On some properties of outranking relations based on a concordance-discordance principle

Denis Bouyssou

Abstract. The purpose of this paper is to study some properties of outranking relations based on a concordance-discordance principle. We show that, whenever the structure of the set of alternatives is sufficiently rich, imposing "nice" transitivity properties on such outranking relations always leads to a somewhat unappealing distribution of "power" among the various attributes. These results directly apply to methods, such as TACTIC, that produce a crisp asymmetric outranking relation. We explore the links between these results and classical ones in the field of Social Choice Theory and show their relevance for users of outranking methods.

I- Introduction

A classical problem in the field of MCDM is to build a preference relation on a set of multiattributed alternatives on the basis of preferences expressed on each attribute and "inter-attribute" information such as weights and/or tradeoffs. A common way (see Keeney and Raiffa (1976)) to do so is to attach a number v(x) to each alternative x and to declare that x is preferred to y if and only if v(x) > v(y). Usually, the number v(x) depends on the evaluations $x_1, x_2, ..., x_n$ of x on the n attributes and we have $v(x) = V(x_1, x_2, ..., x_n)$. When one uses such a method, the preference relation that is built has "nice" transitivity properties. However, the definition of the aggregation function V may not always be an easy task (see, *e.g.*, Roy and Bouyssou (1987)).

Starting with ELECTRE I (see Roy (1968) or, for a presentation in English, Goicoechea *et al.* (1982)), a number of MCDM techniques, the so-called outranking methods, have been proposed that use an alternative way to build a preference relation based on a concordance-discordance principle (see, *e.g.*, Roy and Bertier (1973), Vansnick (1986) and the bibliography of Siskos *et al.* (1983)). In these methods, the preference relation, which is often called an outranking relation, is built through a series of pairwise comparisons. Such pairwise comparisons can be done in many ways. The idea of concordance-discordance consists in declaring that an alternative x is preferred to an alternative y if a "majority" of the attributes supports this assertion (concordance condition) and if the opposition of the other attributes is not "too strong" (non-discordance condition). In this paper we will restrict our attention to methods aiming at building a crisp (*i.e.*, nonfuzzy) and, for reasons to be explained in section 4, asymmetric preference relation[1].

---

[1] A (crisp) binary relation S on a set K is a subset of K^2. Throughout the paper we will classically write a S b instead of $(a, b) \in S$. We say that a binary relation S on a set K is (for all $a, b, c \in K$):
- complete if a S b or b S a,
- asymmetric if a S b implies Not (b S a),
- transitive if a S b and b S c imply a S c,
- negatively transitive if Not(a S b) and Not(b S c) imply Not(a S c),

It is without circuit if for all $k \geq 1$ and all $a_1, a_2, ..., a_k \in K$, $a_1 S a_2$, $a_2 S a_3$, ..., $a_{k-1} S a_k$ imply Not $a_k S a_1$.

In order to be more specific, suppose that we have defined on each attribute i a complete and transitive binary relation R_i allowing to compare in terms of preference the evaluations of the alternatives on this attribute[2]. When comparing two alternatives x and y, it is possible to partition the set of attributes N between attributes favoring x, attributes favoring y and "neutral" attributes, *i.e.*, $P(x, y) = \{i \in N : x_i P_i y_i\}$, $P(y, x) = \{i \in N : y_i P_i x_i\}$ and $I(x, y) = I(y, x) = \{i \in N : x_i I_i y_i\}$. Using the idea of concordance, it is declared that x is preferred to y when the "coalition" of attributes in P(x, y) (or in $P(x, y) \cup I(x, y)$) is considered more important than the "coalition" of attributes in P(y, x). For practical purposes, the importance of a coalition of attributes is usually determined in an additive way after having attached a weight p_i to each attribute. For instance, in TACTIC (Vansnick (1986)), we have, for the concordance part of the method:

$$x \, P \, y \Leftrightarrow \sum_{i \in P(x, y)} p_i > \rho \sum_{i \in P(y, x)} p_i \qquad (1)$$

where ρ is a threshold greater than 1.

It is worth noting that preference relations based on the idea of concordance do not make use of the magnitude of the "differences" between evaluations and are only based on the ordinal information conveyed by the relations R_i.

