Chapter 16

Conjoint Measurement Models for Preference Relations

16.1. Introduction

Conjoint measurement [KRA 71, WAK 89] is concerned with the study of binary relations defined on Cartesian products of sets. Such relations are central in many disciplines, for example:

– multicriteria or multiattribute decision making, in which the preference of the decision maker is a relation that encodes, for each pair of alternatives, the preferred option taking into account all criteria [BEL 01, KEE 76, WIN 86];

– decision under uncertainty, where the preference relation compares alternatives evaluated on several states of nature [FIS 88, GUL 92, SHA 79, WAK 84, WAK 89];

– consumer theory, dealing with preference relations that compare bundles of goods [DEB 59];

– inter-temporal decision making, that uses preference relations for comparing alternatives evaluated at various instants in time [KOO 60, KOO 72, KEE 76]; and

– inequality measurement, that compares distributions of wealth across individuals [ATK 70, BEN 94, BEN 97].

Let $\succsim$ denote a binary relation on a product set $X = X_1 \times X_2 \times \cdots \times X_n$. Conjoint measurement searches for conditions that allow numerical representations of $\succsim$ to be built and possibly guarantee the uniqueness of such representations. The interest of numerical representations is obvious. They not only facilitate the manipulation of preference relations but also, in many cases, the proofs that such representations exist are constructive (or at least provide useful

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indications on how to build them). Very often, the conditions for the existence of a representation can be empirically tested [KRA 71]. All these reasons justify the interest for this theory in many research domains.

16.1.1. *Brief overview of conjoint measurement models*

In most classical models of conjoint measurement, the relation is assumed to be *complete* and *transitive*. The central model is the *additive utility* model in which we have:

$$x \succsim y \Leftrightarrow \sum_{i=1}^{n} u_i(x_i) \geq \sum_{i=1}^{n} u_i(y_i), \tag{16.1}$$

where u_i denotes a real-valued function defined on the set X_i, for all $i = 1, \ldots, n$. x and y denote n-dimensional elements of the product set X i.e. $x = (x_1, \ldots, x_n)$ and $y = (y_1, \ldots, y_n)$.

The axiomatic analysis of this model is now well established and additive utility (also called additive value function) is at the root of many techniques used in decision analysis [FRE 93, KEE 76, WIN 86, WAK 89, POM 00].

This model has two main difficulties, however. The axiomatic analysis of equation (16.1) raises technical questions that are rather subtle yet important. Many systems of axioms have been proposed in order to guarantee the existence of a representation as described by equation (16.1) [KRA 71, WAK 89]. Two cases can be distinguished:

– If X is finite and no upper bound is fixed *a priori* on the number of its elements, Scott and Suppes [SCO 64] have shown that the system of axioms needed consists of an infinite (countable) set of *cancellation* conditions, which guarantee (via the use of the theorem of the alternative) that a system of (finitely many) linear equations possesses at least one solution (see also [KRA 71, chapter 9] and, for more recent contributions, [FIS 96, FIS 97]). These conditions are hardly interpretable or testable.

– The case in which X is infinite is quite different but raises other problems. Non-necessary conditions are usually imposed on X in order to guarantee that the structure of X is 'close' to that of the real numbers and that $\succsim$ behaves consistently with this structure. In one approach, an archimedean axiom is imposed together with solvability conditions [KRA 71, chapter 6]. In another approach, it is assumed that X is a topological space and that $\succsim$ is continuous [DEB 60, WAK 89]. Using such 'structural' assumptions, it is possible to characterize model equation (16.1) by means of a finite number of cancelation conditions (for recent contributions see [GON 96, GON 00, KAR 98]; for an alternative approach extending the technique used in the finite case to the infinite one, see [JAF 74]). In these axiomatic systems, the necessary properties interact with structural, unnecessary assumptions imposed on X [KRA 71, chapter 6], which obscures the understanding of the model and does not allow for completely satisfactory empirical tests [KRA 71, chapter 9]. In addition, the analysis of the two-dimensional case ($n = 2$) differs totally from that of the cases where n is greater than or equal to 3.

As we shall see, it is possible to avoid imposing unnecessary hypotheses (structural assumptions) provided the requirement of an additive representation is abandoned; this is the idea followed by the authors of [KRA 71, chapter 7] when introducing the following *decomposable* model:

$$x \succsim y \Leftrightarrow U(u_1(x_1), u_2(x_2), \ldots, u_n(x_n)) \geq U(u_1(y_1), u_2(y_2), \ldots, u_n(y_n)) \quad (16.2)$$

where U is an increasing function of all its arguments.

There is another type of difficulty with the additive model (16.1) of a more fundamental nature: this model excludes all preference relations that fail to be transitive or complete from consideration. Several authors have now forcefully argued in favor of models tolerating intransitive or incomplete preferences [MAY 54, TVE 69] and there are multiple criteria decision analysis methods that do not exclude such relations [ROY 85, ROY 93].

The *additive difference* model proposed in [TVE 69] is among the first that does not assume transitive preferences; the preference $\succsim$ is supposed to satisfy:

$$x \succsim y \Leftrightarrow \sum_{i=1}^{n} \Phi_i(u_i(x_i) - u_i(y_i)) \geq 0 \quad (16.3)$$

where Φ_i are increasing and odd functions (which implies that the preference $\succsim$ is complete). An axiomatic characterization of this model has been proposed by Fishburn [FIS 92]. Due to the additive form of the representation, Fishburn could not avoid imposing unnecessary structural conditions in his characterization of model (16.3).

More recently, more general additive non-transitive models have been proposed (allowing in particular for incomplete preferences) [BOU 86, FIS 90a, FIS 90b, FIS 91, FIS 92, VIN 91]. They are of the type:

$$x \succsim y \Leftrightarrow \sum_{i=1}^{n} p_i(x_i, y_i) \geq 0 \quad (16.4)$$

where p_i are real-valued functions defined on X_i^2; they may enjoy additional properties (e.g. $p_i(x_i, x_i) = 0 \ \forall i \in \{1, 2, \ldots, n\}$ and for all $x_i \in X_i$).

In the spirit of the decomposable model (16.2) that avoids the difficulties of the axiomatization of the additive models, Goldstein [GOL 91] has proposed a generalization of model (16.4) in which the sum has been substituted by a function G, increasing in its arguments. The underlying model is therefore:

$$x \succsim y \Leftrightarrow G(p_1(x_1, y_1), p_2(x_2, y_2), \ldots, p_n(x_n, y_n)) \geq 0. \quad (16.5)$$

In decision analysis, methods that may lead to intransitive and/or incomplete preference relations have been used for a long time [ROY 68, ROY 73]. They are known as *outranking* methods [ROY 91, ROY 93], and are inspired by social choice procedures, especially the Condorcet voting rule. In a basic version of the ELECTRE method [ROY 68, ROY 73], the outranking relation is obtained as follows:

$$x \succsim y \Leftrightarrow \sum_{\{i : x_i S_i y_i\}} w_i \geq \lambda \quad (16.6)$$

where w_i are weights associated with the criteria, x_i and y_i represent the performance of alternatives x and y on criterion i, S_i is a binary relation that orders the levels on the scale of criterion i and λ is a majority threshold (called *concordance threshold*), generally assigned a value larger than 50% of the sum of the weights. Clearly, binary relations obtained in this way may fail to be transitive or complete. Consider for instance the case where $n = 3$, $p_1 = p_2 = p_3 = \frac{1}{3}$, $x = (3, 2, 1)$, $y = (2, 1, 3)$, $z = (1, 3, 2)$, S_i is the usual order $\geq$ on the set of the real numbers and $\lambda = 60\%$. Denoting by $\succ$ the asymmetric part of $\succsim$ ($a \succ b$ if $a \succsim b$ and not $b \succsim a$) and applying rule (16.6) yields $x \succ y$, $y \succ z$, but not $x \succ z$: i.e. relation $\succ$ is not transitive. Moreover, since $z \succ x$, it has cycles. This is a version of the Condorcet paradox, appearing in a multiple criteria decision making context. In the same perspective, considering $n = 2$, $p_1 = p_2 = \frac{1}{2}$, $x = (2, 1)$, $y = (1, 2)$ and $\lambda = 60\%$, we have that neither $x \succsim y$ nor $y \succsim x$: the relation $\succsim$ is not complete.

As is easily verified, note that outranking relations obtained through equation (16.6) are representable in the additive non-transitive model (16.4), letting:

$$p_i(x_i, y_i) = \begin{cases} w_i - \frac{\lambda}{n} & \text{if } x_i S_i y_i \\ -\frac{\lambda}{n} & \text{otherwise.} \end{cases} \tag{16.7}$$

16.1.2. *Chapter contents*

Our goal is to propose a general framework as well as quite general analytical tools that allow the study of binary relations defined on a Cartesian product in a conjoint measurement perspective. Our framework encompasses most methods that have been proposed in multiple criteria decision analysis to construct a global preference relation.

We consider two main families of models of relations on a product set. To support the reader's intuition, consider the various manners of comparing objects characterized by their description on a set of n attributes. Let $x = (x_1, x_2, \ldots, x_n)$ and $y = (y_1, y_2, \ldots, y_n)$ be two alternatives described by n-dimensional vectors. In a first approach, in view of deciding whether 'x is at least as good as y', we may try to assess the 'value' of either alternative on each attribute and then combine these values in appropriate fashion. It is important to emphasize what we mean by 'value'; the 'value' of alternative x on criterion i is not simply the label describing this alternative on attribute i (which is denoted by x_i) but an assessment that reflects the way this label is perceived by a decision maker in a given decisional context, taking into account their objectives and preferences. Abandoning for the moment classical requirements such as transitivity or completeness, we may consider a model in which:

$$x \succsim y \Leftrightarrow F(u_1(x_1), u_2(x_2), \ldots, u_n(x_n), u_1(y_1), u_2(y_2), \ldots, u_n(y_n)) \geq 0, \tag{16.8}$$

where u_i are real-valued functions on X_i and F is a real-valued function on the product set $\prod_{i=1}^{n} u_i(X_i)^2$.

Another strategy relies on the idea of 'measuring' *differences of preference* between x and y on each attribute separately and then combining these differences in order to determine whether the balance of these is in favor of x or y. This suggests a model in which:

$$x \succsim y \Leftrightarrow G(p_1(x_1, y_1), p_2(x_2, y_2), \ldots, p_n(x_n, y_n)) \geq 0 \tag{16.9}$$

where p_i are real-valued functions on X_i^2 and G is a real-valued function on $\prod_{i=1}^{n} p_i(X_i^2)$.

Of course, the strategies just outlined are not incompatible. It can reasonably be expected that the differences of preference on each criterion can be expressed in terms of values assigned to the alternatives on each criterion. In the model that this suggests, we have:

$$x \succsim y \Leftrightarrow H(\varphi_1(u_1(x_1), u_1(y_1)), \varphi_2(u_2(x_2), u_2(y_2)), \ldots, \varphi_n(u_n(x_n), u_n(y_n))) \geq 0 \tag{16.10}$$

where u_i are real-valued functions on X_i, φ_i are real-valued functions on $u_i(X_i)^2$ and H is a real-valued function on $\prod_{i=1}^{n} \varphi_i(u_i(X_i)^2)$.

As long as no additional property is imposed to the various functions that intervene in the above three models, these models are exceedingly general in the sense that any relation on X (provided that X is finite or denumerable) can be represented in all three models. If X is not denumerable, the generality of the models is only restricted by technical conditions (that are necessary and sufficient).

Consequently, to make these models interesting, we shall impose additional properties on the involved functions. For instance:

– in model (16.8), we shall impose that F is non-decreasing in its first n arguments and non-increasing in its last n arguments;

– in model (16.9), we shall require that G is an odd function or that it is non-decreasing in its n arguments or that p_i is antisymmetric;

– in model (16.10), we shall consider the cases in which H is an odd function or is non-decreasing in its n arguments or the cases in which φ_i are odd functions or functions that are non-decreasing in their first argument and non-increasing in their second one.

By adding such requirements, a large variety of models can be defined. A selection of them will be studied in the sequel. In particular, certain variants are rather close to classical models alluded to in section 16.1.1. Note, however, that our goal is not to characterize exactly classical models but instead to establish general frameworks in which such a characterization could be elaborated. The advantage of general frameworks is to allow for a better understanding of what is common to classical models and what distinguishes them.

Note that the frameworks (16.8), (16.9) and (16.10) rely on fundamental objects that possess nice interpretations in terms of preference and permit the analysis of preference relations on a product set. For understanding of the classical additive value function model, *marginal preference* is the crucial notion. This relation, defined on each factor X_i of the product set X as a projection (in a certain sense) of the global preference $\succsim$ on each attribute, is the relation that is numerically represented by the u_i functions in model (16.1). The process of 'elicitation' of an additive value function model, relies in an essential manner on marginal preferences.

In models (16.8) and (16.9), the central role is no longer played by marginal preferences since these relations do not enjoy, in these models, the properties that facilitate their interpretation in the additive value function model (16.1). In general, they are not transitive or complete. They are ‘too rough’ to allow for a sufficiently detailed analysis of the global preference, as we shall see in the following.

In our three frameworks (16.8), (16.9) and (16.10), the main tool for analyzing the preference relation is the *trace*, a notion that admits different variants. In model (16.8), we shall use the *marginal trace* of the preference on each component X_i; this relation provides an ordering of the labels of the scale X_i of each attribute i. In model (16.9), we shall be concerned with traces on each Cartesian product X_i^2 of each attribute scale with itself; here the trace rank-orders the differences of preference between two alternatives on attribute i. Finally, in model (16.10), both types of traces appear and interact.

The contents of this chapter are the following. In section 16.2, we introduce the main tools for analyzing preference relations: marginal traces on levels and marginal traces on differences. We discuss the position of the more classical marginal preferences w.r.t. these traces. We then show how any preference relation can be represented in any of the three general models introduced above.

