On consistent families of criteria : An axiomatic approach

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Outline

- Introduction: Consistent families in the sense of Roy
- Example: Ranking vectors of constraint satisfaction degrees
- A general model
  - Characterisation
  - Interpretation
  - Back to the example
- Discussion and future work

Introduction

Families of criteria (Roy, 1985)

- Definition of a criterion: $g_i : \mathcal{A} \to \mathbb{R}$
  interpretation of $g_i(a) \geq g_i(b)$: a is at least as good as b
  on point of view i
- n points of view: $i = 1, \ldots, n$
- Notation for the global preference on $\mathcal{A}$ (to be constructed by applying some method): $\succsim$

Consistent families

Properties to be fulfilled

- Exhaustivity:

$$[g_i(a) = g_i(b),\ \forall i\ ] \Rightarrow a \sim b$$

- Consistency:

$$\left.\begin{array}{l} a \succsim b \\ g_i(c) \geq g_i(a) \quad \forall i \\ g_i(b) \geq g_i(d) \quad \forall i \end{array}\right\} \Rightarrow c \succsim d$$

- Non-redundancy

Interpretation and Question

Interpretation . . .

. . . of consistency : The relationship between the criteria and the constructed preference $\succsim$ is supposed to be monotonic

Alternative formulation : respect of dominance

Question :

Is it possible to characterise the preferences that are consistent with a family of criteria ?

Example

Comparing vectors of satisfaction levels

FCSP : Flexible Constraint Satisfaction Problem

- vertices = tasks
- arcs = constraints $C_i, i = 1, \ldots, n$
- solution = a feasible schedule for the tasks
- quality of solution : a vector of satisfaction levels

$$x = (x_1, \ldots, x_n)$$

- problem : comparing the solutions

General problem of "vector optimisation" : no evident complete ordering of the solutions

Dominance

A natural partial order : the dominance relation D (Pareto ordering)

$$x \; D \; y \text{ if } \; x_i \geq y_i, \; \forall i$$

Note: in general x_i belongs to a set X_i that is at least ordered

Question : Characterise "natural and operational methods" leading to a complete ordering of the vectors and preserving dominance (question raised by Dubois and Prade)

A framework

- $X = X_1 \times X_2 \times \ldots \times X_n$: finite set of alternatives
- $N = \{1, 2, \ldots, n\}$: set of attributes
- $x = (x_1, x_2, \ldots, x_n) \in X$
- $x_{-i} = (x_1, \ldots, x_{i-1}, x_{i+1}, \ldots, x_n)$
- $(a_i, x_{-i}) \in X$

- $\succsim$ reflexive binary relation on X interpreted as “large preference”
- $\sim$: symmetric part; $\succ$: asymmetric part

Note : there is no structure assumed on X_i

Model 0

$$x \succsim y \;\Leftrightarrow\; F((u_i(x_i), u_i(y_i)), i = 1, \ldots, n) \geq 0$$

for some functions $u_i : X_i \to \mathbb{R}$
and some function $F : \mathbb{R}^{2n} \to \mathbb{R}$

Note : no properties assumed on functions F and u_i

Result : any relation $\succsim$ on X satisfies Model 0

Note : u_i induces an ordering $\succsim_i^{\pm}$ on X_i :

$$x_i \succsim_i^{\pm} y_i \;\Leftrightarrow\; u_i(x_i) \geq u_i(y_i)$$

This ordering plays no role (is arbitrary) when F has no special property

Model 1

$$x \succsim y \iff F((u_i(x_i), u_i(y_i)), i = 1, \ldots, n) \geq 0$$

for some functions $u_i : X_i \to \mathbb{R}$
and some function $F : \mathbb{R}^{2n} \to \mathbb{R}$
such that

$$F(\nearrow, \searrow)$$

i.e. F is non-decreasing in its first n coordinates
and non-increasing in its last n coordinates:

Property : $\succsim$ is monotonic (consistent) with respect to the orders $\succsim_i^{\pm}$

$$\left.\begin{array}{ll} x \succsim y & \\ u_i(z_i) \geq u_i(x_i) & \forall i \\ u_i(y_i) \geq u_i(w_i) & \forall i \end{array}\right\} \Rightarrow \; z \succsim w$$