This idea of concordance may however be criticized since on some attributes in P(y, x) the difference between y_i and x_i may be so large as to cast a major doubt on the validity of x P y even if these attributes are of limited importance. One simple way to circumvent this problem is to combine the idea of concordance with that of discordance. Taking discordance into account amounts to defining a set D of ordered pairs of evaluations on the various attributes such that $(y_i, x_i) \in D$ for some $i \in N$ implies Not(x P y) regardless of the comparison of the importance of the various coalitions of attributes as modelled using the idea of concordance. This set D is often defined through a binary relation $V_i \subset P_i$ that reads "is very strongly preferred to" such that $(y_i, x_i) \in D \Leftrightarrow y_i V_i x_i$. Thus taking the idea of discordance into account leads to a poorer relation that the one that would have been obtained using concordance alone. For instance, we have in TACTIC:

$$x \, P \, y \Leftrightarrow \sum_{i \in P(x, y)} p_i > \rho \sum_{i \in P(y, x)} p_i \text{ and Not}(y_i V_i x_i) \text{ for all } i \in P(y, x) \qquad (2)$$

The interest of such methods and the way to assess the weights p_i and the relations V_i have been discussed elsewhere (see, *e.g.*, Roy (1971) or Roy and Vincke (1981)). It should be emphasized that, contrary to methods attaching a number to each alternative, the preference

[2] We respectively note P_i and I_i the asymmetric and symmetric parts of R_i *i.e.* [$x_i P_i y_i$ iff $x_i R_i y_i$ and Not $y_i R_i x_i$] and [$x_i I_i y_i$ iff $x_i R_i y_i$ and $y_i R_i x_i$], a similar convention holding for all binary relations used in this paper.

relation that is obtained in that kind of methods may not possess "nice" transitivity properties[3]. The purpose of this paper is to study under what conditions these relations may possess such properties as transitivity or absence of circuit. We present our notations and definitions in the next section. In section 3, we investigate some properties of relations that are only based on the idea of concordance. In section 4, we generalize our results so as to take the idea of discordance into account. In a final section we analyze the links between these results and more classical ones in the Theory of Social Choice and stress their relevance for users of outranking methods..

II- Definitions and Notations

Let N be a set of attributes. To each attribute $i \in N$, we associate a set X_i which will be interpreted as a set of possible levels on attribute i and a binary relation R_i on X_i. Let X be a set of alternatives such that $X \subset \prod_{i \in N} X_i$. When there will be no risk of confusion, x_i will designate the element of X_i being the ith component of $x \in X$.

Throughout the paper we suppose that the following structural conditions hold:

S1 R_i is complete and P_i is transitive,

S2 N is finite and $|N| = n \geq 2$.

Condition S1 implies a minimal consistency requirement on the preferences on each attribute. It should be noticed that S1 is compatible with a semiorder or an interval order structure on each X_i (on these notions see, *e.g.*, Roubens and Vincke (1985)). Condition S2 is hardly restrictive in the context of MCDM.

We will also use a structural condition aiming at introducing a minimal diversity among the alternatives to be compared. For k = 2, 3, ..., we consider:

D(k) for all $i \in N$, there is a set $Y_i \subset X_i$ with $|Y_i| = k$ such that for all $a_i, b_i \in Y_i$ with $a_i \neq b_i$ either $a_i \, P_i \, b_i$ or $b_i \, P_i \, a_i$, furthermore $Y = \prod_{i \in N} Y_i \subset X$.

Condition D(k) implies that on each attribute it possible to find k levels that can be distinguished in terms of strict preference and that all possible combinations of these k levels are in X. This condition may seem very restrictive. It should however be noticed that the alternatives which have to be in X if D(k) holds are very similar to the "artificial" alternatives that are used in many methods to assess inter-attribute information (see, *e.g.*, Keeney and Raiffa (1976) or Roy *et al.* (1986)). Furthermore, the set of alternatives has to be rich enough if we want such properties as transitivity or absence of circuit to be meaningful. Therefore we will use this condition throughout the paper though weaker diversity conditions could sometimes have been used at the cost of a greater complexity. As it will become apparent in the next sections, this

[3] This not to say that these relations are useless for decision-aid purposes. A number of techniques have been devised in order to rank the alternatives or to choose one of them on the basis of such relations, see, *e.g.*, Roy and Vincke (1981).

structural condition plays a crucial part in our proofs in allowing to transfer classical "multi-profile" results into a "single profile" context (see section 5). Most of our results use D(3).

An aggregation procedure in MCDM is a rule allowing to build a preference relation on X on the basis of the relations R_i on X_i and inter-attribute information. As already mentioned, we restrict our attention here to methods producing an asymmetric relation P on X.

A usual condition relating P and the R_i is:

U for all $x, y \in X$, [$x_i \, P_i \, y_i$ for all $i \in N$] $\Rightarrow x \, P \, y$.

Condition U is a hardly controversial unanimity condition for strict preference. A slightly more restrictive unanimity condition is:

U* for all $x, y \in X$, [$x_i \, R_i \, y_i$ for all $i \in N$ and $x_j \, P_j \, y_j$ for some $j \in N$] $\Rightarrow x \, P \, y$.