We briefly describe various specializations of model (16.8) and their axiomatic characterizations. We shall see in section 16.2 that some of these axioms indeed express a fundamental requirement of aggregation procedures, namely that the relation obtained through aggregation should contain the dominance relation. The rest of the section shows how the marginal traces on levels tend to become increasingly similar to marginal preference relations while additional requirements are imposed on the model, driving it closer to the additive value function model.

Section 16.4 studies model (16.9). Much as in the previous section, we characterize several variants of the model. We show that the numerical representations of type (16.9) are well-suited to understand outranking methods.

In section 16.5, we consider the relations that can be described within model (16.10). We characterize some of their variants and analyze the position of some well-known models such as the model of additive differences (16.3) and some outranking methods in this framework.

A brief conclusion summarizes the main advantages of the new concepts for analyzing relations on a product set. Various applications are discussed.

All our results have elementary proofs. We present some which we feel useful for understanding the new concepts. The reader interested in more details is invited to refer to a series of articles in which all proofs are given: [BOU 02b, BOU 04b, BOU 05a, BOU 05b, BOU 09]. These articles contain a complete study of the general, non-denumerable case as well as the proof that our axioms are independent. We shall pay little attention to the latter aspects in this chapter.

16.2. Fundamental relations and trivial models

16.2.1. *Binary relations on a product set*

As far as binary relations are concerned, we adopt the terminology and definitions used in Chapter 2. Hence, we shall use notions such as reflexive, irreflexive, complete, symmetric, asymmetric, transitive, Ferrers and semi-transitive relation with the same meaning. We also assume that the definitions of (complete) weak order, interval order and semiorder are familiar to the reader (see also Chapter 2 for these definitions).

We generally work with binary relations on a product set $X = X_1 \times X_2 \times \ldots \times X_n$. The sets $X_i,\ i = 1, 2, \ldots, n$, may be sets of arbitrary cardinality and n is assumed to be at least equal to 2. The elements of X are n-dimensional vectors: $x \in X$ with $x = (x_1, x_2, \ldots, x_n)$. We interpret them as alternatives described by their values on n attributes.

A binary relation on the set X will usually be denoted by $\succsim$, its asymmetric part by $\succ$ and its symmetric part by $\sim$. A similar convention holds for the asymmetric and symmetric parts of a relation when the symbol $\succsim$ is subscripted or superscripted. Relation $\succsim$ is interpreted as a preference relation and $a \succsim b$ reads: 'a is at least as good as b'.

For any subset I of the set of attributes $\{1, 2, \ldots, n\}$, we denote by X_I (respectively, X_{-I}) the product set $\prod_{i \in I} X_i$ (respectively, $\prod_{i \notin I} X_i$). We denote by (x_I, a_{-I}) the vector $w \in X$ such that $w_i = x_i$ if $i \in I$ and $w_i = a_i$ otherwise. If I is a singleton $\{i\}$, we simply write X_{-i} and (x_i, a_{-i}), abusing notation.

16.2.2. *Independence and marginal preferences*

A preference relation $\succsim$ on a product set X induces relations called *marginal preferences* on the subspaces X_I, for any subset of attributes I. The marginal preference $\succsim_I$ induced by $\succsim$ on X_I is defined for all x_I, y_I by:

$$x_I \succsim_I y_I \Leftrightarrow (x_I, z_{-I}) \succsim (y_I, z_{-I}),\ \text{for all } z_{-I} \in X_{-I}. \tag{16.11}$$

We do not assume in general that preferences have special properties such as completeness or transitivity. Even if $\succsim$ is complete, this property is not necessarily inherited by its marginal preferences $\succsim_I$. Let us define two properties that confer some regularity to marginal preferences.

Definition 16.1. Let $\succsim$ be a preference relation on a product set X and let I be a subset of attributes.

– We say that $\succsim$ is *independent for* I if, for all $x_I,\ y_I \in X_I$,

$$\begin{aligned} &[(x_I, z_{-I}) \succsim (y_I, z_{-I}),\ \text{for some } z_{-I} \in X_{-I}] \\ &\Rightarrow\ [(x_I, w_{-I}) \succsim (y_I, w_{-I}),\ \text{for all } w_{-I} \in X_{-I}]. \end{aligned}$$

– We say that $\succsim$ is *separable for I* if, for all $x_I,\ y_I \in X_I$,

$$[(x_I, z_{-I}) \succ (y_I, z_{-I}),\ \text{for some } z_{-I} \in X_{-I}]$$
$$\Rightarrow\ Not[\,(y_I, w_{-I}) \succ (x_I, w_{-I})\,]\,,\ \text{for all}\ w_{-I} \in X_{-I}.$$

– If $\succsim$ is independent (respectively, separable) for all subset of attributes I, we say that $\succsim$ is independent (respectively, separable). If $\succsim$ is independent (respectively, separable) for all subsets consisting of a single attribute, we say that $\succsim$ is weakly independent (respectively, weakly separable).

Independence is a classical notion in measurement theory. Intuitively, it means that common values on a subset of attributes do not influence preference. It is well known that independence implies weak independence, but not the converse [WAK 89]. Similarly, independence implies separability but the converse is false. Separability is a weakening of the independence property. It is an interesting property since aggregation models based on the $\max$ or $\min$ operator yield preferences that are separable but not independent. Separability prohibits strict reversal of the preferences while letting common values on some attributes vary. Separability entails weak separability but the converse is not true.

Independence and separability are of course related to completeness of marginal preferences. The following results are either well known or obvious.

Proposition 16.1. *Let $\succsim$ be a binary relation on X.*

– If $\succsim$ is complete and independent for attribute i, $\succsim_i$ is complete;

– $\succsim_i$ is complete if and only if $\succsim$ is weakly separable and satisfies the following condition: for all $x_i, y_i \in X_i$ and for all $a_{-i} \in X_{-i}$,

$$(x_i, a_{-i}) \succsim (y_i, a_{-i}) \text{ or } (y_i, a_{-i}) \succsim (x_i, a_{-i}). \tag{16.12}$$

Marginal preferences on each attribute i express the results of the pairwise comparison of levels x_i and y_i when these levels are adjoined common levels on all other attributes (*ceteris paribus* reasoning). We shall see in the next section that marginal preferences $\succsim_i$ do not exploit all the information contained in $\succsim$ relatively to attribute i, contrary to marginal traces on levels.

16.2.3. *Marginal traces on levels*

Various kinds of marginal traces ($\succsim_i^+, \succsim_i^-$ and $\succsim_i^\pm$) on X_i are defined as follows.

Definition 16.2. For all $x_i, y_i \in X_i$, for all $a_{-i} \in X_{-i}$, for all $z \in X$,

$$x_i \succsim_i^+ y_i \Leftrightarrow [(y_i, a_{-i}) \succsim z \Rightarrow (x_i, a_{-i}) \succsim z],$$
$$x_i \succsim_i^- y_i \Leftrightarrow [z \succsim (x_i, a_{-i}) \Rightarrow z \succsim (y_i, a_{-i})],$$
$$x_i \succsim_i^\pm y_i \Leftrightarrow \begin{cases} (y_i, a_{-i}) \succsim z \Rightarrow (x_i, a_{-i}) \succsim z, \\ \text{and} \\ z \succsim (x_i, a_{-i}) \Rightarrow z \succsim (y_i, a_{-i}). \end{cases}$$

These definitions clarify the difference between marginal preferences and marginal traces. Marginal traces use all the information available in $\succsim$ in order to compare x_i with y_i. These two levels in X_i are adjoined the same evaluations on X_{-i} and one observes how such alternatives compare with all other alternatives. In contrast, marginal preference results from the comparison of alternatives, evaluated by level x_i on attribute i, with alternatives that are evaluated by level y_i. Both alternatives are adjoined the same evaluations on X_{-i} (*ceteris paribus* comparison). The latter mode of comparison does not take into account the behavior of such alternatives with respect to others. Under a very weak hypothesis, namely reflexivity of $\succsim$, we have indeed that $x_i \succsim_i^+ y_i$ (or $x_i \succsim_i^- y_i$) entails $x_i \succsim_i y_i$. This is readily verified starting e.g. from $(y_i, a_{-i}) \succsim (y_i, a_{-i})$. Applying the definition of $\succsim_i^+$, we obtain $(x_i, a_{-i}) \succsim (y_i, a_{-i})$. Similarly, starting from $(x_i, a_{-i}) \succsim (x_i, a_{-i})$ and using the definition of $\succsim_i^-$, we obtain the other entailment.

Using their definitions, it is not difficult to see that $\succsim_i^+, \succsim_i^-$ and $\succsim_i^\pm$ are reflexive and transitive relations.

According to our conventions, we denote the asymmetric (respectively, symmetric) part of $\succsim_i^+$ by $\succ_i^+$ (respectively, $\sim_i^+$) and similarly for $\succsim_i^-$ and $\succsim_i^\pm$. In the following lemma we note a few links between marginal traces and the preference relation $\succsim$. These properties, which will be used in the sequel, describe the 'responsiveness' of the preference with respect to the traces. The proof of this lemma is left to the reader.

Lemma 16.1. *For all $i \in \{1, \dots, n\}$ and $x, y, z, w \in X$:*

1) $[x \succsim y, z_i \succsim_i^+ x_i] \Rightarrow (z_i, x_{-i}) \succsim y$,

2) $[x \succsim y, y_i \succsim_i^- w_i] \Rightarrow x \succsim (w_i, y_{-i})$,

3) $$[z_i \succsim_i^\pm x_i, y_i \succsim_i^\pm w_i] \Rightarrow \begin{cases} x \succsim y \Rightarrow (z_i, x_{-i}) \succsim (w_i, y_{-i}), \\ \textit{and} \\ x \succ y \Rightarrow (z_i, x_{-i}) \succ (w_i, y_{-i}), \end{cases}$$

4) $$[z_i \sim_i^\pm x_i, y_i \sim_i^\pm w_i, \forall\, i \in \{1, \dots, n\}] \Rightarrow \begin{cases} x \succsim y \Leftrightarrow z \succsim w, \\ \textit{and} \\ x \succ y \Leftrightarrow z \succ w. \end{cases}$$

Marginal traces are not necessarily complete relations. When this is the case, this has important consequences as we shall see in section 16.3.

16.2.4. *Marginal traces on differences*

Wakker [WAK 88, WAK 89] has demonstrated the importance of traces on differences for understanding conjoint measurement models. We introduce two relations on preference differences $\succsim_i^*$ and $\succsim_i^{**}$ for each attribute i. These relations compare pairs of levels; they are subsets of $X_i^2 \times X_i^2$.

Definition 16.3. For all $x_i, y_i, z_i, w_i \in X_i$,

$$(x_i, y_i) \succsim_i^* (z_i, w_i) \quad \text{if and only if}$$
$$\forall\, a_{-i}, b_{-i} \in X_{-i}, (z_i, a_{-i}) \succsim (w_i, b_{-i}) \Rightarrow (x_i, a_{-i}) \succsim (y_i, b_{-i});$$

$$(x_i, y_i) \succsim_i^{**} (z_i, w_i) \quad \text{if and only if} \quad [(x_i, y_i) \succsim_i^{*} (z_i, w_i) \text{ and } (w_i, z_i) \succsim_i^{*} (y_i, x_i)].$$

Intuitively, we interpret $(x_i, y_i) \succsim_i^* (z_i, w_i)$ as stating that the difference of preference between levels x_i and y_i is at least as large as that between z_i and w_i. By definition, $\succsim_i^*$ is reflexive and transitive while, in contrast, there is no necessary link between (x_i, y_i) and the 'opposite' difference (y_i, x_i); that is why we introduce relation $\succsim_i^{**}$.

As for marginal traces on levels, the preference relation $\succsim$ is monotone with respect to marginal traces on differences. Moreover, traces on levels and traces on differences are not unrelated. The following lemmas describe the former and the latter links; their elementary proof is left to the reader.

Lemma 16.2. *For all $x, y \in X$ and all $z_i, w_i \in X_i$,*

1) $\succsim$ is independent if and only if $(x_i, x_i) \sim_i^* (y_i, y_i)$, $\forall\, i \in \{1, \ldots, n\}$,

2) $[x \succsim y$ and $(z_i, w_i) \succsim_i^* (x_i, y_i)] \Rightarrow (z_i, x_{-i}) \succsim (w_i, y_{-i})$,

3) $[(z_i, w_i) \sim_i^* (x_i, y_i), \forall\, i \in \{1, \ldots, n\}] \Rightarrow [x \succsim y \Leftrightarrow z \succsim w]$,

4) $[x \succ y$ and $(z_i, w_i) \succsim_i^{**} (x_i, y_i)] \Rightarrow (z_i, x_{-i}) \succ (w_i, y_{-i})$,

5) $$[(z_i, w_i) \sim_i^{**} (x_i, y_i), \forall\, i \in \{1, \ldots, n\}] \Rightarrow \begin{cases} [x \succsim y \Leftrightarrow z \succsim w] \\ \textit{and} \\ [x \succ y \Leftrightarrow z \succ w], \end{cases}$$

Lemma 16.3. *For all $i \in \{1, \ldots, n\}$ and all $x_i, y_i \in X_i$,*

1) $x_i \succsim_i^+ y_i \Leftrightarrow [(x_i, w_i) \succsim_i^* (y_i, w_i), \forall\, w_i \in X_i]$,

2) $x_i \succsim_i^- y_i \Leftrightarrow [(w_i, y_i) \succsim_i^* (w_i, x_i), \forall\, w_i \in X_i]$,

3) $x_i \succsim_i^\pm y_i \Leftrightarrow [(x_i, w_i) \succsim_i^{**} (y_i, w_i), \forall\, w_i \in X_i]$,

4) $[\ell_i \succsim_i^+ x_i$ and $(x_i, y_i) \succsim_i^* (z_i, w_i)] \Rightarrow (\ell_i, y_i) \succsim_i^* (z_i, w_i)$,

5) $[y_i \succsim_i^- \ell_i$ and $(x_i, y_i) \succsim_i^* (z_i, w_i)] \Rightarrow (x_i, \ell_i) \succsim_i^* (z_i, w_i)$,

6) $[z_i \succsim_i^+ \ell_i$ and $(x_i, y_i) \succsim_i^* (z_i, w_i)] \Rightarrow (x_i, y_i) \succsim_i^* (\ell_i, w_i)$,

7) $[\ell_i \succsim_i^- w_i$ and $(x_i, y_i) \succsim_i^* (z_i, w_i)] \Rightarrow (x_i, y_i) \succsim_i^* (z_i, \ell_i)$,

8) $[x_i \sim_i^+ z_i$ and $y_i \sim_i^- w_i] \Rightarrow (x_i, y_i) \sim_i^* (z_i, w_i)$,

9) $[x_i \sim_i^\pm z_i$ and $y_i \sim_i^\pm w_i] \Rightarrow (x_i, y_i) \sim_i^{**} (z_i, w_i)$.

