U} \rangle$ is representable in the noncompensatory sorting model with veto iff it is linear and 3-graded with veto

Uniqueness

Remark

- the representation of $\langle \mathscr{A}, \mathscr{U} \rangle$ in the noncompensatory sorting model with veto may be non-unique even if all attributes are influent
- the necessary and sufficient conditions for uniqueness are known and are stringent (all attributes should be influent for the partition obtained after having removed the levels that are "obviously" in $\mathscr{V}_i$)

skip other extensions

Extensions

3v-graded$_i$

$$\left.\begin{array}{c}(x_i, a_{-i}) \in \mathscr{A} \\ \text{and} \\ (y_i, a_{-i}) \in \mathscr{A} \\ \text{and} \\ (y_i, b_{-i}) \in \mathscr{A} \\ \text{and} \\ (z_i, c_{-i}) \in \mathscr{A}\end{array}\right\} \Rightarrow$$

$$\left\{\begin{array}{c}(x_i, b_{-i}) \in \mathscr{A} \\ \text{or} \\ (z_i, a_{-i}) \in \mathscr{A}\end{array}\right.$$

3v-graded$^*_i$

$$\left.\begin{array}{c}(x_i, a_{-i}) \in \mathscr{A} \\ \text{and} \\ (z_i, c_{-i}) \in \mathscr{A} \\ \text{and} \\ (y_i, b_{-i}) \in \mathscr{A}\end{array}\right\} \Rightarrow$$

$$\left\{\begin{array}{c}(x_i, b_{-i}) \in \mathscr{A} \\ \text{or} \\ (x_i, c_{-i}) \in \mathscr{A} \\ \text{or} \\ (z_i, b_{-i}) \in \mathscr{A}\end{array}\right.$$

Observations

Lemma

1. *Conditions* 3v-graded$_i$ *and* 3v-graded$_i^*$ *are independent*
2. *In presence of* linear$_i$, *conditions* 3v-graded$_i$ *and* 3v-graded$_i^*$ *are equivalent*
3. [3v-graded$_i$ *and* 3v-graded$_i^*$] *do not imply* linear$_i$

Model

The generalized noncompensatory model with veto

- for all $i \in N$ there are disjoint sets $\mathscr{A}_i, \mathscr{V}_i \subseteq X_i$
- there is a subset $\mathscr{F}$ of 2^N

such that, for all $x \in X$,

$$x \in \mathscr{A} \Leftrightarrow \left[\{i \in N : x_i \in \mathscr{A}_i\} \in \mathscr{F} \text{ and } \{i \in N : x_i \in \mathscr{V}_i\} = \varnothing\right]$$

Interpretation

- $x \in X$ belongs to $\mathscr{A}$ if it has a *combination* of elements in $\mathscr{A}_i$ that is “typical” of $\mathscr{A}$
- a repulsive evaluation for $\mathscr{A}$ is able to destroy this typicalness

Result

Theorem

A partition $\langle \mathscr{A}, \mathscr{U} \rangle$ *has a representation in the generalized noncompensatory model with veto iff it is 3-graded with veto and 3-graded* with veto*

Bonuses

- using a simple duality argument, it is possible to characterize models with "bonuses" instead of "vetoes" (some levels are "compulsive" for $\mathscr{A}$)

Implications

Theory

- the use of conjoint measurement methods is *very* enlightening
  - conjoint measurement is not restricted to models using preference relations (Nakamura, JMP, 2004)
- a characterization of (our version of) ELECTRE TRI (pessimistic)
  - within a general framework (model $(D \nearrow\nearrow)$)
  - revealing the specific features of (our version of) ELECTRE TRI when compared with other methods
  - using simple and testable conditions
  - leading to some new interpretations (Sugeno integral)
- new models (generalized noncompensatory sorting model and generalized noncompensatory model with veto)

Implications

Practice

- difference between pessimistic and optimistic versions of ELECTRE TRI
  - maybe seen as a "problem" of our models...
  - ...but may also be seen as a "problem" with ELECTRE TRI optimistic
- uniqueness is a problem in the noncompensatory sorting model with veto
  - interactive elicitation methods should be prepared to deal with this

Extensions

Extension to r categories?

- is it possible?
  - YES!
- is it easy?
  - NO!
  - ... but the principles remain unchanged
  - each of the twofold partitions $\langle C^{\geq k}, C^{<k} \rangle$ induced by $\langle C^1, C^2, \ldots, C^r \rangle$ should satisfy the axioms seen today...
  - ... plus consistency requirements on $\langle C^{\geq k}, C^{<k} \rangle$

Current research

- model (Add)
- model (Su)

Bibliography

References

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