As soon as all weights p_i are strictly positive it is easy to see that, when a relation P is obtained by (1), it satisfies U* and, thus, U. We are not aware of any MCDM aggregation procedure that does not satisfy U. However, as exemplified by ELECTRE I (see Roy (1968)), some procedures may fail to satisfy U*.

The following condition aims at capturing a crucial property underlying the idea of concordance:

NC for all $x, y, z, w \in X$, [$x_i \, R_i \, y_i \Leftrightarrow z_i \, R_i \, w_i$ and $y_i \, R_i \, x_i \Leftrightarrow w_i \, R_i \, z_i$ for all $i \in N$] $\Rightarrow$ [$x \, P \, y \Rightarrow z \, P \, w$].

Condition NC has been introduced by Fishburn (1975 and 1976) under the name of noncompensation. It has been studied by Bouyssou and Vansnick (1986) and Bouyssou (1986). This condition implies that preference among two alternatives x and y only depends on the subsets of N for which $x_i \, R_i \, y_i$ and $y_i \, R_i \, x_i$ and seems at the heart of the idea of concordance. It is not difficult to see that when a relation P is obtained by (1) it satisfies condition NC.

Condition NC does not convey any notion of monotonicity which seems a crucial part of any aggregation procedure. Combining NC with an idea of monotonicity, we obtain:

M for all $x, y, z, w \in X$, [$x_i \, P_i \, y_i \Rightarrow z_i \, P_i \, w_i$ and $x_i \, I_i \, y_i \Rightarrow z_i \, R_i \, w_i$ for all $i \in N$] $\Rightarrow$ [$x \, P \, y \Rightarrow z \, P \, w$].

Condition M implies that, when x is strictly preferred to y, if the set of attributes for which there is a strict preference for x over y is enlarged then x remains strictly preferred to y regardless of what is happening on the other attributes. It is easy to see that condition M implies condition NC. When a binary relation P is obtained by (1), it is not difficult to see that it satisfies condition M.

Suppose that $\underline{P}$ is a preference relation obtained using the idea of concordance. Taking discordance into account leads to a relation $P \subset \underline{P}$, *i.e.* a relation in which a number of preferences have been deleted. Generalizing the conditions we just introduced, we obtain:

WNC for all $x, y, z, w \in X$, [$x_i \, R_i \, y_i \Leftrightarrow z_i \, R_i \, w_i$ and $y_i \, R_i \, x_i \Leftrightarrow w_i \, R_i \, z_i$ for all $i \in N$] $\Rightarrow$ [$x \, P \, y \Rightarrow$ Not($w \, P \, z$)].

WM for all $x, y, z, w \in X$, [$x_i \, P_i \, y_i \Rightarrow z_i \, P_i \, w_i$ and $x_i \, I_i \, y_i \Rightarrow z_i \, R_i \, w_i$ for all $i \in N$] $\Rightarrow$ [$x \, P \, y \Rightarrow$ Not($w \, P \, z$)].

It is easy to see that [WM $\Rightarrow$ WNC] and that if a binary relation $\underline{P}$ satisfies condition NC (resp. M) then all binary relations P such that $P \subset \underline{P}$ satisfy condition WNC (resp. WM)[4].

III- Properties of outranking relations based on concordance

The purpose of this section is to investigate some consequences of conditions NC and M when they are coupled with some particular properties of P. It is easy to see that a binary relation P built using relation (1) satisfies U and M (and, thus NC). Simple numerical examples inspired from Condorcet's paradox show that, in general, such a relation will not be transitive and will have circuits. Two simple ways to avoid these phenomena can be envisaged.

The first one is to chose the threshold ρ very close to one so that x P y is obtained only when all the attributes are unanimous to support this proposition. Such a solution is however extremely ineffective and leads to a relation P that is very poor.

The second one consists in giving a very large weight to a particular criterion so that the relation P more or less coincides with the preference relation on that criterion. This is not a very attractive solution however since it amounts to amounts to giving much power to a single criterion. Of course many other ways to obtain a relation P with "nice" transitivity properties can be envisaged. The following results show that, when the structure of X is sufficiently rich, imposing "nice" transitivity properties on P always leads to a somewhat unappealing distribution of power among the attributes. We sum up our results in:

Theorem 1. Given S1-S2,

(a) [D(3), NC, U, P is asymmetric and negatively transitive] $\Rightarrow$
there is an $i \in N$ such that for all $x, y \in X$, $x_i\ P_i\ y_i \Rightarrow x\ P\ y$.

(b) [D(3), NC, U, P is asymmetric and transitive] $\Rightarrow$
there is a unique $O \subset N$ such that for all $x, y \in X$:
$x_i\ P_i\ y_i$ for all $i \in O \Rightarrow x\ P\ y$
$x_j\ P_j\ y_j$ for some $j \in O \Rightarrow$ Not(y P x).