Marginal traces on differences are not generally complete. When they are, this has interesting consequences that will be studied in section 16.4.

16.2.5. *Three models for general relations on a Cartesian product*

Provided the cardinal of X is not larger than that of the set of real numbers, every binary relation on X can be represented in the three models described by equations (16.8–16.10).

As we shall see in the proof of the following proposition, marginal traces on levels play a fundamental role for representation (16.8). Marginal traces on differences play a similar role in representation (16.9) and both types of traces are important for model (16.10). The importance of this role will be strengthened when we impose the completeness of the traces in the following three sections.

We use the notation $[u_i(x_i)]$ to denote the n-components vector $(u_1(x_1), \dots, u_n(x_n))$.

Proposition 16.2. *Trivial representations on product sets*
Let $\succsim$ be a binary relation on the set $X = \prod_{i=1}^n X_i$, the cardinal of which is at most that of $\mathbb{R}$.

1) There are real-valued functions u_i on X_i and a real-valued function F defined on $[\prod_{i=1}^n u_i(X_i)]^2$ such that, for all $x, y \in X$,

$$x \succsim y \Leftrightarrow F([u_i(x_i)]; [u_i(y_i)]) \geq 0. \tag{$L0$}$$

2) There are real-valued functions p_i on X_i^2 and a real-valued function G defined on $\prod_{i=1}^n p_i(X_i^2)$ such that, for all $x, y \in X$,

$$x \succsim y \Leftrightarrow G([p_i(x_i, y_i)]) \geq 0. \tag{$D0$}$$

3) There exist real-valued functions u_i on X_i, real-valued functions φ_i on $u_i(X_i)^2$ and a real-valued function H defined on $\prod_{i=1}^n \varphi_i(u_i(X_i)^2)$ such that, for all $x, y \in X$,

$$x \succsim y \Leftrightarrow H([\varphi_i(u_i(x_i), u_i(y_i))]) \geq 0. \tag{$L0D0$}$$

Proof. Part (1). Let $i \in \{1, \dots, n\}$. By construction, $\sim_i^{\pm}$ is an equivalence relation since it is reflexive, symmetric and transitive. Since X_i has at most the cardinality of $\mathbb{R}$, there exists a function u_i from X_i to $\mathbb{R}$ such that for all $x_i, y_i \in X_i$:

$$x_i \sim_i^{\pm} y_i \Leftrightarrow u_i(x_i) = u_i(y_i). \tag{16.13}$$

For all $i \in \{1, \dots, n\}$, let u_i be a function that satisfies equation (16.13). We define F from $[\prod_{i=1}^n u_i(X_i)]^2$ to $\mathbb{R}$ by:

$$F([u_i(x_i)]; [u_i(y_i)]) = \begin{cases} +1 & \text{if } x \succsim y, \\ -1 & \text{otherwise.} \end{cases} \tag{16.14}$$

Lemma 16.1(4) guarantees that F is well defined.

Part (2). Since $\sim_i^{**}$ is an equivalence relation and in view of the cardinality of X_i, for all i there is a function p_i from X_i^2 to $\mathbb{R}$ that separates the equivalence classes of $\sim_i^{**}$, i.e. that is such that for all $x_i, y_i, z_i, w_i \in X_i$:

$$(x_i, y_i) \sim_i^{**} (z_i, w_i) \Leftrightarrow p_i(x_i, y_i) = p_i(z_i, w_i). \tag{16.15}$$

Using lemma 16.2(5), the following function G is well defined:

$$G([p_i(x_i, y_i)]) = \begin{cases} +1 & \text{if } x \succsim y, \\ -1 & \text{otherwise.} \end{cases} \tag{16.16}$$

Part (3). Let us consider, for all i, a function u_i that satisfies equation (16.13) and a function p_i that satisfies equation 16.15. We define φ_i on $u_i(X_i)^2$ by:

$$\varphi_i(u_i(x_i), u_i(y_i)) = p_i(x_i, y_i) \tag{16.17}$$

for all $x_i, y_i \in X_i$. Let us show that φ_i is welldefined i.e. that $u_i(x_i) = u_i(z_i)$ and $u_i(y_i) = u_i(w_i)$ imply $p_i(x_i, y_i) = p_i(z_i, w_i)$. By construction, we have $x_i \sim_i^{\pm} z_i$ and $y_i \sim_i^{\pm} w_i$; lemma 16.3(9) yields $(x_i, y_i) \sim_i^{**} (z_i, w_i)$, hence $p_i(x_i, y_i) = p_i(z_i, w_i)$.

Finally, we define H on $\prod_{i=1}^{n} \varphi_i(u_i(X_i), u_i(X_i))$ by:

$$H([\varphi_i(u_i(x_i), u_i(y_i))]) = \begin{cases} +1 & \text{if } x \succsim y, \\ -1 & \text{otherwise.} \end{cases} \tag{16.18}$$

Using lemma 16.2(3), we see that H is well defined.$\diamond$

Remark 16.1. The limitation on the cardinality of X imposed in proposition 16.2 is not a necessary condition. This condition can be weakened in the following way. For model ($L0$), it is sufficient that the number of equivalence classes of the relations $\sim_i^{\pm}$ is not larger than the cardinal of $\mathbb{R}$; in the same way, for model ($D0$), it is necessary and sufficient to impose the same restriction on the number of equivalence classes of relations $\sim_i^{**}$. For model ($L0D0$), the two previous restrictions are required.

16.3. Models using marginal traces on levels

16.3.1. *Definition of the models*

In model ($L0$), the role of u_i consists only of associating a numerical 'label' to each equivalence class of relation $\succsim_i^{\pm}$. The role of F is only to determine whether the profiles $[(u_i(x_i))]$, $[(u_i(y_i))]$ correspond to a preference (see definition of F in equation (16.14)) or not. Things become more interesting when additional properties are imposed on F. We consider the following models:

– model ($L1$), obtained by imposing $F([u_i(x_i)]; [u_i(x_i)]) \geq 0$ on model ($L0$); and

– model ($L2$), obtained by imposing $F([u_i(x_i)]; [u_i(y_i)]) = -F([u_i(y_i)]; [u_i(x_i)])$ on model ($L1$).

Moreover, in each of the models ($L0$), ($L1$) and ($L2$), we consider the consequences of imposing that F is non-decreasing (respectively, increasing) in its first n arguments and non-increasing (respectively, decreasing) in its last n arguments. The resulting eight new models are defined in Table 16.1.

A number of implications between these models result immediately from their definitions. We do not detail them here. We note in the following proposition a number of consequences of the properties of F introduced to define models ($L1$) and ($L2$).

Proposition 16.3. *A binary relation $\succsim$ on a product set $X = \prod_{i=1}^{n} X_i$, the cardinal of which is bounded by that of $\mathbb{R}$, can be represented in*

$(L0)$	$x \succsim y \Leftrightarrow F([u_i(x_i)];[u_i(y_i)]) \geq 0$
$(L1)$	$(L0)$ with $F([u_i(x_i)];[u_i(x_i)]) \geq 0$
$(L2)$	$(L0)$ with $F([u_i(x_i)];[u_i(y_i)]) = -F([u_i(y_i)];[u_i(x_i)])$
	....
$(L3)$	$(L0)$ with F non-decreasing, non-increasing,
$(L4)$	$(L0)$ with F increasing, decreasing,
	....
$(L5)$	$(L1)$ with F non-decreasing, non-increasing,
$(L6)$	$(L1)$ with F increasing, decreasing,
	....
$(L7)$	$(L2)$ with F non-decreasing, non-increasing,
$(L8)$	$(L2)$ with F increasing, decreasing,

Table 16.1. *Models using traces on levels*

1) model ($L1$) if and only if $\succsim$ is reflexive;

2) model ($L2$) if and only if $\succsim$ is complete.

Proof. Reflexivity and completeness of $\succsim$ are clear consequences of models ($L1$) and ($L2$), respectively. Reflexivity of $\succsim$ is evidently sufficient for model ($L1$). It remains to be shown that completeness is a sufficient condition for model ($L2$). This is readily done by reconsidering the construction of the representation of $\succsim$ in the proof of proposition 16.2; we simply change the definition of F, equation (16.14), to:

$$F([u_i(x_i)];[u_i(y_i)]) = \begin{cases} +1 & \text{if } x \succ y, \\ 0 & \text{if } x \sim y, \\ -1 & \text{otherwise.} \end{cases} \tag{16.19}$$

Using the completeness of $\succsim$, we readily verify that F is still well defined and satisfies

$$F([u_i(x_i)];[u_i(y_i)]) = -F([u_i(y_i)];[u_i(x_i)]).$$

$\diamond$

In the next section, we introduce properties that are intimately connected to the monotonicity of F. Interestingly, the same properties ensure the completeness of marginal traces.

16.3.2. *Completeness of marginal traces and monotonicity of F*

We introduce the following three axioms for each dimension i.

Definition 16.4. Conditions $AC1$, $AC2$ and $AC3$
Let $\succsim$ be a binary relation on $X = \prod_{i=1}^{n} X_i$. For $i \in \{1, \ldots, n\}$, we say that relation $\succsim$

satisfies: $AC1_i$ if

$$\left.\begin{array}{c} x \succsim y \\ \text{and} \\ z \succsim w \end{array}\right\} \Rightarrow \left\{\begin{array}{l} (z_i, x_{-i}) \succsim y \\ \text{or} \\ (x_i, z_{-i}) \succsim w, \end{array}\right.$$

$AC2_i$ if

$$\left.\begin{array}{c} x \succsim y \\ \text{and} \\ z \succsim w \end{array}\right\} \Rightarrow \left\{\begin{array}{l} x \succsim (w_i, y_{-i}) \\ \text{or} \\ z \succsim (y_i, w_{-i}), \end{array}\right.$$

and $AC3_i$ if

$$\left.\begin{array}{r} z \succsim (x_i, a_{-i}) \\ \text{and} \\ (x_i, b_{-i}) \succsim y \end{array}\right\} \Rightarrow \left\{\begin{array}{l} z \succsim (w_i, a_{-i}) \\ \text{or} \\ (w_i, b_{-i}) \succsim y, \end{array}\right.$$

for all $x, y, z, w \in X$, for all $a_{-i}, b_{-i} \in X_{-i}$ and for all $x_i, w_i \in X_i$.

We say also that $\succsim$ satisfies $AC1$ (respectively, $AC2$, $AC3$) if it satisfies $AC1_i$ (respectively, $AC2_i$, $AC3_i$) for all $i \in \{1, \dots, n\}$. We use $AC123$ as short-hand for the conjunction of properties $AC1$, $AC2$ and $AC3$.

These three conditions are called *cancelation conditions,* which is classical terminology in conjoint measurement theory. The denomination of the axioms comes from the fact that these axioms express 'intrA-Criterion' cancelation conditions (in contrast to axioms RC – 'inteR-Criterion' cancelation conditions; see section 16.4). Conditions $AC1$, $AC2$ and $AC3$ were initially introduced in [BOU 99, BOU 97] and then used in [GRE 02].

Condition $AC1_i$ suggests that the elements of X_i can be ordered taking into account 'upward dominance': 'x_i upward dominates z_i' means that if $(z_i, c_{-i}) \succsim w$, then $(x_i, c_{-i}) \succsim w$. Condition $AC2_i$ has a similar interpretation taking into account 'downward dominance': 'y_i downward dominates w_i' if $x \succsim (y_i, c_{-i})$ entails $x \succsim (w_i, c_{-i})$. Condition $AC3_i$ ensures that it is possible to rank-order the elements of X_i taking into account both upward and downward dominance; these are not incompatible. It can be shown [BOU 04b, appendix A] that $AC1$, $AC2$ and $AC3$ are logically independent axioms.

Conditions $AC1$, $AC2$, $AC3$ have consequences on marginal traces. We describe them in the following proposition.

Lemma 16.4. *Completeness of marginal traces*
Let $\succsim$ be a binary relation on X. We have:

1) $\succsim_i^+$ is complete if and only if $\succsim$ verifies $AC1_i$;

2) $\succsim_i^-$ is complete if and only if $\succsim$ verifies $AC2_i$;

3) [Not $x_i \succsim_i^+ y_i \Rightarrow y_i \succsim_i^- x_i$] if and only if $\succsim$ verifies $AC3_i$;

4) $\succsim_i^\pm$ is complete if and only if $\succsim$ verifies $AC1_i, AC2_i$ and $AC3_i$.

Proof. To prove part (1), it is sufficient to observe that the negation of $AC1_i$ is equivalent to the negation of the completeness of $\succsim_i^+$. Part (2) is proven in a similar way.

Part (3). Assume that Not $x_i \succsim_i^+ y_i$; then there exist $z \in X$ and $a_{-i} \in X_{-i}$ such that $(y_i, a_{-i}) \succsim z$ and Not $(x_i, a_{-i}) \succsim z$. If $w \succsim (y_i, b_{-i})$, then $AC3_i$ entails $(x_i, a_{-i}) \succsim z$ or $w \succsim (x_i, b_{-i})$. Since by hypothesis, Not $(x_i, a_{-i}) \succsim z$, we must have $w \succsim (x_i, b_{-i})$ hence $\succsim_i^- x_i$. The converse implication results from the fact that the negation of $AC3_i$ is equivalent to the existence of $x_i, y_i \in X_i$ such that Not $y_i \succsim_i^+ x_i$ and Not $x_i \succsim_i^- y_i$.

