Notes on bipolar outranking

Denis Bouyssou *
CNRS – LAMSADE

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1 Introduction

We consider a finite set of alternatives $A = \{a_1, a_2, \ldots, a_m\}$ evaluated on a family of n criteria $F = \{g_1, g_2, \ldots, g_n\}$. Let $N = \{1, 2, \ldots, n\}$. To each criterion $g_i \in F$ is assigned a positive weight w_i. It is supposed wlog that the weights are normalized so that $\sum_{i=1}^{n} w_i = 1$.

Let $X_i = \{g_i(a) : a \in A\}$ be the set of evaluations of the alternatives on the ith criterion. It is supposed that a semi-order P_i (i.e., an asymmetric, Ferrers and semitransitive binary relation) is defined on X_i. We denote by I_i the symmetric complement of P_i. Let $S_i = P_i \cup I_i$. The relations P_i (resp. I_i) models strict preference (resp. indifference) on the ith criterion.

In order to model discordance, we introduce a second semiorder V_i on X_i. It is supposed that $V_i \subseteq P_i$ and that there is a weak order compatible with both P_i and V_i (i.e., there is a weak order $\succsim_i$ on X_i such that $[\alpha \succsim_i \beta$ and $\beta \; P_i \; \gamma$ implies $\alpha \; P_i \; \gamma]$ and $[\alpha \succsim_i \beta$ and $\delta \; P_i \; \alpha$ implies $\delta \; P_i \; \beta]$, with similar relations holding with V_i instead of P_i).

2 Concordance

We first consider the case in which the relations V_i are all empty, i.e., the outranking relation is only based on concordance.

We suppose that the family of criteria is used to build a bipolar *concordance relation* on the set of alternatives. This leads to build three binary relations on A:

* LAMSADE, Université Paris Dauphine, Place du Maréchal de Lattre de Tassigny, F-75775 Paris Cedex 16, France, tel: +33 1 44 05 48 98, fax: +33 1 44 05 40 91, e-mail: `bouyssou@lamsade.dauphine.fr`.

- *TS* indicating that the assertion "a is at least as good as b" is true,
- *FS* indicating that the assertion "a is at least as good as b" is false,
- *US* indicating that the assertion "a is at least as good as b" is unknown,

(T is for "true", F for "false", and U for "unknown").

Let $a, b \in A$. We use the following notation:

$$s_i^{ab} = \begin{cases} w_i & \text{if } g_i(a)\ S_i\ g_i(b), \\ 0 & \text{otherwise,} \end{cases}$$

$$i_i^{ab} = \begin{cases} w_i & \text{if } g_i(a)\ I_i\ g_i(b), \\ 0 & \text{otherwise,} \end{cases}$$

$$p_i^{ab} = \begin{cases} w_i & \text{if } g_i(a)\ P_i\ g_i(b), \\ 0 & \text{otherwise,} \end{cases}$$

$$i^{ab} = \sum_{i=1}^{n} i_i^{ab}, \quad p^{ab} = \sum_{i=1}^{n} p_i^{ab}, \quad s^{ab} = \sum_{i=1}^{n} s_i^{ab}.$$

We obviously have, for all $a, b \in A$,

$$\begin{aligned} s_i^{ab} &= p_i^{ab} + i_i^{ab}, \\ p_i^{ab} + i_i^{ab} + p_i^{ba} &= w_i, \\ w_i - s_i^{ab} &= p_i^{ba}, \\ s_i^{ab} + s_i^{ba} &= p_i^{ab} + 2i_i^{ab} + p_i^{ba} \geq w_i, \\ s^{ab} + s^{ba} &\geq 1, \\ s^{aa} &= 1. \end{aligned}$$

Let $\lambda \in [0.5, 1[$ be the concordance threshold.

We define the three relations, letting, for all $a, b \in A$,

$$\begin{aligned} a\ TS\ b &\iff s^{ab} > \lambda, \\ a\ FS\ b &\iff s^{ab} < 1 - \lambda, \\ a\ US\ b &\iff 1 - \lambda \leq s^{ab} \leq \lambda. \end{aligned} \tag{1}$$

It is clear that the three relations *TS*, *US* and *FS* must be pairwise disjoint. Observe that, since $s^{aa} = 1$, the relation *TS* is always reflexive. Hence, both *US* and *FS* are irreflexive. Let $\{a, b\}$ be a pair of distinct alternatives. We have one and only one of the 9 possibilities:

- $a\ TS\ b$ and $b\ TS\ a$,
- $a\ TS\ b$ and $b\ US\ a$,
- $a\ TS\ b$ and $b\ FS\ a$,
- $a\ US\ b$ and $b\ TS\ a$,
- $a\ US\ b$ and $b\ US\ a$,
- $a\ US\ b$ and $b\ FS\ a$,
- $a\ FS\ b$ and $b\ TS\ a$,
- $a\ FS\ b$ and $b\ US\ a$,
- $a\ FS\ b$ and $b\ FS\ a$.