(c) [D(n), M, P is without circuit] $\Rightarrow$
there is an $i \in N$ such that for all $x, y \in X$, $x_i\ P_i\ y_i \Rightarrow$ Not(y P x).

In proving theorem 1, the following definitions will be useful. We say that $A \subset N$ is

- decisive if, for all $x, y \in X$, $x_i\ P_i\ y_i$ for all $i \in A \Rightarrow x\ P\ y$,
- almost decisive if, for all $x, y \in X$, $x_i\ P_i\ y_i$ for all $i \in A$ and $y_j\ P_j\ x_j$ for all $j \notin A \Rightarrow x\ P\ y$,
- semi-decisive if, for all $x, y \in X$, $x_i\ P_i\ y_i$ for all $i \in A \Rightarrow$ Not(y P x),
- almost semi-decisive if, for all $x, y \in X$, $x_i\ P_i\ y_i$ for all $i \in A$ and $y_j\ P_j\ x_j$ for all $j \notin A \Rightarrow$ Not(y P x).

We state without proof the following obvious facts as:

[4] It is worth noting that this would not be the case if discordance were applied to a binary relation satisfying NC but not asymmetric, as this is the case for example in ELECTRE I.

Lemma 1.

(a) M $\Rightarrow$ NC $\Rightarrow$ [x P y (resp. Not(y P x) for some x, y $\in$ X, such that x_i P_i y_i for all i $\in$ A and y_j P_j x_j for all j $\notin$ A $\Rightarrow$ A is almost decisive (resp. almost semi-decisive)]

(b) M $\Rightarrow$ [A is almost semi-decisive $\Rightarrow$ A is semi-decisive].

We also have:

Lemma 2.

For all A $\subset$ N, [S1, S2, NC, U, D(3), P is asymmetric and transitive] $\Rightarrow$ [A is almost decisive (resp. almost semi decisive) $\Rightarrow$ A is decisive (resp. semi decisive)].

Proof of lemma 2.

If A = N, the conclusion follows. If not, use D(3) with $Y_i = \{a_i, b_i, c_i\}$ and a_i P_i b_i and b_i P_i c_i to construct the following alternatives:

	A	N\A
x	a_i	d_i
y	b_i	a_i
z	c_i	e_i

where a_i P_i d_i and a_i P_i e_i. Such alternatives are in X by D(3). By U we get y P z.

Suppose that A is almost decisive. We thus have x P y and the transitivity of P leads to x P z, independently of the comparison of d_i and e_i on the attributes of N\A. In view NC, this proves that A is decisive.

Suppose now that A is almost semi decisive. If z P x, the transitivity of P implies y P x contradicting the fact that A is almost semi-decisive. We thus have Not(z P x), independently of the comparison of d_i and e_i on the attributes of N\A. In view NC this proves that A is semi-decisive. □

Proof of theorem 1.

Proof of part (a).

We prove that if A is decisive and |A| >1 then some proper subset of A is decisive. In order to do so let B be a proper subset of a decisive set A and use D(3) with $Y_i = \{a_i, b_i, c_i\}$ and a_i P_i b_i and b_i P_i c_i to construct the following alternatives:

	B	A\B	N\A
x	a_i	c_i	b_i
y	b_i	a_i	c_i
z	c_i	b_i	a_i

A being decisive, we have y P z. If x P z then NC implies that B is almost decisive and, by lemma 2, decisive (since asymmetry and negative transitivity imply transitivity). If Not(x P z) then Not(y P x) would imply by negative transitivity Not(y P z) violating the fact that A is decisive. Thus, we have y P x. Given NC and lemma 2, we conclude in this case that A\B is decisive.

By U we know that N is decisive. Repeating the previous argument leads to the conclusion that a singleton is decisive, completing the proof of part (a).

Proof of part (b).

By U, we know that N decisive. Given S2, there exists at least one decisive set of minimal cardinality. Let O be one of them. We have $x_i \; P_i \; y_i$ for all $i \in O \Rightarrow x \; P \; y$.
If $|O| = 1$, then we obviously have that $x_j \; P_j \; y_j$ for some $j \in O \Rightarrow$ Not(y P x).
If not consider $\{i\} \subset O$ and use D(3) with $Y_i = \{a_i, b_i, c_i\}$ and $a_i \; P_i \; b_i$ and $b_i \; P_i \; c_i$ to construct the following alternatives:

	$\{i\}$	$O\backslash\{i\}$	$N\backslash O$
x	c_i	a_i	b_i
y	a_i	b_i	c_i
z	b_i	c_i	a_i