Part (4) is a direct consequence of the first three parts. $\diamond$

Conditions $AC1$, $AC2$ and $AC3$ together imply that the marginal traces $\succsim_i^\pm$ induced by $\succsim$ are (complete) weak orders. We can expect that these axioms have consequences on marginal preferences $\succsim_i$. Note, however, that marginal preferences and marginal traces on levels do not generally coincide, even under conditions $AC123$. The following results are given without proofs (these can be found in [BOU 04b, proposition 3]).

Proposition 16.4. *Properties of marginal preferences*
We have:

1) If $\succsim$ is reflexive and verifies $AC1_i$ or $AC2_i$ for all $i \in \{1, \ldots, n\}$, then $\succsim$ is weakly separable and satisfies condition (16.12).

2) If $\succsim$ is reflexive and verifies $AC1_i$ or $AC2_i$ then $\succsim_i$ is an interval order.

3) If, in addition, $\succsim$ satisfies $AC3_i$, then $\succsim_i$ is a semiorder.

From part (1), using proposition 16.1, we infer that $\succsim_i$ is complete as soon as $\succsim$ is reflexive and verifies $AC1_i$ or $AC2_i$.

We know that if $\succsim$ is reflexive and satisfies $AC123$, the marginal traces $\succsim_i^\pm$ are weak orders (lemma 16.4(4)). Under the same conditions, part (3) of the previous proposition tells us that marginal preferences $\succsim_i$ are semiorders. This suggests that marginal traces and preferences are distinct relations, which is confirmed by examples in [BOU 04b]; we shall see conditions ensuring that these relations are identical below. If they are distinct, we have seen that $x_i \succsim_i^\pm y_i$ entails $x_i \succsim_i y_i$ as soon as $\succsim$ is reflexive. Since under $AC123$, $\succsim_i^\pm$ and $\succsim_i$ are complete, this means that under these conditions $\succsim_i^\pm$ is more discriminant than $\succsim_i$ (in the sense that $\sim_i^\pm \subseteq \sim_i$: more pairs are indifferent with respect to marginal preference than to marginal trace).

Axioms $AC123$ are not only related to the completeness of marginal traces but also to the monotonicity properties of the function F that appears in models of type (16.8). In the next proposition, we establish a characterization of models ($L5$) and ($L6$). We prove the result only for the case where X is a countable set.

Proposition 16.5. *Characterization of ($L5$) and ($L6$)*
Let $\succsim$ be a binary relation on the countable set $X = \prod_{i=1}^n X_i$. We have that $\succsim$ verifies model ($L6$) if and only if $\succsim$ is reflexive and satisfies $AC1$, $AC2$ and $AC3$. Models ($L5$) and ($L6$) are equivalent.

Proof. Model ($L5$) is a particular case of model ($L1$); hence in that model the preference relation $\succsim$ is reflexive (proposition 16.3(1)). It is easily checked that any relation representable in model ($L5$) verifies $AC123$. Conversely, if $\succsim$ is reflexive and verifies $AC123$, we can construct a numerical representation that follows model ($L6$). As function u_i, we select a numerical representation of the weak order $\succsim_i^{\pm}$, i.e. $\forall\, x_i, y_i \in X_i$, we have:

$$x_i \succsim_i^{\pm} y_i \Leftrightarrow u_i(x_i) \geq u_i(y_i). \tag{16.20}$$

Such a representation does exist since we have assumed that X is a countable set. We then define F on $[\prod_{i=1}^{n} u_i(X_i)]^2$ by setting:

$$F([u_i(x_i)];[u_i(y_i)]) = \begin{cases} +\exp(\sum_{i=1}^{n} (u_i(x_i) - u_i(y_i))) & \text{if } x \succsim y, \\ -\exp(\sum_{i=1}^{n} (u_i(y_i) - u_i(x_i))) & \text{otherwise.} \end{cases} \tag{16.21}$$

That F is well defined results from lemma 16.1(4). The fact that F is increasing in its first n arguments and decreasing in its last n arguments is a consequence of the definition of F and of lemma 16.1(3). $\diamond$

The case in which X is not denumerable does not raise serious difficulties. A necessary and sufficient condition for its representability is that the marginal traces of $\succsim$ are representable on the real numbers, which is equivalent to imposing an ‘order-density’ condition. We say that $\succsim_i^{\pm}$ satisfies the ‘order-density’ condition $OD_i^{\pm}$ if there is a denumerable subset $Y_i \subseteq X_i$ such that $\forall x_i, z_i \in X_i$,

$$x_i \succ_i^{\pm} z_i \;\Rightarrow\; \exists\, y_i \in Y_i \text{ such that } x_i \succsim_i^{\pm} y_i \succsim_i^{\pm} z_i. \tag{16.22}$$

Conditional to this additional condition imposed on $\succsim$ for all $i \in \{1, \ldots, n\}$ is that the characterization of the above models remains valid.

Note also that the slightly more general case of models ($L3$) and ($L4$) is dealt with very similarly. These models are equivalent and the preferences that can be represented in these models are those that verify $AC1$, $AC2$ and $AC3$ (they need not be reflexive).

16.3.3. *Model ($L8$) and strict monotonicity w.r.t. traces*

In order to obtain a characterization of the more constrained model in Table 16.1, we introduce two new axioms that are effective only when the preference relation is complete. These axioms follow the scheme of the classical ‘triple cancelation’ axioms that are used in the characterization of additive value function models. That is the reason why we denote them by the acronym TAC (Triple intrA-Criteria annulation).

Definition 16.5. Conditions $TAC1$, $TAC2$
We say that $\succsim$ satisfies

$TAC1_i$ if

$$\left.\begin{array}{c} (x_i, a_{-i}) \succsim y \\ \text{and} \\ y \succsim (z_i, a_{-i}) \\ \text{and} \\ (z_i, b_{-i}) \succsim w \end{array}\right\} \Rightarrow (x_i, b_{-i}) \succsim w,$$

and $TAC2_i$ if

$$\left.\begin{array}{r}(x_i, a_{-i}) \succsim y \\ \text{and} \\ y \succsim (z_{-i}, a_{-i}) \\ \text{and} \\ w \succsim (x_i, b_{-i})\end{array}\right\} \Rightarrow w \succsim (z_i, b_{-i}),$$

for all $x_i, z_i \in X_i$, for all $a_{-i}, b_{-i} \in X_{-i}$ and for all $y, w \in X$.

We say that $\succsim$ satisfies $TAC1$ (respectively, $TAC2$) if it satisfies $TAC1_i$ (respectively, $TAC2_i$) for all $i \in \{1, \dots, n\}$. We use also $TAC12$ as short-hand for $TAC1$ and $TAC2$.

The first two conditions in the premise of $TAC1_i$ and $TAC2_i$ suggest that level x_i is not lower than level z_i. $TAC1_i$ (respectively, $TAC2_i$) entail that x_i should then upward (respectively, downward) dominate z_i.

We give without proof a few consequences of $TAC1$ and $TAC2$. These axioms will only be imposed to complete relations; without this hypothesis, they have rather limited power.

Lemma 16.5. *Strictly positive responsiveness to the traces on levels*
If $\succsim$ is a complete binary relation on $X = \prod_{i=1}^{n} X_i$ then:

1) $TAC1_i \Rightarrow [AC1_i$ and $AC3_i]$

2) $TAC2_i \Rightarrow [AC2_i$ and $AC3_i]$

3) $TAC1_i$ is equivalent to the completeness of relation $\succsim_i^{\pm}$ together with the condition:

$$[x \succsim y \text{ and } z_i \succ_i^{+} x_i] \Rightarrow (z_i, x_{-i}) \succ y. \tag{16.23}$$

4) $TAC2_i$ is equivalent to the completeness of relation $\succsim_i^{\pm}$ together with the condition:

$$[x \succsim y \text{ and } y_i \succ_i^{-} w_i] \Rightarrow x \succ (w_i, y_{-i}). \tag{16.24}$$

5) If $TAC1_i$ or $TAC2_i$, then $\succsim$ is independent for $\{i\}$ and $\succsim_i$ is a weak order. Moreover, if we have $TAC12$ then $\succsim_i = \succsim_i^{\pm}$.

As we can see, as soon as $\succsim$ is complete, the conjunction of $TAC1_i$ and $TAC2_i$ guarantees that $\succsim$ responds in a strictly increasing manner to the marginal trace $\succ_i^{\pm}$. These properties also imply that $\succsim$ is weakly independent on criterion $\{i\}$ and that the marginal preference $\succsim_i$ is a weak order and identical to the marginal trace $\succsim_i^{\pm}$. We do not examine in detail here the relationship between $TAC1_i$, $TAC2_i$ on the one hand and $AC1_i$, $AC2_i$, $AC3_i$ on the other. We shall return to this in section 16.3.6. It can be shown [BOU 04b, appendix A] that for a complete relation, $TAC1$ and $TAC2$ are logically independent properties.

Note that the above system of axioms does not imply that the preference $\succsim$ has strong properties such as transitivity or even semi-transitivity or the Ferrers property. In these models (even in the more constrained i.e. model ($L8$)), the preference cannot even be supposed to be an interval order. The previous results lead directly to the characterization of model ($L8$).

Proposition 16.6. *Characterization of ($L8$)*
Let $\succsim$ be a binary relation on the denumerable set $X = \prod_{i=1}^{n} X_i$. The relation $\succsim$ verifies model ($L8$) if and only if $\succsim$ is complete and satisfies $TAC1$ and $TAC2$.

Proof. The proof follows exactly the same scheme as that of proposition 16.5. The only difference lies in the definition of function F which has to be altered in order to take into account the completeness of $\succsim$. We define F on $[\prod_{i=1}^{n} u_i(X_i)]^2$, substituting equation (16.21) by:

$$F([u_i(x_i)];[u_i(y_i)]) = \begin{cases} +\exp(\sum_{i=1}^{n}(u_i(x_i) - u_i(y_i))) & \text{if } x \succsim y, \\ 0 & \text{if } x \sim y, \\ -\exp(\sum_{i=1}^{n}(u_i(y_i) - u_i(x_i))) & \text{otherwise.} \end{cases} \tag{16.25}$$

Parts (3) and (4) of lemma 16.5 entail that F is strictly increasing (respectively, decreasing) in its first (respectively, last) n arguments since, in this construction, the u_i have been chosen to be numerical representations of the weak orders $\succsim_i^{\pm}$. $\diamond$

16.3.4. *Complete characterization of the models on levels*

To be complete, we give without proof [see BOU 04b] a characterization of all the models on levels described in Table 16.1. We limit ourselves to the case in which the set X is denumerable. The non-denumerable case can be dealt with without major difficulty by imposing order density conditions on the traces, starting from model ($L4$).

Theorem 16.1. *Models based on traces on levels*
Let $\succsim$ be a binary relation on the denumerable set $X = \prod_{i=1}^{n} X_i$. This relation can be represented in

1) model ($L1$) if and only if $\succsim$ is reflexive;

2) model ($L2$) if and only if $\succsim$ is complete;

3) model ($L4$) if and only if $\succsim$ verifies $AC1$, $AC2$ and $AC3$; models ($L3$) and ($L4$) are equivalent;

4) model ($L6$) if and only if $\succsim$ is reflexive and verifies $AC1$, $AC2$ and $AC3$; models ($L5$) and ($L6$) are equivalent;

5) model ($L7$) if and only if $\succsim$ is complete and verifies $AC1$, $AC2$ and $AC3$;

6) model ($L8$) if and only if $\succsim$ is complete and verifies $TAC1$ and $TAC2$.

Let us observe that increasing or non-decreasing (respectively, decreasing or non-increasing) do not make a difference in our models unless function F is also supposed to be antisymmetric (i.e. $F([u_i(x_i)]; [u_i(y_i)]) = -F([u_i(y_i)]; [u_i(x_i)])$). In this case, the value ‘0’ plays a special role, which is to represent indifference. This is what led us to differentiate the increasing case from the non-decreasing one.

16.3.4.1. *Uniqueness and regular representations*

All these models have obviously rather poor properties regarding uniqueness of numerical representation. A large variety of functions can of course be used for F as well as for the u_i. Nevertheless, it is not difficult to determine necessary and sufficient conditions that these functions must fulfill. Let us consider, for instance, model ($L6$). Our proof of proposition 16.5 shows that it is always possible to use functions u_i that verify:

$$x_i \succsim_i^{\pm} y_i \Leftrightarrow u_i(x_i) \geq u_i(y_i). \tag{16.26}$$

Let us refer to a representation in which the functions u_i verify equation (16.26) as *regular*. According to our proof, any strictly increasing transformation of a function u_i verifying this condition can also be used and yields another valid representation. Other choices can be made, however. It is easy to see that any function u_i that satisfies

$$x_i \succ_i^{\pm} y_i \Rightarrow u_i(x_i) > u_i(y_i) \tag{16.27}$$

can be used in a representation of $\succsim$ in model ($L6$).

Regarding function F, we can substitute the exponential of the sum of the differences of the $2n$ arguments, that appears in equation (16.21), by any real-valued positive function defined on $\mathbb{R}^{2n}$ (or at least on the subset $[\prod_{i=1}^n u_i(X_i)]^2$) that is increasing in its first n arguments and decreasing in its last n ones. It is also clear that only such functions can be used.

The representations described above are the only possible ones for model ($L6$). It is easy to adapt the reasoning that we have just used to cover all the models considered here [BOU 04b].

16.3.5. *Relations compatible with dominance*

Why should we be particularly interested in models ($L5$), ($L6$) and ($L8$)? The major reason is related to the application of conjoint measurement models to multiple criteria decision analysis. In this field of application the preference is usually constructed; it is not known *a priori*. The process of constructing the preference relies upon data (that are the evaluations of the alternatives on the various attributes recognized as relevant for the decision) and their interpretation in terms of preference on each criterion.

