On consistent families of criteria:
An axiomatic approach

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Outline

• Introduction: Consistent families in the sense of Roy

• Example: Ranking vectors of constraint satisfaction degrees

• A general model
  – Characterisation
  – Interpretation
  – Back to the example

• Discussion and future work
Introduction

Families of criteria (Roy, 1985)

• Definition of a criterion: \( g_i : A \rightarrow \mathbb{R} \)

  interpretation of \( g_i(a) \geq g_i(b) \): \( a \) is at least as good as \( b \)
  on point of view \( i \)

• \( n \) points of view: \( i = 1, \ldots, n \)

• Notation for the global preference on \( A \) (to be constructed by applying some method): \( \succeq \)
Consistent families

Properties to be fulfilled

- Exhaustivity:
  \[ g_i(a) = g_i(b), \forall i \] \implies a \sim b

- Consistency:
  \[
  \begin{align*}
  a \succeq b \\
  g_i(c) \geq g_i(a), \forall i \\
  g_i(b) \geq g_i(d), \forall i
  \end{align*}
  \]
  \implies c \succeq d

- Non-redundancy
Interpretation and Question

Interpretation . . .

. . . of consistency: The relationship between the criteria and the constructed preference $\preceq$ is supposed to be monotonic

Alternative formulation: respect of dominance

Question:

Is it possible to characterise the preferences that are consistent with a family of criteria?
Example

Comparing vectors of satisfaction levels

**FCSP :** Flexible Constraint Satisfaction Problem

- vertices = tasks
- arcs = constraints $C_i, i = 1, \ldots, n$
- solution = a feasible schedule for the tasks
- quality of solution : a vector of satisfaction levels

$$x = (x_1, \ldots, x_n)$$

- problem : comparing the solutions

General problem of “vector optimisation” : no evident complete ordering of the solutions
Dominance

A natural partial order: the dominance relation $D$ (Pareto ordering)

$$x D y \text{ if } x_i \geq y_i, \forall i$$

Note: in general $x_i$ belongs to a set $X_i$ that is at least ordered

**Question**: Characterise “natural and operational methods” leading to a complete ordering of the vectors and preserving dominance (question raised by Dubois and Prade)
A framework

- $X = X_1 \times X_2 \times \ldots \times X_n$: finite set of alternatives
- $N = \{1, 2, \ldots, n\}$: set of attributes
- $x = (x_1, x_2, \ldots, x_n) \in X$
- $x_{-i} = (x_1, \ldots, x_{i-1}, x_{i+1}, \ldots, x_n)$
- $(a_i, x_{-i}) \in X$

- $\succeq$ reflexive binary relation on $X$ interpreted as “large preference”
- $\sim$: symmetric part; $\succ$: asymmetric part

Note: there is no structure assumed on $X_i$
Model 0

\[ x \succcurlyeq y \iff F((u_i(x_i), u_i(y_i)), i = 1, \ldots, n) \geq 0 \]

for some functions \( u_i : X_i \rightarrow \mathbb{R} \)
and some function \( F : \mathbb{R}^{2n} \rightarrow \mathbb{R} \)

Note: no properties assumed on functions \( F \) and \( u_i \)

**Result:** any relation \( \succcurlyeq \) on \( X \) satisfies Model 0

Note: \( u_i \) induces an ordering \( \succcurlyeq_i^\pm \) on \( X_i \):

\[ x_i \succcurlyeq_i^\pm y_i \iff u_i(x_i) \geq u_i(y_i) \]

This ordering plays no role (is arbitrary) when \( F \) has no special property
Model 1

\[ x \succeq y \iff F((u_i(x_i), u_i(y_i)), i = 1, \ldots, n) \geq 0 \]

for some functions \( u_i : X_i \to \mathbb{R} \)
and some function \( F : \mathbb{R}^{2n} \to \mathbb{R} \)
such that

\[ F(\% , \& ) \]

i.e. \( F \) is non-decreasing in its first \( n \) coordinates
and non-increasing in its last \( n \) coordinates:

\textbf{Property :} \( \succeq \) is monotonic (consistent) with respect to the orders \( \succeq_i \)
Indeed:

\[
\begin{align*}
x & \preceq y \\
u_i(z_i) & \geq u_i(x_i) \quad \forall i \\
u_i(y_i) & \geq u_i(w_i) \quad \forall i
\end{align*}
\]

\[
\Rightarrow z \preceq w
\]

\[
\begin{align*}
F(u_i(x_i), u_i(y_i)) & \geq 0 \\
F(\text{min}, \&) \\
\Rightarrow F(u_i(z_i), u_i(w_i)) & \geq 0
\end{align*}
\]
Characterisation of Model 1

**Result**: the relations that satisfy Model 1 are those verifying properties AC123

\[ AC_{1_{i}} : \]
\[
(x_{i}, a_{-i}) \preceq (y_{i}, b_{-i}) \quad \text{and} \quad (z_{i}, a_{-i}) \preceq (y_{i}, b_{-i}) \]
\[ \Rightarrow \left\{ \begin{array}{l}
(z_{i}, c_{-i}) \preceq (w_{i}, d_{-i}) \\
(x_{i}, c_{-i}) \preceq (w_{i}, d_{-i})
\end{array} \right. \]

\[ \text{or} \]
\[
(x_{i}, a_{-i}) \preceq (y_{i}, b_{-i}) \quad \text{and} \quad (z_{i}, c_{-i}) \preceq (w_{i}, d_{-i}) \]

**Definition of \( \preceq_{i}^{+} \):**

\[ x_{i} \preceq_{i}^{+} y_{i} \iff [(y_{i}, a_{-i}) \preceq z \Rightarrow (x_{i}, a_{-i}) \preceq z] \]

**Result 1**: \( AC_{1_{i}} \) iff \( \preceq_{i}^{+} \) is a complete preorder
Result 2: AC2_\text{i} \iff \preceq_\text{i}^- is a complete preorder with

\[ x_i \preceq_\text{i}^- y_i \iff [z \succeq (x_i, a_{-i}) \Rightarrow z \succeq (y_i, a_{-i})], \]

Result 3: AC123_\text{i} \iff \preceq_\text{i}^{\pm} is a complete preorder with

\[ x_i \preceq_\text{i}^{\pm} y_i \iff [x_i \preceq_\text{i}^{+} y_i \text{ and } x_i \preceq_\text{i}^{-} y_i], \]

Consequence:
If \succeq satisfies AC123,

- the relations \preceq_\text{i}^{\pm} are complete preorders
- \( u_i \) can be chosen to be any numerical representation of \( \preceq_\text{i}^{\pm} \)
- \( F(u_i(x_i), u_i(y_i)) \) is easily built
Model 2

A model for a preference \( \succsim \) that is a **complete preorder**:

\[
\begin{align*}
x \succsim y & \iff u(x) = u(y) \\
& \iff G(u_i(x_i), i = 1, \ldots, n) \geq G(u_i(y_i), i = 1, \ldots, n)
\end{align*}
\]

for some functions \( u_i : X_i \to \mathbb{R} \)
and some function \( G : \mathbb{R}^n \to \mathbb{R} \)

Model 2 \( \iff \) Model 0 with:

\[
F(u_i(x_i), u_i(y_i)) = G(u_i(x_i), i = 1, \ldots, n) - G(u_i(y_i), i = 1, \ldots, n)
\]
Model 1 ∧ 2

Result:

≿ is a complete preorder and satisfies AC123

⇔

∃ u_i : X_i → \mathbb{R}, ∃ G : \mathbb{R}^n → \mathbb{R}

G( % ) (non-decreasing)

such that

x ≿ y ⇔ G(u_i(x_i), i = 1, \ldots, n) \geq G(u_i(y_i), i = 1, \ldots, n)

Examples:

• G = \sum_i u_i(x_i) (MAU)

• G = \max_i \{u_i(x_i)\}
Back to the example FCSP

... and coherent families

What if the $X_i$ are ordered *a priori*?

If a preference $\succeq$ is built that is a complete preorder and respects dominance, then the relations $\succeq_i^{\pm}$ that can be deduced from $\succeq$ are compatible with the complete preorders given *a priori*. 
Conclusions and future research

• Framework in which specific methods can be further specified (max, MAU, … )

• There are further interesting “sub frameworks” : e.g. the case where $F$ (or $G$) is strictly monotonic (already characterised)

• Advantages of such a framework :
  – understanding
  – suggests methods of elicitation
  – allows for choosing between several specific procedures in a “family”