O being decisive, we have y P z. If x P z, then, given NC, $O\backslash\{i\}$ is almost decisive and thus decisive, by lemma 2, violating the fact that O is a decisive set of minimal cardinality . We thus have Not(x P z). But this implies Not(x P y) since x P y, y P z and Not(x P z) would contradict the transitivity of P. Given NC, Not(x P y) implies that $\{i\}$ is almost semi-decisive and thus semi-decisive by lemma 2. Therefore all singletons in O are semi-decisive.
The proof of (b) is completed observing that O is necessarily unique. In fact suppose that there are two sets O and O' with $O \neq O'$ satisfying the conclusions of (b). Consider the following alternatives which by D(3) are in X:

	O	$O'\backslash O$	$N\backslash O \cup O'$
x	a_i	b_i	a_i
y	b_i	a_i	a_i

We have, by construction, x P y and Not(y P x), a contradiction.

Proof of part (c).

Suppose, in contradiction with the thesis, that no singleton is semi-decisive. Given M and in view of part (b) of lemma 1, this implies that no singleton is almost semi-decisive, *i.e.* that:
for all $i \in N$, $x_i \; P_i \; y_i$ and $y_j \; P_j \; x_j$ for all $j \neq i$ imply y P x.
We use D(n) to construct the following n alternatives with $Y_i = \{x_i^1, x_i^2, ..., x_i^n\}$ and $x_i^1 \; P_i \; x_i^2$, $x_i^2 \; P_i \; x_i^3 ...$, $x_i^{n-1} P_i \; x_i^n$,

	$\{1\}$	$\{2\}$	$\{3\}$	…	$\{n\}$
x^1	x_1^1	x_2^n	x_3^{n-1}		x_n^2
x^2	x_1^2	x_2^1	x_3^n		x_n^3
x^3	x_1^3	x_2^2	x_3^1		x_n^4
…					
x^n	x_1^n	x_2^{n-1}	x_3^{n-2}		x_n^1

We have $x^1 \; P \; x^2$, $x^2 \; P \; x^3$, ..., $x^n \; P \; x^1$, which violates the fact that P has no circuits and completes the proof of part (c). □

Theorem 1 and its proof have strong connections with classical results in Social Choice Theory that will be explored in section 5. We briefly comment here each part of theorem 1.
(a). This result says that if a concordance relation is a weak order then some attribute dictates its strict preferences to all others. Given D(3), this distribution of "power" among the various attributes seems very undesirable. Very similar results have been proved by Fishburn (1975), Plott *et al.* (1975), Roberts (1980), Pollack (1979), Parks (1976) and Kemp and Ng (1976).

As its proof suggests, this result appears as a MCDM counterpart of Arrow's theorem that takes advantage of the structure of cartesian product of the set of alternatives[5]. Fishburn (1975) proves that if U is replaced by U* then P is obtained lexicographically *i.e* there is a linear order >> on N such that for all $x, y \in X$:

$x \, P \, y \Leftrightarrow [x_i \, P_i \, y_i$ for some $i \in N$ and for every $k \in N$ such that $y_k \, P_k \, x_k$ there is a $j \in N$ such that $j >> k$ and $x_j \, P_j \, y_j]$.

When D(3) is strengthened to D(4), it is easy to see that it is possible to replace [P is asymmetric and negatively transitive] by [P is asymmetric and for all $x, y, z, w \in X$, $x \, P \, y$ and $y \, P \, z \Rightarrow x \, P \, w$ or $w \, P \, z$] or by [P is asymmetric and for all $x, y, z, w \in X$, $x \, P \, y$ and $z \, P \, w \Rightarrow x \, P \, w$ or $z \, P \, y$] without altering the conclusion. Thus asking for a semi-order or even an interval order instead of a weak order as the result of an aggregation satisfying U and NC does not change the situation.

(b). This result is a single profile counterpart of a result of Weymark (1983). It says the transitivity of P together with D(3), NC and U generates what is usually called an "oligarchy" of attributes. A good example of this situation is offered by the (strict) dominance relation defined by:

$x \, P \, y \Leftrightarrow x_i \, P_i \, y_i$ for all $i \in N$.

With this relation the oligarchy is the entire set N. Smaller oligarchies lead to a richer relation P at the cost of giving more "power" to a smaller number of attributes.

(c). This result is a single profile counterpart of a result of Blau and Deb (1977). It shows that an asymmetric concordance relation without circuit gives much power to a single attribute when the structure of X is very rich[6].

IV- Properties of outranking relations based on concordance-discordance

Suppose that $\underline{P}$ is a preference relation on X obtained using an idea of concordance. Taking discordance into account leads to a relation $P \subset \underline{P}$. In general, discordance may impoverish $\underline{P}$ in rather an uncontrollable way. For instance, the transitivity of $\underline{P}$ may be destroyed in P. On the contrary, if $\underline{P}$ has circuits, it may happen than discordance destroys all these circuits.