We emphasize that we have not assumed any *a priori* structure on the sets X_i. We did not suppose that they are sets of numbers; they may be ordered sets or even nominal scales. The interpretation of the evaluations of the alternatives in terms of preference requires at least the definition of an ordering of the elements of X_i, an order that would correspond to the direction of increasing preference of the decision maker on the viewpoint attached to that attribute. The set X_i endowed with this interpretation is what we call a *criterion* [ROY 93].

We expect of course the existence of certain logical connections between the criteria and global preference. *Respect of dominance* is such a natural connection [ROY 85, ROY 93] and [VIN 89]. (This notion of dominance must not be confused with that introduced just after definition 16.4. The latter only deals with the relative positions of the levels on the scale of a single attribute. We called it ‘upward dominance’ and ‘downward dominance’ due to the lack of a more appropriate term.) In conjoint measurement theory, no order is *a priori* postulated on the sets X_i. Would it exist, such an order should be compatible with global preference. We can therefore formulate the principle of the respect of dominance in a conjoint measurement context as follows.

Definition 16.6. A reflexive binary relation $\succsim$ on a set $X = \prod_{i=1}^{n} X_i$ is *compatible with a dominance relation* if for all $i \in \{1,\dots,n\}$, there is a weak order S_i on X_i such that for all $x, y \in X$ and all $z_i, w_i \in X_i$,

$$[x \succsim y, z_i S_i x_i \text{ and } y_i S_i w_i \text{ for all } i \in \{1,\dots,n\}] \Rightarrow z \succsim w. \tag{16.28}$$

We say that this compatibility is *strict* if the conclusion of condition (16.28) is modified in $z \succ w$ as soon as, for some $j \in \{1,\dots,n\}$, $z_j P_j x_j$ or $y_j P_j w_j$ (where P_j denotes the asymmetric part of S_j).

This definition requires a comment. It could be thought that a reasonable definition of the compatibility with a dominance relation would require the fulfillment of the following condition instead of condition (16.28):

$$[x_i S_i y_i \text{ for all } i \in \{1,\dots,n\}] \Rightarrow x \succsim y. \tag{16.29}$$

The reader will easily be convinced that defining compatibility in this way would make this notion too weak in case the preference relation cannot be supposed transitive. Indeed, if $\succsim$ has cycles in its asymmetric part, it is possible that this relation verifies condition (16.28) while there exist alternatives $x, y, z \in X$ such that $x \Delta y$, $y \succ z$ and $z \succ x$ (where the dominance relation $x \Delta y$ is defined by $[x_i S_i y_i$ for all $i \in \{1,\dots,n\}]$). In such a case, the non-dominated alternatives (w.r.t. relation Δ) need not always be considered as good choices in a multiple criteria choice decision problem since x could be non-dominated while there would exist an alternative z such that $z \succ x$.

Definition 16.6 avoids this drawback since, using condition (16.28), $x \Delta y$ and $y \succ z$ imply $x \succsim z$, which contradicts $z \succ x$.

In view of the results in section 16.3.2, establishing a link between relations $\succsim_i^{\pm}$ and the monotonicity of F, we can expect that when a preference $\succsim$ is compatible with a dominance relation, the relations S_i in definition 16.6 are related to the marginal traces $\succsim_i^{\pm}$. It is indeed the case as shown in the next proposition (in which we limit ourselves to reflexive preference relations; the case of asymmetric relations could be treated similarly).

Proposition 16.7. *Compatibility with dominance*
A reflexive binary relation $\succsim$ on a set $X = \prod_{i=1}^{n} X_i$ is compatible *with a dominance relation if and only if it satisfies $AC1$, $AC2$ and $AC3$. In such a case, S_i is compatible with $\succsim_i^{\pm}$ in the following sense:*

$$x_i \succ_i^{\pm} y_i \;\Rightarrow\; \textit{Not } y_i S_i x_i. \tag{16.30}$$

Proof. The necessity of $AC1$, $AC2$ and $AC3$ is almost immediate. Consider the case of $AC1$, the cases of the other axioms being similar. Assume that $(x_i, a_{-i}) \succsim y$ and $(z_i, b_{-i}) \succsim w$. Relation S_i being complete, we have either $x_i S_i z_i$ or $z_i S_i x_i$. If we have $z_i S_i x_i$ then, using the definition of compatibility with dominance, $(x_i, a_{-i}) \succsim y$ entails $(z_i, a_{-i}) \succsim y$. If we have $x_i S_i z_i$, then $(z_i, b_{-i}) \succsim w$ entails $(x_i, b_{-i}) \succsim w$. As a consequence, $AC1$ is verified.

The fact that $AC1$, $AC2$ and $AC3$ are sufficient conditions is clear. We can indeed take $S_i = \succsim_i^{\pm}$ for all $i \in \{1, \ldots, n\}$. Under $AC123$, the relations $\succsim_i^{\pm}$ are complete weak orders (lemma 16.4(4)) and, using lemma 16.1(3), we get equation (16.28).

To show equation (16.30), let us suppose on the contrary that there exist $x_i, y_i \in X_i$ with $x_i \succ_i^{\pm} y_i$ and $y_i S_i x_i$. From the former relation we deduce that there exist either $a_{-i} \in X_{-i}$ and $z \in X$ such that $(x_i, a_{-i}) \succsim z$ and Not $(y_i, a_{-i}) \succsim z$, or $b_{-i} \in X_{-i}$ and $w \in X$ such that $w \succsim (y_i, b_{-i})$ and Not $w \succsim (x_i, b_{-i})$. In both cases, using $y_i S_i x_i$ and applying equation (16.28) leads to a contradiction. $\diamond$

From this result we deduce, when the preference $\succsim$ is compatible with a dominance relation, that $\succsim_i^{\pm}$ cannot be finer than S_i. In other words, $S_i \subseteq \succsim_i^{\pm}$. From a practical point of view, if we consider that a global preference $\succsim$ compatible with a dominance relation is the result of the aggregation of relations S_i defining the criteria, we understand that $\succsim$ cannot induce a trace on X_i that would contradict S_i; $\succsim$ cannot even create a preference where S_i only sees indifference. Even although, for a reflexive preference satisfying $AC123$, we cannot guarantee the uniqueness of the relations S_i, we see that such relations are strongly constrained: S_i can only be a weak order included in $\succsim_i^{\pm}$.

With the previous proposition, model ($L6$) (or the equivalent model ($L5$)) can be seen as a natural framework for describing preferences compatible with a dominance relation. This prompts the question of a similar framework for preferences that are *strictly* compatible with a dominance relation. Surprisingly, the natural framework for such preferences is not model ($L8$). This model imposes complete preferences which is not, as we shall see, a necessary condition for strict dominance.

16.3.6. *Strict compatibility with dominance*

Strict compatibility with dominance requires, of course, stronger axioms than AC_1, AC_2, AC_3. We refer to the following strengthening of $AC3$ as $AC4$.

Definition 16.7. Condition $AC4$
We say that $\succsim$ satisfies $AC4_i$ if $\succsim$ verifies $AC3_i$ and if, whenever one of the consequences in $AC3_i$ is false, then the other consequence is strictly satisfied, i.e. with $\succ$ instead of $\succsim$. We say that $\succsim$ satisfies $AC4$ if it satisfies $AC4_i$ for all $i \in \{1, \ldots, n\}$.

The following lemma that we state without proof [see BOU 04b] collects a few consequences of $AC4$.

Lemma 16.6. *Consequences of* $AC4$
If $\succsim$ *is a relation on* X*, we have:*

1) If $\succsim$ is reflexive, $AC4_i$ is equivalent to the completeness of $\succsim_i^{\pm}$ and the conjunction of the following two conditions:

$$[x \succsim y \text{ and } z_i \succ_i^{\pm} x_i] \Rightarrow (z_i, x_{-i}) \succ y, \tag{16.31}$$

$$[x \succsim y \text{ and } y_i \succ_i^{\pm} w_i] \Rightarrow x \succ (w_i, y_{-i}). \tag{16.32}$$

2) If $\succsim$ is reflexive and satisfies $AC4_i$ then
- $\succsim$ is independent for $\{i\}$,
- $\succsim_i$ is a weak order and
- $\succsim_i = \succsim_i^{\pm}$.

3) If $\succsim$ is complete, $[TAC1_i$ and $TAC2_i] \Leftrightarrow AC4_i$.

As soon as $\succsim$ is reflexive, condition $AC4$ (which, by definition, is stronger than $AC3$) also entails $AC1$ and $AC2$ since it implies the completeness of relations $\succsim_i^{\pm}$ (lemmas 16.6(1) and 16.4(4)). If $\succsim$ is complete, $AC4$ is equivalent to $TAC1$ and $TAC2$, which also provides (see proposition 16.6) an alternative characterization of model ($L8$): $\succsim$ satisfies ($L8$) if and only if $\succsim$ is complete and verifies $AC4$.

$AC4$ has the advantage over $TAC1$ and $TAC2$ that it implies a strictly positive response to marginal traces even when $\succsim$ is incomplete. It is the condition that we look for in view of obtaining a characterization of strict compatibility with dominance.

Proposition 16.8. *Strict compatibility with dominance*
A reflexive binary relation $\succsim$ on a set $X = \prod_{i=1}^{n} X_i$ is strictly compatible *with a dominance relation if and only if it satisfies $AC4$. In such a case, the relations S_i are uniquely determined and $S_i = \succsim_i^{\pm}$, for all i.*

The proof of this proposition is similar to that of proposition 16.7; [see BOU 04b].

Let us observe that the conditions ensuring strict compatibility with a dominance relation do not, however, guarantee that $\succsim$ possesses 'nice' properties such as completeness or transitivity. It is straightforward, using examples inspired by Condorcet's paradox [e.g. SEN 86], to build a binary relation $\succsim$ that is strictly compatible with a dominance relation and has circuits in its asymmetric part (building for example $\succsim$ via the majority rule applied to relations S_i).

16.3.7. *The case of weak orders*

Visiting more classical models of preferences, i.e. models in which the preference is a weak order, we examine how this hypothesis combines with our axioms. When $\succsim$ is a weak order, the marginal trace $\succsim_i^{\pm}$ is identical to the marginal preference $\succsim_i$. We give the following results without proof [see BOU 04b].

Lemma 16.7. *Case of a weak order*
If $\succsim$ is a weak order on the set $X = \prod_{i=1}^{n} X_i$, we have:

1) [$\succsim$ *is weakly separable*] $\Leftrightarrow$ [$\succsim$ *satisfies* $AC1$] $\Leftrightarrow$ [$\succsim$ *satisfies* $AC2$] $\Leftrightarrow$ [$\succsim$ *satisfies* $AC3$]*; and*

2) [$\succsim$ *is weakly independent*] $\Leftrightarrow$ [$\succsim$ *satisfies* $AC4$] $\Leftrightarrow$ [$\succsim$ *satisfies* $TAC1$ *and* $TAC2$].

In the case of weakly independent weak orders, we can neglect considering marginal traces; we do not need tools more refined than marginal preferences for analyzing preferences when these are weakly independent weak orders. Note that the case of weak orders is highly specific: see [BOU 04b, appendix A] for examples of weakly separable (even weakly independent) semiorders which violate $AC1$, $AC2$ and $AC3$. In this slightly less constrained case, weak separability is not equivalent to $AC1$, $AC2$ or $AC3$.

Using these observations, it is easy to prove the following proposition.

Proposition 16.9. *Let* $\succsim$ *be a weak order on a denumerable set* $X = \prod_{i=1}^{n} X_i$*. There exist real-valued functions* u_i *defined on* X_i *and a real-valued function* U *on* $\prod_{i=1}^{n} u_i(X_i)$ *such that for all* $x, y \in X$*,*

$$x \succsim y \Leftrightarrow U(u_1(x_1), \dots, u_n(x_n)) \geq U(u_1(y_1), \dots, u_n(y_n)) \geq 0. \tag{16.33}$$

Function U *in equation (16.33) can be chosen to be:*

1) non-decreasing *in all its arguments if and only if* $\succsim$ *is weakly separable; and*

2) increasing *in all its arguments if and only if* $\succsim$ *is weakly independent.*

Proof. We start with applying Cantor's classical result [CAN 95]: any weak order $\succsim$ on a denumerable set X admits a numerical representation, i.e. there exists a function $f : X \to \mathbb{R}$ such that $x \succsim y \Leftrightarrow f(x) \geq f(y)$. In the general case, a factorization of f as $U(u_1(x_1), \dots, u_n(x_n))$ obtains, as in the proof of proposition 16.2(1), the following. We choose functions u_i that separate the equivalence classes of $\succsim_i^{\pm}$ (see condition (16.13): $x_i \sim_i^{\pm} y_i \Leftrightarrow u(x_i) = u_i(y_i)$) and we define U setting $f(x) = U(u_1(x_1), \dots, u_n(x_n))$. In the weakly separable and weakly independent cases, u_i will be a numerical representation of the marginal preference, the weak order $\succsim_i$ or the marginal trace $\succsim_i^{\pm}$ which is equivalent here. We define U as before. Combining the results of lemmas 16.4, 16.6 and 16.7 we show that U is non-decreasing (respectively, increasing) in each of its arguments. $\diamond$

The non-denumerable case requires the adjunction of the usual hypothesis limiting the cardinality of X and guaranteeing the existence of numerical representations for the weak orders $\succsim$ and $\succsim_i$ (order-density condition).

While the case of a representation with an increasing function U is well known in the literature [KRA 71, theorem 7.1], the result in the case of non-decreasing U generalizes a theorem obtained by [BLA 78] under the hypothesis that $X \subseteq \mathbb{R}^n$.

16.3.8. *Examples*

Models (16.1), (16.3) and (16.6) enter into the framework of our models using traces on levels. Among them, the additive value function model (16.1) is the only one in which the preference is a weak order. However, all three models have marginal traces $\succsim_i^{\pm}$ that are weak orders.

In contrast, in the additive non-transitive model (16.4), the marginal traces of the preference relation are not necessarily complete. Postulating the latter condition in this model drives us closer to Tversky's additive differences model (16.3).