Because we know that $s^{ab} + s^{ba} \geq 1$, the following three possibilities are excluded:

- $a\ US\ b$ and $b\ FS\ a$,
- $a\ FS\ b$ and $b\ US\ a$,
- $a\ FS\ b$ and $b\ FS\ a$.

Indeed, e.g., $a\ US\ b$ and $b\ FS\ a$ imply $s^{ab} \leq \lambda$ and $s^{ba} < 1 - \lambda$, so that $s^{ab} + s^{ba} < 1$, a contradiction.

Hence we have only 6 possibilities for each pair of distinct alternatives:

$$\begin{aligned} &a\ TS\ b \text{ and } b\ TS\ a,\\ &a\ TS\ b \text{ and } b\ US\ a,\\ &a\ TS\ b \text{ and } b\ FS\ a,\\ &a\ US\ b \text{ and } b\ TS\ a,\\ &a\ US\ b \text{ and } b\ US\ a,\\ &a\ FS\ b \text{ and } b\ TS\ a. \end{aligned} \tag{2}$$

Our main result in this section is as follows.

Proposition 1

Let A be a finite set. Let F, U and T be three binary relations on A that are pairwise disjoint and such that T is reflexive. Suppose that these three relations are such that (2) is satisfied. It is possible to find a family of criteria (together with semiorders P_i) and a concordance threshold such that applying equation (1) to this family of criteria with this threshold will lead to $TS = T$, $US = U$ and $FS = F$. Furthermore, on each criterion of this family, it is possible to take S_i to be a weak order.

PROOF
Let $M = m(m-1)/2$ be the number of pairs of distinct alternatives in A. For each of these M pairs of distinct alternatives, we introduce four criteria defined as follows [1]

1. If a *TS* b and b *TS* a. We introduce four criteria on which S_i is a weak order that are such that, abusing notation in an obvious way:

   (a) $[a, b]xyz$,

   (b) $[a, b]zyx$,

   (c) $xyz[a, b]$,

   (d) $zyx[a, b]$,

   where for instance the criterion such that $[a, b]xyz$ means that we have a and b indifferent on this criterion and are both strictly preferred to all other alternatives, these ones being ordered in an arbitrary way. Similarly, $[a, b]zyx$ means that we have a and b indifferent on this criterion and are both strictly preferred to all other alternatives, these ones being ordered in a way that is the opposite of the one chosen for the first criterion.

2. If a *TS* b and b *US* a. We introduce four criteria on which S_i is a weak order that are such that

   (a) $abxyz$,

   (b) $abxyz$,

   (c) $zyxab$,

   (d) $zyxba$.

3. If a *TS* b and b *FS* a. We introduce four criteria on which S_i is a weak order that are such that

   (a) $abxyz$,

   (b) $abxyz$,

   (c) $zyxab$,

   (d) $zyxab$.

4. If a *US* b and b *US* a. We introduce four criteria on which S_i is a weak order that are such that

[1] The case [a *US* b and b *TS* a] is clearly symmetric to case 2; similarly, the case [a *FS* b and b *TS* a] is symmetric to case 3.

(a) $axyzb$,

(b) $azyxb$,

(c) $bxyza$,

(d) $bzyxa$.

We give an equal weight to all of the $4M$ criteria. Let $\{a, b\}$ be any pair of distinct alternatives. On all the $4M - 4$ criteria that are not linked to that pair, there are $2M - 2$ criteria such that $a\ P_i\ b$ and $2M - 2$ criteria such that $b\ P_i\ a$. Let $k = (2M - 2)/4M$. It is easy to see that:

1. If a *TS* b and b *TS* a, we have
$$s^{ab} = k + 4/4M, \qquad s^{ba} = k + 4/4M.$$

2. If a *TS* b and b *US* a, we have
$$s^{ab} = k + 3/4M, \qquad s^{ba} = k + 1/4M.$$

3. If a *TS* b and b *FS* a, we have
$$s^{ab} = k + 4/4M, \qquad s^{ba} = k.$$

4. If a *US* b and b *US* a, we have
$$s^{ab} = k + 2/4M, \qquad s^{ba} = k + 2/4M.$$

Take the threshold $\lambda = k + 5/8M = 1/2 + 1/8M$, so that $1 - \lambda = k + 1/8M = 1/2 - 1/8M$. Observe that we have $0.5 \le \lambda < 1$, as required.