The only thing that seems reasonable to ask on P is the absence of circuit. This is the case if $\underline{P}$ has no circuit[7]. But, as we mentioned, it may also happen that P has no circuit because all the

[5] It should be noted that a similar result holds if we suppose that we are working on a binary relation R that is complete and transitive when the last part of NC is changed to [$x \, R \, y \Rightarrow z \, R \, w$], which is more in line with the usual presentation of Arrow's theorem.

[6] Such a rich structure for X can be criticized. Using D(3), a similar result could be obtained strengthening condition M to a condition of "positive responsiveness" leading to a single profile counterpart of a result of Mas-Colell and Sonnenschein (1972).

[7] Note that this is not so if discordance is applied to a relation that is not asymmetric. In that case, discordance may create as well as destroy circuits in the asymmetric part of the relation. This the reason why we restrict our attention here to asymmetric relations.

circuits in P have been destroyed by discordance. It is therefore easy to see that imposing that P has no circuit coupled with WNC or WM will not create results similar to those of section 3. One simple way, among others, to obtain similar results is to limit the extent of discordance. For instance we may impose that there is no discordance between the elements of $Y = \prod_{i \in N} Y_i$ where the sets Y_i are those used in the diversity condition D(k). This amounts to supposing that, though the elements of Y_i can be distinguished in terms of strict preference, they are sufficiently "close" to one another not to generate discordance effects. This hypothesis seems in line with the way "artificial" alternatives are introduced in X in order to assess inter-attribute information (see, *e.g.*, Roy *et al.* (1986)). Therefore we reformulate conditions WM as:

WM* for all $x, y, z, w \in X$, $[x_i P_i y_i \Rightarrow z_i P_i w_i$ and $x_i I_i y_i \Rightarrow z_i R_i w_i$ for all $i \in N] \Rightarrow$ $[x P y \Rightarrow \text{Not}(w P z)$, furthermore if $z, w \in Y$ then $z P w]$.

Though this condition may seem *ad hoc*, it has a simple interpretation. If a relation P has been obtained by applying discordance to a relation satisfying M then it satisfies WM* if there is no discordance between the elements of Y. We have the following:

Theorem 2.

Given S1-S2,

[D(n), WM*, P is without circuit] $\Rightarrow$

there is an $i \in N$ such that for all $x, y \in X$, $x_i P_i y_i \Rightarrow \text{Not}(y P x)$.

Proof of theorem 2.

It is easy to see that if P satisfies WM* on X then it satisfies M on Y. Since P has no circuit on Y, we know from theorem 1 (c) that there is an $i \in N$ such that for all $x, y \in Y$, $x_i P_i y_i \Rightarrow \text{Not}(y P x)$. Suppose now that for some $z, w \in X$, we have $z_i P_i w_i$ and $w P z$. From D(3), we know that there are $r, s \in Y$ such that $r_i R_i s_i \Leftrightarrow z_i R_i w_i$ and $s_i R_i r_i \Leftrightarrow w_i R_i z_i$ for all $i \in N$. Therefore, WM* implies that $s P r$, a contradiction. This completes the proof. □

V- Relation with Social Choice Theory.

As already argued, the results of section 3 are transpositions of classical results in the field of Social Choice Theory. In order to understand better the extent of this transposition, it is worth recalling here the classical result of Arrow.

A central theme in Social Choice Theory is to study how the preferences of several individuals can be aggregated in a "reasonable" way. Let X be a set of objects called "alternatives" and N a finite set, the $|N| = n$ elements of N being interpreted as "voters". We define $\mathcal{R}_X$ as the set of all binary relations on X. A Social Aggregation Procedure is a function:

$$G : E \subset [\mathcal{R}_X]^n \to F \subset \mathcal{R}_X$$

$$(R_1, R_2, ..., R_n) \mapsto G(R_1, R_2, ..., R_n) = R$$

associating[8] a binary relation on X to n-uples of binary relations on X.

[8] in the sequel, it is understood that $R = G(R_1, R_2, ..., R_n)$ and $R' = G(R_1', R_2', ..., R_n')$. As before, P denotes the asymmetric part of R.