Let us briefly review the three models cited above, for the aim of illustration.

The additive value function model (16.1) belongs to model ($L8$), the more constrained of our models based on levels. In addition, the preferences representable by an additive value function are weak orders. In view of lemma 16.6, marginal traces and marginal preferences are identical and are weak orders. The functions u_i that appear in (16.1) are numerical representations of the marginal preferences (or traces). The preference reacts in a strictly positive way to any progress of an alternative on any marginal trace.

Tversky's additive differences model (16.3) tolerates intransitive preferences. Like the additive value function it belongs to the more constrained class of models ($L8$). Lemma 16.6 applies also to this model, in which marginal traces and preferences are identical; the functions u_i that appear in (16.3) are numerical representations of these marginal preferences (or traces). We shall turn again to this model in section 16.5.2 since it is also based on the traces of differences (represented by the functions Φ_i).

Although the models based on levels are not the most adequate for describing relations obtained by outranking methods (a basic version of which is described by condition (16.6)), such relations nevertheless possess marginal traces that are weak orders. The preference relations representable in model (16.6) belong to class ($L5$) or ($L6$). The asymmetric part of their marginal preferences $\succ_i$ is usually empty. Indeed, the marginal preference on dimension j does not discriminate at all between levels unless the weight p_j of criterion j is 'dominant', i.e. if $\sum_{i=1}^n w_i \geq \lambda$, while $\sum_{i;i\neq j} w_i < \lambda$.

At this stage, it may come as a surprise to see that the additive value function model and the additive differences model belong to the same class ($L8$) of models on the levels. In particular, for those models, there is no distinction between marginal preferences and traces. Does this mean that the only interesting class of models on the levels is ($L8$), if we except the models inspired by the majoritarian methods in Social Choice (such as the ELECTRE methods)? If the answer were positive, the more refined analysis made here (which consists of carefully distinguishing marginal traces from marginal preferences) would lose a great deal of its interest. As well as the fact that our approach allows us to understand important issues such as the respect of a dominance relation (section 16.3.5), there exist models that are both genuinely interesting and cannot be described satisfactorily in terms of marginal preferences. Let us consider for instance a preference $\succsim$ which is representable in an additive value function model with a threshold:

$$\begin{array}{lll} x \succ y & \Leftrightarrow & \sum_{i=1}^n u_i(x_i) \geq \sum_{i=1}^n u_i(y_i) + \varepsilon \\ x \sim y & \Leftrightarrow & \left|\sum_{i=1}^n u_i(x_i) - \sum_{i=1}^n u_i(y_i)\right| \leq \varepsilon, \end{array} \quad (16.34)$$

where ε is a positive number representing a threshold above which a difference of preference becomes noticeable; differences that do not reach this threshold escape perception and lead to an indifference judgement ($\sim$). The preferences $\succsim$ that can be described by such a model are not weak orders but semiorders. The asymmetric part $\succ$ of the preference is transitive, while indifference $\sim$ is not [LUC 56, PIR 97]. Such a model can be used e.g. for describing a statistical test for the comparison of means (taking into account that, in this context, relation $\succsim$ should not be interpreted as a preference but rather as a comparative judgement on two quantities). It is impossible to analyze such a relation in terms of marginal preferences. Indeed, the latter can be represented by

$$x_i \succsim_i y_i \Leftrightarrow u_i(x_i) \geq u_i(y_i) - \varepsilon,$$

which implies that each marginal preference relation $\succsim_i$ is a semiorder. Generally, marginal traces are more discriminant. They are weak orders; if the set of alternatives is sufficiently rich (it is the case, for instance, when the image sets $u_i(X_i)$ are intervals of the real line), they can be represented by the functions u_i (i.e. $x_i \succsim_i^{\pm} y_i \Leftrightarrow u_i(x_i) \geq u_i(y_i)$). In this model, preference $\succsim$ is complete and its marginal traces are complete; hence it belongs to model ($L7$). It is likely that the reason why such models have received little attention is related to the fact that the dominant additive value function model does not require tools more refined than marginal preferences for its analysis. In the next section, we are interested in another fundamental tool for analyzing preferences: traces on differences.

Before closing this section, there is a final issue to be discussed. In the last part of this section, devoted to preferences that are weak orders (section 16.3.7), we distinguished weakly separable and weakly independent weak orders. The reader may wonder if there are interesting preference relations that are weak orders, weakly separable but not weakly independent. The answer is definitely positive. Consider for instance the additive value function model (16.1) and substitute the sum by a 'minimum' or a 'maximum' operator. We then obtain a weak order that is weakly separable but not independent. Indeed, let $(X_i) = [0, 10]$ and $u_i(x_i) = x_i$ for $i = 1, 2$. Preference $\succsim$ compares the alternatives only taking into consideration their 'weak point', that is $x \succsim y$ if and only if $\min x_i \geq \min y_i$. Clearly, marginal traces and marginal preferences are identical and correspond to the usual order of the real numbers of the interval $[0, 10]$. Let $x = (3, 5)$ and $y = (7, 3)$; we have $x \sim y$, but preference $\succsim$ does not strictly react if e.g. we raise the level of x on the second dimension. Even if we set x_2 to 10, we still have $(3, 10)$ indifferent to $(7, 3)$.

Other decision rules of practical importance, such as 'LexiMin' or 'LexiMax', the Choquet integral, the Sugeno integral (see section 17.5) lead in general to weak orders that are weakly separable but not weakly independent.

16.4. Models using marginal traces on differences

In this section we study preference models obtained in a similar manner to those in the previous section; we simply substitute marginal traces on levels by marginal traces on differences.

16.4.1. *Models definition*

We start from the trivial model ($D0$) based on marginal traces and introduced in section 16.2.5, in which:

$$x \succsim y \Leftrightarrow G([p_i(x_i, y_i)]) \geq 0.$$

We define the following variants:

– model ($D1$), by imposing that $p_i(x_i, x_i) = 0$ on ($D0$);

– model ($D2$), by imposing that each p_i is antisymmetric, i.e. $p_i(x_i, y_i) = -p_i(y_i, x_i)$, on ($D1$); and

– model ($D3$), by imposing that G is odd, i.e. $G(\mathbf{x}) = -G(-\mathbf{x})$, on ($D2$).

In the same way as in section 16.3, we also consider the models obtained by assuming in each variant ($D0$), ($D1$), ($D2$) and ($D3$), that G is non-decreasing or increasing in each of its n arguments which yields twelve models as defined in Table 16.2.

($D0$)	$x \succsim y \Leftrightarrow G([p_i(x_i, y_i)]) \geq 0$
($D1$)	($D0$) with $p_i(x_i, x_i) = 0$
($D2$)	($D1$) with $p_i(x_i, y_i) = -p_i(y_i, x_i)$
($D3$)	($D2$) with G odd
	....
($D4$)	($D0$) with G non-decreasing
($D8$)	($D0$) with G increasing
	....
($D5$)	($D1$) with G non-decreasing
($D9$)	($D1$) with G increasing
	....
($D6$)	($D2$) with G non-decreasing
($D10$)	($D2$) with G increasing
	....
($D7$)	($D3$) with G non-decreasing
($D11$)	($D3$) with G increasing

Table 16.2. *Models using traces on differences*

There are obvious implications linking these models; we do not detail them. As well as these implications, the properties of G in models ($D1$), ($D2$) and ($D3$) entail simple properties of the relations representable in these models. We shall lean on these properties to characterize the models.

Proposition 16.10. *Characterization of ($D1$), ($D2$) and ($D3$)*
A binary relation $\succsim$ on a product set $X = \prod_{i=1}^{n} X_i$ having at most the cardinality of $\mathbb{R}$ can be represented in

1) model ($D1$) or model ($D2$) if and only if $\succsim$ is independent; and

2) model ($D3$) if and only if $\succsim$ is independent and complete.

Proof. Part (1). We have $p_i(x_i, x_i) = 0$ in model ($D1$), which implies that $(x_i, a_{-i}) \succsim (x_i, b_{-i}) \Leftrightarrow G(0, (p_j(a_j, b_j))_{j \neq i}) \geq 0 \Leftrightarrow (y_i, a_{-i}) \succsim (y_i, b_{-i})$. As a consequence, $\succsim$ is independent as soon as $\succsim$ is representable in model ($D1$).

Assume conversely that $\succsim$ is independent and let us construct a representation of $\succsim$ in model ($D2$). We reconsider the construction of a representation described in the proof of part (2) of proposition 16.2, and slightly modify it. The alteration is related to the specification of functions p_i. These functions separate the equivalence classes of $\sim_i^{**}$: $(x_i, y_i) \sim_i^{**} (z_i, w_i) \Leftrightarrow p_i(x_i, y_i) = p_i(z_i, w_i)$. Nothing prevents us from imposing on p_i the verification of $p_i(x_i, x_i) = 0$ for a certain $x_i \in X_i$. Since $\succsim$ is independent, $(x_i, x_i) \sim_i^{**} (y_i, y_i)$ for all $y_i \in X_i$ and hence $p_i(y_i, y_i) = 0$ for all $y_i \in X_i$. We can also impose on p_i the verification of $p_i(x_i, y_i) = -p_i(y_i, x_i)$. Finally, G can be defined by equation (16.16) in the same way as for the trivial model, i.e.

$$G([p_i(x_i, y_i)]) = \begin{cases} +1 & \text{if } x \succsim y, \\ -1 & \text{otherwise.} \end{cases}$$

Clearly, G is well-defined and yields a representation of $\succsim$ in model ($D2$).

Part (2). The completeness of $\succsim$ is a direct consequence of the definition of model ($D3$); since model ($D3$) implies model ($D1$), $\succsim$ is independent. Reciprocally, let us assume that $\succsim$ is independent and complete. If this is the case, we use the same functions p_i as in part (1), but we change the definition of G as follows:

$$G([p_i(x_i, y_i)]) = \begin{cases} +1 & \text{if } x \succ y, \\ 0 & \text{if } x \sim y, \\ -1 & \text{otherwise.} \end{cases} \qquad (16.35)$$

We show, using independence of $\succsim$ that G is well defined. Since $\succsim$ is complete, function G is odd. $\diamond$

The monotonicity properties of G are linked with specific axioms, rather similar to those defined in section 16.3.2. We introduce them in the next section.

16.4.2. *Completeness of marginal traces on differences and monotonicity of* G

There are two axioms for each attribute i. As with $AC1$, $AC2$ and $AC3$, these axioms appear as cancelation conditions. Their denomination, $RC1$, $RC2$ recalls the fact that they are 'inteR-Criteria' cancelation conditions.

Definition 16.8. Conditions $RC1$ and $RC2$
Let $\succsim$ be a binary relation on the set $X = \prod_{i=1}^{n} X_i$. We say that this relation satisfies axiom: $RC1_i$ if

$$\left.\begin{array}{r}(x_i, a_{-i}) \succsim (y_i, b_{-i}) \\ \text{and} \\ (z_i, c_{-i}) \succsim (w_i, d_{-i})\end{array}\right\} \Rightarrow \left\{\begin{array}{l}(x_i, c_{-i}) \succsim (y_i, d_{-i}) \\ \text{or} \\ (z_i, a_{-i}) \succsim (w_i, b_{-i}),\end{array}\right.$$

and $RC2_i$ if

$$\left.\begin{array}{r}(x_i, a_{-i}) \succsim (y_i, b_{-i}) \\ \text{and} \\ (y_i, c_{-i}) \succsim (x_i, d_{-i})\end{array}\right\} \Rightarrow \left\{\begin{array}{l}(z_i, a_{-i}) \succsim (w_i, b_{-i}) \\ \text{or} \\ (w_i, c_{-i}) \succsim (z_i, d_{-i}),\end{array}\right.$$

for all $x_i, y_i, z_i, w_i \in X_i$ and for all $a_{-i}, b_{-i}, c_{-i}, d_{-i} \in X_{-i}$. We say that $\succsim$ satisfies $RC1$ (respectively, $RC2$) if it satisfies $RC1_i$ (respectively, $RC2_i$) for all $i \in \{1, \ldots, n\}$. We shall sometimes use $RC12$ for the conjunction of conditions $RC1$ and $RC2$.

Condition $RC1_i$ suggests that (x_i, y_i) corresponds to a difference of preference at least as large as (z_i, w_i) or vice versa. It is easily seen that assuming both Not $(x_i, y_i) \succsim_i^* (z_i, w_i)$ and Not $(z_i, w_i) \succsim_i^* (x_i, y_i)$ leads to a violation of $RC1_i$. From this we can see that $RC1_i$ is equivalent to the completeness of $\succsim_i^*$. The second axiom, $RC2_i$, suggests that the 'opposite' differences (x_i, y_i) and (y_i, x_i) are linked. In terms of the marginal trace on differences $\succsim_i^*$, this axiom tells us if the preference difference between x_i and y_i is not at least as large as that between z_i and w_i, then the difference between y_i and x_i is at least as large as that between w_i and z_i.

These observations are collected in the next lemma whose proof immediately results from the definitions and is omitted.

Lemma 16.8. *Completeness of the traces on differences*
We have:

1) $[\succsim_i^*$ is complete$]$ if and only if $RC1_i$;

2) $RC2_i$ if and only if [for all $x_i, y_i, z_i, w_i \in X_i,$ Not $(x_i, y_i) \succsim_i^* (z_i, w_i) \Rightarrow (y_i, x_i) \succsim_i^* (w_i, z_i)$]; and

3) $[\succsim_i^{**}$ is complete$]$ if and only if $[RC1_i$ and $RC2_i]$.

Condition $RC1$ has been introduced in [BOU 86] under the name *weak cancelation*. The extension of condition $RC1$ to subsets of attributes (instead of singletons) is of fundamental importance in [VIN 91] where this condition receives the name of *independence*. Condition $RC2$ was first proposed in [BOU 99, BOU 97, BOU 09].

We note below two easy yet important consequences of $RC1$ and $RC2$ [BOU 05b].