Indeed :

$$\left.\begin{array}{l} F(u_i(x_i), u_i(y_i)) \geq 0 \\ F(\; \nearrow \; , \; \searrow \;) \end{array}\right\} \Rightarrow \; F(u_i(z_i), u_i(w_i)) \geq 0$$

Characterisation of Model 1

Result : the relations that satisfy Model 1 are those verifying properties AC123

AC1$_i$:

$$\left.\begin{array}{r}(x_i, a_{-i}) \succsim (y_i, b_{-i}) \\ \text{and} \\ (z_i, c_{-i}) \succsim (w_i, d_{-i})\end{array}\right\} \Rightarrow \left\{\begin{array}{l}(z_i, a_{-i}) \succsim (y_i, b_{-i}) \\ \textit{or} \\ (x_i, c_{-i}) \succsim (w_i, d_{-i}),\end{array}\right.$$

Definition of $\succsim_i^+$:

$$x_i \succsim_i^+ y_i \Leftrightarrow [(y_i, a_{-i}) \succsim z \Rightarrow (x_i, a_{-i}) \succsim z]$$

Result 1 : AC1$_i$ iff $\succsim_i^+$ is a complete preorder

Result 2 : AC2$_i$ iff $\succsim_i^-$ is a complete preorder
with

$$x_i \succsim_i^- y_i \Leftrightarrow [z \succsim (x_i, a_{-i}) \Rightarrow z \succsim (y_i, a_{-i})],$$

Result 3 : AC123$_i$ iff $\succsim_i^\pm$ is a complete preorder
with

$$x_i \succsim_i^\pm y_i \Leftrightarrow [x_i \succsim_i^+ y_i \text{ and } x_i \succsim_i^- y_i],$$

Consequence :
If $\succsim$ satisfies AC123,

- the relations $\succsim_i^\pm$ are complete preorders
- u_i can be chosen to be any numerical representation of $\succsim_i^\pm$
- $F(u_i(x_i), u_i(y_i))$ is easily built

Model 2

A model for a preference $\succsim$ that is a **complete preorder** :

$$x \succsim y \quad \Leftrightarrow \quad u(x) = u(y)$$
$$\Leftrightarrow \quad G(u_i(x_i),\ i = 1, \ldots, n) \ \geq\ G(u_i(y_i), i = 1, \ldots, n)$$

for some functions $u_i : X_i \to \mathbb{R}$
and some function $G : \mathbb{R}^n \to \mathbb{R}$

Model 2 $\Leftrightarrow$ Model 0 with :

$$F(u_i(x_i), u_i(y_i)) \ = \ G(u_i(x_i),\ i = 1, \ldots, n) \ - \ G(u_i(y_i), i = 1, \ldots, n)$$

Model $1 \wedge 2$

Result :

$$\succsim \text{ is a complete preorder and satisfies AC123}$$

$$\Leftrightarrow$$

$$\exists\, u_i : X_i \;\rightarrow\; \mathbb{R}, \qquad \exists\, G : \mathbb{R}^n \;\rightarrow\; \mathbb{R}$$

$$G(\nearrow) \text{ (non-decreasing)}$$

$$\text{such that}$$

$$x \succsim y \;\Leftrightarrow\; G(u_i(x_i),\; i = 1, \ldots, n) \;\geq\; G(u_i(y_i), i = 1, \ldots, n)$$

Examples :

- $G = \sum_i u_i(x_i)$ (MAU)
- $G = \max_i \{u_i(x_i)\}$

Back to the example FCSP

. . . and coherent families

What if the X_i are ordered *a priori* ?

If a preference $\succsim$ is built that is a complete preorder and respects dominance, then the relations $\succsim_i^{\pm}$ that can be deduced from $\succsim$ are compatible with the complete preorders given *a priori*.

Conclusions and future research

- Framework in which specific methods can be further specified (max, MAU, …)
- There are further interesting “sub frameworks” : e.g. the case where F (or G) is **strictly** monotonic (already characterised)
- Advantages of such a framework :
  - understanding
  - suggests methods of elicitation
  - allows for choosing between several specific procedures in a “family”