Let R_X be the set all all complete and transitive binary relations on X. We introduce the following conditions:

D $E = [R_X]^n$ [Domain]

C $F = R_X$ [Codomain]

I for all $(R_1, R_2, ..., R_n), (R_1', R_2', ..., R_n') \in E$, and for all $x, y \in X$,
$[x\ R_i\ y \Leftrightarrow x\ R_i'\ y$ and $y\ R_i\ x \Leftrightarrow y\ R_i'\ x$ for all $i \in N] \Rightarrow$
$[x\ R\ y \Leftrightarrow x\ R'\ y$ and $y\ R\ x \Leftrightarrow y\ R'\ x]$ [Independence]

UN for all $(R_1, R_2, ..., R_n)$ E, and for all $x, y \in X$, [Unanimity]
$[x\ P_i\ y$ for all $i \in N] \Rightarrow x\ P\ y$

ND for all $i \in N$, [Nondictatorship]
$x\ P_i\ y$ and Not$(x\ P\ y)$ for some $(R_1, R_2, ..., R_n) \in E$ and some $x, y \in X$

In this context we have the following:

Theorem (Arrow (1963)): When $|X| \geq 3$, there is no Social Aggregation Procedure satisfying D, C, I, UN and ND.

The relations between Arrow's result and part (a) of theorem 1 have been explored by many authors (*e.g.*, Fishburn (1975), Roberts (1980), Pollack (1979), Parks (1976) and Kemp and Ng (1976)). It will suffice to recall here that, in the framework of Arrow's theorem, all possible profiles of weak orders are in the domain of the Social Aggregation Procedure and that condition I relates the result of the aggregation between two different profiles. This multi-profile formulation is at the crux of the "impossibility" result. On the contrary, in theorem 1 only one particular profile of preference relation on each attribute is used. But D(k) requires the set X to contain alternatives for which the preferences on each attribute are conflictual. This diversity together with NC lead to the results of theorem 1. Thus, condition D(k) appears as the counterpart of condition D in the single-profile context.

Both theorems use a similar unanimity condition (U and UN) and there is an obvious correspondence between condition C in Arrow's theorem and the requirement that P be asymmetric and negatively transitive in theorem 1 (a). It is worth observing that when I is coupled with UN, C and D, it implies the following "neutrality" condition:

for all $(R_1, R_2, ..., R_n), (R_1', R_2', ..., R_n') \in E$, and for all $x, y, z, w \in X$,
$[x\ R_i\ y \Leftrightarrow z\ R_i'\ w$ and $y\ R_i\ x \Leftrightarrow w\ R_i'\ z$ for all $i \in N] \Rightarrow$
$[x\ R\ y \Leftrightarrow z\ R'\ w$ and $y\ R\ x \Leftrightarrow w\ R'\ z]$,
which is an analogue of NC with several profiles.

Parts (b) and (c) of theorem 1 have similar multi-profile correspondents in the Theory of Social Choice (see Weymark (1983) and Blau and Deb (1977)).

Consider, for instance, the concordance part of TACTIC as defined by (1). Given a set of alternatives X evaluated on a set of attributes N, once the weights p_i and the threshold ρ have been chosen, it is possible to see (1) as a way to aggregate any n-uples of weak orders defined on X. It is easy to see that this aggregation satisfies conditions D, UN and I and consequently violates either C or ND. Thus, Arrow's result applies directly to the context of MCDM and may lead the reader to question the interest of the reformulation presented in theorem 1.

Apart from bringing to the attention of the MCDM community several results that are less famous than Arrow's theorem, it seems to us that the single-profile approach is of special interest in the context of MCDM. We refer to Sen (1986), Fishburn (1987), Roberts (1980) and Pollack (1979) for a thorough analysis of the comparison between the multi-profile and the single-profile approaches in Social Choice Theory[9]. We will only emphasize here what seem to be the advantages of a single profile formulation in the context of MCDM.

It should first be observed that the multi-profile format, when applied to MCDM, does not make use of the fact that alternatives are multiattributed, which, in our opinion, is a fundamental characteristic of MCDM. Second, it is worth noting that the proof of theorem 1 is, by far, simpler than the corresponding proofs for the multi-profile case. Third, in order to apply a multi-profile result to MCDM, one has to suppose that all inter-attribute information remains unchanged when aggregating different profiles, whereas it is well-known, in practice, that the modelling of such information crucially depends on the particular problem that is under study. Finally, whereas a multi-profile result is of direct interest to the analyst wishing to create a method that "works" in all cases, it is of limited interest to the user of the method who is always confronted to a single profile, *i.e.* to a given set of alternatives with given evaluations on several attributes. Define a MCDM problem as a set of alternatives evaluated on several attributes. Suppose that you want to informally sum up multi-profile results for a user of MCDM techniques. A "multi-profile explanation" could sound like:

"You have applied an outranking method based on a concordance-discordance principle with given weights and threshold to a particular problem. If the concordance relation you obtain has ‘nice properties’, you should check whether your choice of weights and threshold does not amount to giving much power to a single attribute. If this is not the case, it can be proved that it is possible to find at least one problem with the same number of alternatives and attributes as your problem such that, if you apply the same method with the same weights and threshold, to this new problem, the resulting concordance relation will not have ‘nice properties’".