Lemma 16.9. *Consequences of $RC1$ and $RC2$*
We have the following:

1) if $\succsim$ satisfies $RC1_i$ then $\succsim$ is weakly separable for i; and

2) if $\succsim$ satisfies $RC2$ then $\succsim$ is independent and either reflexive or irreflexive.

Axioms $RC1$ and $RC2$ allow us to analyze all the remaining models with the exception of the more constrained model ($D11$). We observe that the properties of non-decreasingness and increasingness with respect to the traces on differences do not lead to different models except in the more constrained case (models ($D7$) and ($D11$)).

Proposition 16.11. *Characterization of models ($D4$) to ($D10$)*
A binary relation $\succsim$ on a denumerable set $X = \prod_{i=1}^{n} X_i$ can be represented in

1) model ($D4$) or model ($D8$) if and only if $\succsim$ satisfies $RC1$;

2) model ($D5$) or model ($D9$) if and only if $\succsim$ is independent and satisfies $RC1$;

3) model ($D6$) or model ($D10$) if and only if $\succsim$ satisfies $RC1$ and $RC2$;

4) model ($D7$) if and only if $\succsim$ is complete and satisfies $RC1$ and $RC2$.

Proof. Part (1). Model ($D4$) verifies $RC1$. Assume that $(x_i, a_{-i}) \succsim (y_i, b_{-i})$ and $(z_i, c_{-i}) \succsim (w_i, d_{-i})$. Using model ($D4$) we have:

$$G(p_i(x_i, y_i), (p_j(a_j, b_j))_{j \neq i}) \geq 0 \text{ and}$$
$$G(p_i(z_i, w_i), (p_j(c_j, d_j))_{j \neq i}) \geq 0.$$

If $p_i(x_i, y_i) \geq p_i(z_i, w_i)$ then, using the non-decreasingness of G, we obtain $G(p_i(x_i, y_i),$ $(p_j(c_j, d_j))_{j \neq i}) \geq 0$, hence $(x_i, c_{-i}) \succsim (y_i, d_{-i})$. If $p_i(z_i, w_i) > p_i(x_i, y_i)$, we have $G(p_i(z_i, w_i),$ $(p_j(a_j, b_j))_{j \neq i}) \geq 0$, hence $(z_i, a_{-i}) \succsim (w_i, b_{-i})$. Consequently, $RC1$ is verified.

The second part of the proof constructs a representation in model ($D8$) of a relation $\succsim$ provided it verifies $RC1$. Using $RC1$, we know that $\succsim_i^*$ is a weak order. As function p_i, we choose a numerical representation of $\succsim_i^*$ (which exists since X_i has been supposed to be denumerable): $(x_i, y_i) \succsim_i^* (z_i, w_i) \Leftrightarrow p_i(x_i, y_i) \geq p_i(z_i, w_i)$. We then define G on $p_i(X_i^2)$ as follows:

$$G([p_i(x_i, y_i)]) = \begin{cases} + \exp(\sum_{i=1}^{n} p_i(x_i, y_i)) & \text{if } x \succsim y, \\ - \exp(- \sum_{i=1}^{n} p_i(x_i, y_i)) & \text{otherwise.} \end{cases} \quad (16.36)$$

We see that G is well defined using lemma 16.2(3) and the definition of the p_i. To show that G is increasing, let us assume that $p_i(z_i, w_i) > p_i(x_{,i}, y_i)$, i.e. that $(z_i, w_i) \succ_i^* (x_i, y_i)$. If $x \succsim y$, lemma 16.2(2) implies that $(z_i, x_{-i}) \succsim (w_i, y_{-i})$ and the conclusion follows from the definition of G. If Not $x \succsim y$, we have either Not $(z_i, x_{-i}) \succsim (w_i, y_{-i})$ or $(z_i, x_{-i}) \succsim (w_i, y_{-i})$. In both cases the conclusion follows from the definition of G.

Part (2). Since model ($D5$) implies models ($D1$) and ($D4$), the necessity of the independence condition and of $RC1$ is straightforward. Under these hypotheses, we can build a representation of $\succsim$ in model ($D9$), as in part (1), with the exception that we require that p_i verifies $p_i(x_i, x_i) = 0$ (which is made possible as a consequence of the independence property; see lemma 16.2(1)).

Part (3). We readily check that if $\succsim$ is representable in model ($D6$), it satisfies $RC1$ and $RC2$. For $RC1$, it is a consequence of the fact that model ($D6$) implies model ($D4$). For $RC2$, we can proceed as for part (1) for $RC1$. The necessity of conditions $RC1$ and $RC2$ is thus proven.

Under the hypothesis that $\succsim$ satisfies $RC1$ and $RC2$, we can construct a representation of $\succsim$ in model ($D10$) as follows. By lemma 16.8(3), we know that relations $\succsim_i^*$ and $\succsim_i^{**}$ are weak orders. Since sets X_i are supposed to be denumerable, there exist functions $q_i : X_i \to \mathbb{R}$ that represent $\succsim_i^*$; we choose one such function for each i and we define p_i through $p_i(x_i, y_i) = q_i(x_i, y_i) - q_i(y_i, x_i)$. It is clear that these functions p_i are antisymmetric and provide numerical representations of relations $\succsim_i^{**}$. Using these functions p_i, we define G through equation (16.36). Lemma 16.2(5) shows that this definition makes sense. To show that G is increasing, let us assume that $p_i(z_i, w_i) > p_i(x_i, y_i)$, i.e. that $(z_i, w_i) \succ_i^{**} (x_i, y_i)$. This construction implies that $(z_i, w_i) \succsim_i^* (x_i, y_i)$. The increasingness of G can then be proven as in part (1).

Part (4). The necessity of the completeness of $\succsim$ results from proposition 16.10(2) and from the fact that model ($D7$) implies model ($D3$). The necessity of $RC1$ and $RC2$ is a consequence of the fact that model ($D7$) implies model ($D6$) and of part (3). Making these hypotheses on $\succsim$, a representation of $\succsim$ in model ($D7$) is obtained as for model ($D10$). The only difference lies in the definition of function G. We define G as follows:

$$G([p_i(x_i, y_i)]) = \begin{cases} +\exp(\sum_{i=1}^n p_i(x_i, y_i)) & \text{if } x \succ y, \\ 0 & \text{if } x \sim y, \\ -\exp(-\sum_{i=1}^n p_i(x_i, y_i)) & \text{otherwise.} \end{cases} \quad (16.37)$$

Since $\succsim$ is complete, G is odd; G is well defined as a consequence of the definition of the p_i and of lemma 16.2(5). It is non-decreasing due to lemma 16.2, parts (2) and (4). $\diamond$

16.4.3. *Characterization of model ($D11$)*

Distinguishing between models ($D7$) and ($D11$) requires the introduction of a new axiom. It is similar to axioms $TAC1$ and $TAC2$, introduced in section 16.3.2, for studying the models based on traces on levels. Here, axiom TC will only deliver its full power for complete preferences. It is useful for characterizing the model in which increasingness with respect to marginal traces on differences is distinguished from non-decreasingness.

Definition 16.9. Condition TC
Let $\succsim$ be a binary relation on the set $X = \prod_{i=1}^n X_i$. We say that this relation satisfies axiom:
TC_i if

$$\left.\begin{array}{r} (x_i, a_{-i}) \succsim (y_i, b_{-i}) \\ \text{and} \\ (z_i, b_{-i}) \succsim (w_i, a_{-i}) \\ \text{and} \\ (w_i, c_{-i}) \succsim (z_i, d_{-i}) \end{array}\right\} \Rightarrow (x_i, c_{-i}) \succsim (y_i, d_{-i}),$$

for all $x_i, y_i, z_i, w_i \in X_i$ and for all $a_{-i}, b_{-i}, c_{-i}, d_{-i} \in X_{-i}$. We say that $\succsim$ satisfies TC if it satisfies TC_i for all $i \in \{1, \ldots, n\}$.

Condition TC_i (*Triple Cancelation*) is a classical cancelation condition that has often been used [KRA 71, WAK 89] in the analysis of the additive value function model (16.1) or the additive utility model. In the next lemma, we state without proof two properties involving TC. See [WAK 88, WAK 89] for a detailed analysis of this axiom, including its interpretation in terms of differences of preference.

Lemma 16.10. *Strict monotonicity with respect to traces on differences*

1) If $\succsim$ is complete, TC_i implies $RC1_i$ and $RC2_i$.

2) If $\succsim$ is complete and verifies TC_i, we have: $[x \succsim y$ and $(z_i, w_i) \succ_i^{**} (x_i, y_i)] \Rightarrow (z_i, x_{-i}) \succ (w_i, y_{-i})$.

The second of the above properties clearly underlines that TC is related to the strict monotonicity of $\succsim$ with respect to its traces $\succsim_i^{**}$ (as soon as $\succsim$ is complete). It shows that TC is the missing link that will allow us to characterize model ($D11$).

Proposition 16.12. *Characterization of model ($D11$)*
A binary relation $\succsim$ on a denumerable product set $X = \prod_{i=1}^n X_i$ is representable in model ($D11$) if and only if $\succsim$ is complete and satisfies TC.

Proof. The necessity of these conditions is straightforward. Assuming that $\succsim$ is complete and verifies TC, we obtain by lemma 16.10(1) that $\succsim$ verifies $RC1$ and $RC2$. We thus define p_i and G as in the proof of part (4) of proposition 16.11. The increasingness of G is a consequence of lemma 16.10(2). $\diamond$

For the reader's convenience, we summarize the characterization of all the models based on marginal traces on differences in Table 16.3.

16.4.4. *Remarks*

16.4.4.1. *Goldstein's model*

Models ($D8$) and ($D4$) were introduced by Goldstein [GOL 91] as particular cases of his 'decomposable model with thresholds'; the equivalence of models ($D8$) and ($D4$) had been noticed.

16.4.4.2. *Marginal preferences*

Which role is played by marginal preferences $\succsim_i$ in the models based on traces on differences? They certainly do not play a central role but some monotonicity properties linking them to the global preference $\succsim$ can nevertheless be established. We present some of them, without proof, in the next proposition.

Proposition 16.13. *Properties of models using differences*

1) If $\succsim$ is representable in model ($D5$) then: $[x_i \succ_i y_i$ for all $i] \Rightarrow$ Not $y \succsim x$.

2) If $\succsim$ is representable in model ($D6$) then:

Model	Definition	Conditions
$(D0)$	$x \succsim y \Leftrightarrow G([p_i(x_i, y_i)]) \geq 0$	$\varnothing$
$(D1)$ $\Updownarrow$ $(D2)$	$(D0)$ with $p_i(x_i, x_i) = 0$; $(D0)$ with p_i antisymmetric	independent
$(D3)$	$(D0)$ with p_i antisymmetric and G odd	complete, independent
$(D8) \Leftrightarrow (D4)$	$(D0)$ with $G(\nearrow\nearrow)$	$RC1$
$(D9) \Leftrightarrow (D5)$	$(D1)$ with $G(\nearrow\nearrow)$	$RC1$, independent
$(D10) \Leftrightarrow (D6)$	$(D2)$ with $G(\nearrow\nearrow)$	$RC12$
$(D7)$	$(D3)$ with $G(\nearrow)$	complete, $RC12$
$(D11)$	$(D3)$ with $G(\nearrow\nearrow)$	complete, TC

Table 16.3. *Characterization of the models using traces on differences ($\nearrow$: non-decreasing, $\nearrow\nearrow$: increasing)*

- *$\succsim_i$ is complete; and*
- *$[x_i \succ_i y_i$ for all $i] \Rightarrow [x \succsim y]$.*

3) If $\succsim$ is representable in model $(D11)$ then:
- *$[x_i \succsim_i y_i$ for all $i] \Rightarrow [x \succsim y]$; and*
- *$[x_i \succsim_i y_i$ for all i and there exists $j \in \{1, \dots, n\}$ such that $x_j \succ_j y_j] \Rightarrow [x \succ y]$.*

The reader might feel somewhat disappointed while looking at the monotonicity properties of our models, except for model $(D11)$. One must however keep in mind that we address preferences that are not necessarily transitive or complete. In such a framework, properties that could be seen as natural requirements for preferences could simply be undesirable. For example, when the marginal indifference relations $\sim_i$ are not transitive, it may be inadequate to require a property such as:

$$[x_i \sim_i y_i \text{ for all } i] \Rightarrow [x \sim y].$$

Were such a property verified, it would forbid that tiny but actual differences on several criteria, none of which yield a preference when taken separately, could interact or 'cooperate ' and yield global preference. Let us consider, for example, comparing triplets $x = (x_1, x_2, x_3)$ of numbers x_i belonging to the $[0, 1]$ interval. We decide to compare these triplets using the following majoritarian method: $x \succsim y$ if and only if $x_i \geq y_i$ for at least 2 values of index i out of 3. We clearly have, on each dimension i, that $\succsim_i = \sim_i$ i.e. that there is no strict marginal preference, all pairs of levels being indifferent. Indeed, $(x_i, z_{-i}) \sim (y_i, z_{-i})$ for all x_i, y_i and z_{-i}. However, the global preference relation $\succsim$ is not reduced to indifference between all pairs of triplets (for example, $1 \succsim_i 0$ for all $i = 1, 2, 3$, but $(1, 1, 1) \succ (0, 0, 0)$).

For broader views on this topic, see [GIL 95] or [PIR 97]. As emphasized in section 16.3, marginal preferences are not a sufficiently refined tool to analyze preferences that are not necessarily transitive or complete; we have to use the marginal traces $\succsim_i^{\pm}$ instead. In the example introduced above, the traces $\succsim_i^{\pm}$ are, on each dimension i, the natural order on the $[0, 1]$ interval. The monotonicity properties of the preference with respect to marginal traces have been described in lemmas 16.1 and 16.5(4).