We have in each of the above cases,

$$k + 4/4M > \lambda = k + 5/8M,$$
$$k + 3/4M > \lambda = k + 5/8M \text{ and } 1 - \lambda = k + 1/8M \le k + 1/4M \le \lambda = k + 5/8M,$$
$$k + 4/4M > \lambda = k + 5/8M \text{ and } k < 1 - \lambda = k + 1/8M,$$
$$1 - \lambda = k + 1/8M \le k + 2/4M \le \lambda = k + 5/8M.$$

Hence, we have the desired result. □

Remark 2
The construction used above is not unique (e.g., there are clearly several possible values for λ in the above construction). It is unlikely to be minimal. •

3 Outranking

We now introduce discordance into the picture. The definition of the three relations is modified as follows.

We define the three outranking relations, letting, for all $a, b \in A$,

$$\begin{aligned} a\ \mathit{TS}\ b &\iff [s^{ab} \geq \lambda \text{ and } \{i \in N : g_i(b)\ V_i\ g_i(a)\} = \varnothing], \\ a\ \mathit{FS}\ b &\iff s^{ab} < 1 - \lambda, \text{ or } \{i \in N : g_i(b)\ V_i\ g_i(a)\} \neq \varnothing], \\ a\ \mathit{US}\ b &\iff 1 - \lambda \leq s^{ab} < \lambda \text{ and } \{i \in N : g_i(b)\ V_i\ g_i(a)\} = \varnothing]. \end{aligned} \tag{3}$$

The three relations TS, US and FS must be pairwise disjoint. Since $s^{aa} = 1$, the relation TS is always reflexive. Hence, both US and FS are irreflexive.

Let $\{a, b\}$ be a pair of distinct alternatives. We have one and only one of the 9 possibilities:

- $a\ \mathit{TS}\ b$ and $b\ \mathit{TS}\ a$,
- $a\ \mathit{TS}\ b$ and $b\ \mathit{US}\ a$,
- $a\ \mathit{TS}\ b$ and $b\ \mathit{FS}\ a$,
- $a\ \mathit{US}\ b$ and $b\ \mathit{TS}\ a$,
- $a\ \mathit{US}\ b$ and $b\ \mathit{US}\ a$,
- $a\ \mathit{US}\ b$ and $b\ \mathit{FS}\ a$,
- $a\ \mathit{FS}\ b$ and $b\ \mathit{TS}\ a$,
- $a\ \mathit{FS}\ b$ and $b\ \mathit{US}\ a$,
- $a\ \mathit{FS}\ b$ and $b\ \mathit{FS}\ a$.

This time, it is not difficult to see that any of these 9 possibilities may occur. Our main result in this section is as follows.

Proposition 3
Let A be a finite set. Let F, U and T be three binary relations on A that are pairwise disjoint and such that T is reflexive. It is possible to find a family of criteria (together with semiorders P_i and V_i such that $V_i \subseteq P_i$ and there is weak order compatible with both P_i and V_i) and a concordance threshold such that applying equation (3) to this family of criteria with this threshold will lead to $\mathit{TS} = T$, $\mathit{US} = U$ and $\mathit{FS} = F$. Furthermore, on each criterion of this family, it is possible to have S_i to be a weak order.

Proof

We use the same construction as in the above proof. For all the cases examined above, we take $V_i = \varnothing$. There are only two more cases to consider (the case [a *FS* b and b *US* a] is clearly symmetric to the case [a *US* b and b *FS* a]). We deal with them introducing veto effects on top of case 4 in the above proof.

1. If a *US* b and b *FS* a. We introduce four criteria on which S_i is a weak order that are such that
   (a) $axyzb$ and $a\ V_i\ b$,
   (b) $azyxb$,
   (c) $bxyza$,
   (d) $bzyxa$.
2. If a *FS* b and b *FS* a. We introduce four criteria on which S_i is a weak order that are such that
   (a) $axyzb$ and $a\ V_i\ b$,
   (b) $azyxb$,
   (c) $bxyza$ and $b\ V_i\ a$,
   (d) $bzyxa$.

In any of these new cases, S_i is a weak order. The relation V_i is always a semiorder that is included in P_i and compatible with the weak order S_i. Using the same reasoning as above, taking the threshold $\lambda = k + 5/8M = 1/2 + 1/8M$ gives the desired result. □

References

D. Bouyssou. Outranking relations: Do they have special properties? *Journal of Multi-Criteria Decision Analysis*, 5:99–111, 1996.