An informal explanation of single-profile results could be:

"You have applied an outranking method based on a concordance-discordance principle to a particular problem. If the concordance relation you obtain has ‘nice properties’, you should check whether your choice of weights and threshold does not amount to giving much power to a single attribute. If this is not the case, it can be proved that if you add some alternatives to your problem, very similar to those you have used to assess your weights, the concordance relation will loose its nice properties unless you modify the weights and threshold so as to give much power to a single attribute."

[9] In particular, Roberts (1980) studies to what extent multi-profile results have single-profile analogues.

The choice of a particular presentation is, to a certain extent a matter of taste. However, if the emphasis is on the user of the method and if it is agreed that modelling inter-attribute information often implies to take into consideration "artificial" alternatives and is highly specific to a particular problem, it seems that the single-profile approach we have used has definite advantages.

References

Arrow K.J. (1963), *Social Choice and Individual Values*, Wiley, New-York.

Blau J.H. and R. Deb (1977), Social decision functions and the veto, *Econometrica*, **45**, 871-879.

Bouyssou D. (1986), Some remarks on the notion of compensation in MCDM, *European Journal of Operational Research*, **26**, 150-160.

Bouyssou D.and J.C. Vansnick (1986), Noncompensatory and generalized noncompensatory preference structures, *Theory and Decision*, **21**, 251-266.

Fishburn P.C. (1975), Axioms for lexicographic preferences, *Review of Economic Studies*, **42**, 415-419.

Fishburn P.C. (1976), Noncompensatory preferences, *Synthese*, **33**, 393-403.

Fishburn P.C. (1987), *Interprofile Conditions and Impossibility*, Harwood Academic Publishers, Chur.

Goicoechea A., D.R. Hansen and L. Duckstein (1982), *Multiobjective Decision Analysis with Engineering and Business Applications*, Wiley, New-York.

Keeney R.L. and H. Raiffa (1976), *Decisions with Multiple Objectives, Preferences and Value Tradeoffs*, Wiley, New-York.

Kemp M.C. and Y.K. Ng (1976), On the existence of social welfare functions, social orderings and social decision functions, *Economica*, **43**, 59-66.

Mas-Colell A. and H. Sonnenschein (1972), General Possibility theorems for group decisions, *Review of Economic Studies*, **39**, 185-192.

Parks R.P. (1976), An impossibility theorem for fixed preferences: a dictatorial Bergson-Samuelson welfare function, *Review of Economic Studies*, **43**, 447-450.

Plott C.R., J.T. Little and R.P. Parks (1975), Individual choice when objects have 'ordinal' properties, *Review of Economic Studies*, **32**, 403-413.

Pollack R.A. (1979), Bergson-Samuelson social welfare functions and the theory of social choice, *Quarterly Journal of Economics*, **93**, 73-90.

Roberts K.W. (1980), Social choice theory: the single-profile and multi-profile approaches, *Review of Economic Studies*, **47**, 441-450.

Roubens M. and Ph. Vincke (1985), *Preference Modelling*, Springer Verlag, Heidelberg.

Roy B. (1968), Classement et choix en présence de points de vue multiples (la méthode ELECTRE), *RIRO*, **8**, 57-75.

Roy B. (1971), Problems and methods with multiple objective functions, *Mathematical Programming*, **1**, 239-266.

Roy B. and P. Bertier (1973), La méthode ELECTRE II - Une application au media-planning, in M. Ross (Ed.), *OR'72*, 291-302, North Holland, Amsterdam.

Roy B. and D. Bouyssou (1987), *Procédures d'agrégation multicritère conduisant à un critère unique de synthèse*, Working Paper, LAMSADE, Université de Paris-Dauphine.

Roy B., M. Présent and D. Silhol (1986), A programming method for determining which Paris metro station should be renovated, *European Journal of Operational Research*, **24**, 318-334.

Roy B. and Ph. Vincke (1981), Multicriteria analysis: survey and new directions, *European Journal of Operational Research*, **8**, 207-218.

Sen A.K. (1986), Social Choice Theory, in K.J. Arrow and M.D. Intriligator (Eds.), *Handbook of Mathematical Economics Vol. III*, 1073-1181, North Holland, Amsterdam.

Siskos J., G. Wascher and H.M. Winkels (1983), *A bibliography on outranking approaches (1966-1982)*, Working Paper, LAMSADE, Université de Paris-Dauphine.

Vansnick J.C. (1986), On the problem of weights in multiple criteria decision making - The noncompensatory approach, *European Journal of Operational Research*, **24**, 288-294.

Weymark J.A. (1983), Arrow's theorem with social quasi-orderings, *Public Choice*, **42**, 235-246.