16.4.4.3. *Uniqueness of the representation*

Regarding the models on levels, the uniqueness properties of the representations described in propositions 16.11 and 16.12 are quite weak. In model ($D8$), for instance, we may always take any numerical representation of the weak order $\succsim_i^*$ (at least, in the finite or countable case) for $p_i(x_i, y_i)$. Regarding the models on levels, we shall call a representation in which p_i is a numerical representation of $\succsim_i^*$, for all i, *regular*. Other choices can be made, but it is necessary (and sufficient) that p_i satisfies the condition:

$$(x_i, y_i) \succ_i^* (z_i, w_i) \Rightarrow p_i(x_i, y_i) > p_i(z_i, w_i). \tag{16.38}$$

In other terms, the chosen numerical representation must be at least as discriminant as relation $\succ_i^*$. In more constrained models such as ($D7$) or ($D10$), a similar condition, involving $\succ_i^{**}$ instead of $\succ_i^*$, is needed. For more details, see [BOU 05b, lemma 5.5].

16.4.5. *Examples*

Among all the models described in the introduction, the only one that does not use traces on differences is the decomposable model (16.2), since this model aggregates the levels of each alternative independently of other alternatives. We briefly review the other models.

Let us start with the additive non-transitive preference model (16.4), which we recall here:

$$x \succsim y \Leftrightarrow \sum_{i=1}^{n} p_i(x_i, y_i) \geq 0.$$

If we do not assume any property of functions p_i, the appropriate model is ($D8$) (equivalent to ($D4$)); the p_i functions represent the traces $\succsim_i^*$ that are weak orders; and function G, which reduces to addition of its n components, is strictly increasing. Assuming additional properties of functions p_i, such as $p_i(x_i, x_i) = 0$ or antisymmetry, leads us to models ($D9$) (equivalent to ($D5$)) and ($D11$), respectively. In the latter model, p_i represents the weak order $\succsim_i^{**}$ instead of representing $\succsim_i^*$ (function G is odd).

Tversky's model of additive differences (16.3) is a particular case of the latter model. Functions p_i reduce to algebraic differences $u_i(x_i) - u_i(y_i)$ of marginal value functions that represent the traces on levels. This is therefore a model which is based both on traces on differences and on traces on levels. Such models will be investigated in the next section.

Rewriting the additive value function model (16.1) as

$$x \succsim y \Leftrightarrow \sum_{i=1}^{n}(u_i(x_i) - u_i(y_i)) \geq 0,$$

we observe that it is a particular case of the additive differences model, in which functions Φ_i reduce to identity. The differences of marginal value functions $(u_i(x_i) - u_i(y_i))$ represent the traces $\succsim_i^{**}$.

The additive value function model sharply differentiates differences of preference since each value of the difference $(u_i(x_i) - u_i(y_i))$ corresponds to a specific equivalence class of relation $\succsim_i^{**}$. In contrast, outranking methods obtained by means of condition (16.6) distinguish differences of preference in a very rough manner. In the case of the majoritarian model (16.6), p_i represents $\succsim_i^{*}$ and distinguishes only two classes of differences of preference, as shown by equation (16.7). Either difference (x_i, y_i) is 'positive', in which case the whole weight of criterion i is assigned to this difference (diminished by a fraction of the majority threshold), or else this difference is 'negative' in which case it counts for nothing. Notice that equation (16.7) provides a representation of the preference obtained by the majoritarian method in model ($D8$) while the properties of such a preference would allow it to be represented in model ($D10$). Relations $\succsim_i^{**}$ have three equivalence classes and can be represented by function:

$$p_i(x_i, y_i) = \begin{cases} w_i & \text{if} \quad x_i > y_i \\ 0 & \text{if} \quad x_i = y_i \\ -w_i & \text{if} \quad x_i < y_i. \end{cases} \tag{16.39}$$

We then define G as:

$$G(p_1, \ldots, p_n) = 1 - \sum_{i:p_i<0} p_i - \lambda. \tag{16.40}$$

Using this representation, we obtain the same relation as that defined by condition (16.6). Indeed, assuming normalized weights ($\sum w_i = 1$), we see that G computes (in a somewhat bizarre way) the sum of the weights of the criteria in which difference (x_i, y_i) is 'positive', diminished by threshold λ.

These elementary observations open the way to a characterization of majoritarian methods within the framework of model ($D10$). These methods are characterized by traces on differences $\succsim_i^{**}$ that distinguish no more than three classes of differences of preference [BOU 01, BOU 05a, BOU 07].

The ELECTRE methods, as they appear in literature [ROY 68, ROY 73, ROY 91, ROY 93], involve an additional element with respect to pure majoritarian methods. In order to decide whether x is preferred to y (x 'outranks' y), we 'weigh' the arguments in favor of x which corresponds to the majoritarian model (16.6). If this weight is large enough, we then verify that no 'strong argument' opposes the statement that x is preferred to y. By 'strong argument', we mean a difference (x_i, y_i) on some criterion i that is 'very negative', in disfavor of x. If x_i and y_i represent numerical assessments of alternatives on criterion i, a 'very negative difference' may for instance result from trespassing a threshold ν_i, called *veto threshold*; we cannot state that x is preferred to y if, on at least one criterion i, we have:

$$x_i < y_i - \nu_i.$$

We observe that the idea of a 'very negative difference' introduced a third class of preference differences in $\succsim_i^*$, corresponding to a 'veto'. Relations $\succsim_i^*$ can therefore be represented by

$$p_i(x_i, y_i) = \begin{cases} w_i & \text{if} \quad x_i \geq y_i \\ 0 & \text{if} \quad y_i - \nu_i \leq x_i < y_i \\ -M & \text{if} \quad x_i < y_i - \nu_i, \end{cases} \tag{16.41}$$

where M is a large positive number. We define G as:

$$G(p_1, \ldots, p_n) = \sum_i p_i - \lambda. \tag{16.42}$$

We easily verify that $x \succsim y$ if and only if the sum of the weights of the criteria on which x is at least as good as y passes λ and there is no criterion on which the level of x goes beyond that of y by more than the veto threshold (the value assigned to $-M$ is such that it prevents the λ threshold being reached as soon as it appears in any of the terms p_i).

A relation $\succsim$ obtained through the above-defined *majoritarian rule with veto* can be represented in model ($D10$). Relations $\succsim_i^*$ distinguish at most three classes of preference differences; relations $\succsim_i^{**}$ at most five. Such preference relations can be fully characterized within model ($D10$) [BOU 08, GRE 01a].

These examples show that models using traces on differences are well suited for describing and understanding outranking methods. We shall return to these models at the end of the following section where we shall show how relations obtained by comparing differences can generally be related to the description of the alternatives by levels on attributes. (We have assumed above that the X_i are sets of real numbers endowed with their natural order which was supposed to be compatible with the decision maker's preferences).

16.5. Models using both marginal traces on levels and on differences

After studying models based on marginal traces on levels and those based on marginal traces on differences in the previous sections, it is quite natural to discuss models based on both types of traces. This is done by expressing the differences of preference in terms of the traces on the levels.

We recall the definition of the general model ($L0D0$) presented in section 16.1.1; in this model, the preference relation $\succsim$ is defined as follows:

$$x \succsim y \Leftrightarrow H([\varphi_i(u_i(x_i), u_i(y_i))]) \geq 0. \tag{$L0D0$}$$

This model can be seen as a particular case of model ($D0$), in which functions $p_i(x_i, y_i)$ have been substituted by functions $\varphi_i(u_i(x_i), u_i(y_i))$. It is also possible to view it as a generalization of the additive differences model (16.3) in which the simple addition and subtraction operations have been substituted by general, appropriately monotonic, functions.

A model in which $p_i(x_i, y_i)$ is substituted by $\varphi_i(u_i(x_i), u_i(y_i))$ corresponds to each of the twelve models ($D0$) to ($D11$) studied in section 16.3, without imposing any additional property.

This allows us to define models ($L0D0$) to ($L0D11$). These 'new' models have in fact very little interest since they are equivalent (if the cardinality of the set of alternatives X is not larger than that of the real numbers) to the corresponding models based on traces on differences ($D0$) to ($D11$). They simply provide another representation of the same models. Indeed, starting with a given function $p_i(x_i, y_i)$ defined on $X_i \times X_i$, it is always possible to factorize it by means of a real-valued function u_i defined on X_i. The only condition that u_i must fulfill is to separate the elements of X_i that belong to different equivalence classes of the marginal trace $\succsim_i^{\pm}$. Notice that we do not assume the completeness of the traces on levels $\succsim_i^{\pm}$ (at the moment). More formally, the functions u_i must verify the following condition:

$$u_i(x_i) = u_i(y_i) \;\Rightarrow\; x_i \sim_i^{\pm} y_i.$$

For any function u_i satisfying this basic requirement and for any given function p_i, we define unambiguously the function φ_i on subset $u_i(X_i) \times u_i(X_i)$ of $\mathbb{R}^2$ by setting:

$$p_i(x_i, y_i) = \varphi_i(u_i(x_i), u_i(y_i)).$$

Consequently, starting from any representation $G([p_i(x_i, y_i)])$ of a relation $\succsim$ in one of the models based on traces on differences, we automatically obtain a representation of this relation in the corresponding model based on traces on differences and levels. This is done by substituting $p_i(x_i, y_i)$ by the function $\varphi_i(u_i(x_i), u_i(y_i))$ we have just defined. Let us note that function H is identical to G. Notice also that this substitution can be done without problem only when the cardinality of X does not exceed that of $\mathbb{R}$, and if no additional requirement is imposed on φ_i. At this stage, we do not even assume that φ_i is monotonic in its two arguments.

To make φ_i more similar to subtraction, we consider two variants of each of the twelve models ($L0D0$) to ($L0D11$). In the first variant we impose that φ_i is non-decreasing in its first argument and non-increasing in its second argument. This leads to models ($L1D0$) to ($L1D11$). In the other variant, we impose that functions φ_i must be increasing in their first argument and decreasing in their second argument. This yields models ($L2D0$) to ($L2D11$).

In summary, we have now defined $3 \times 12 = 36$ new models (see Table 16.4) using both marginal traces on levels and marginal traces on differences. Skipping the first twelve models that are not interesting as already mentioned, we study the others in the rest of this section after discussing the relationships between traces on differences and traces on levels.

$(L0D0)$	$x \succsim y \Leftrightarrow H([\varphi_i(u_i(x_i), u_i(y_i))]) \geq 0$
$(L0D1)$	$(L0D0)$ with $\varphi_i(u(x_i), u_i(x_i)) = 0$
$(L0D2)$	$(L0D1)$ with φ_i antisymmetric
$(L0D3)$	$(L0D2)$ with H odd
	....
$(L0D4)$	$(L0D0)$ with H non-decreasing
$(L0D5)$	$(L0D0)$ with H increasing
	....
$(L0D6)$	$(L0D1)$ with H non-decreasing
$(L0D7)$	$(L0D1)$ with H increasing
	....
$(L0D8)$	$(L0D2)$ with H non-decreasing
$(L0D9)$	$(L0D2)$ with H increasing
	....
$(L0D10)$	$(L0D3)$ with H non-decreasing
$(L0D11)$	$(L0D3)$ with H increasing

Table 16.4. *Models based both on traces on levels and on differences. Models* $(L1Dx)$ *correspond to models* $(N0Dx)$ *where* $\varphi_i(\nearrow, \searrow)$*; models* $(L2Dx)$ *correspond to models* $(L0Dx)$ *where* $\varphi_i(\nearrow\nearrow, \searrow\searrow)$

16.5.1. *Relationships between traces on differences and on levels*

The traces on differences $\succsim_i^*$ and $\succsim_i^{**}$ are binary relations on the product set $X_i \times X_i$. We may define their own traces on levels in the usual way. For $\succsim_i^*$, we denote

– the left (respectively, right, left-right) trace on the first dimension by $(\succsim_i^*)_1^+$ (respectively, $(\succsim_i^*)_1^-$, $(\succsim_i^*)_1^\pm$);

– the left (respectively, right, left-right) trace on the second dimension by $(\succsim_i^*)_2^+$ (respectively, $(\succsim_i^*)_2^-$, $(\succsim_i^*)_2^\pm$).

Their definition is a straightforward transposition of definition 16.3 applied to $\succsim_i^*$ instead of $\succsim$ as follows.

Definition 16.10. Left and right traces of the traces on differences
Let $\succsim$ be a preference relation on the product set X and $\succsim_i^*$ its trace on differences relative to the ith dimension. The traces of $\succsim_i^*$ are defined as follows. For all $x_i, y_i \in X_i$,

1) $x_i\,(\succsim_i^*)_1^+\;y_i$ if $\forall s_i, t_i, z_i \in X_i, (y_i, s_i) \succsim_i^* (z_i, t_i) \Rightarrow (x_i, s_i) \succsim_i^* (z_i, t_i)$;
2) $x_i\,(\succsim_i^*)_1^-\;y_i$ if $\forall s_i, t_i, z_i \in X_i, (z_i, t_i) \succsim_i^* (x_i, s_i) \Rightarrow (z_i, t_i) \succsim_i^* (y_i, s_i)$;
3) $x_i\,(\succsim_i^*)_2^+\;y_i$ if $\forall s_i, t_i, z_i \in X_i, (s_i, y_i) \succsim_i^* (t_i, z_i) \Rightarrow (s_i, x_i) \succsim_i^* (t_i, z_i)$;
4) $x_i\,(\succsim_i^*)_2^-\;y_i$ if $\forall s_i, t_i, z_i \in X_i, (t_i, z_i) \succsim_i^* (s_i, x_i) \Rightarrow (t_i, z_i) \succsim_i^* (s_i, y_i)$.

The traces of $\succsim_i^{**}$ are defined similarly.

Are there relationships between these traces and the traces on levels of $\succsim$? The answer is positive as suggested by lemma 16.3. Referring to definitions 16.2 and 16.3 of $\succsim_i^+$, $\succsim_i^-$ and $\succsim_i^*$, it is easy to see that the traces on levels $\succsim_i^+$ and $\succsim_i^-$ can be defined in terms of $\succsim_i^*$ as follows:

$$\begin{array}{lll} x_i \succsim_i^+ y_i & \text{if and only if} & \forall z_i \in X_i,\ (x_i, z_i) \succsim_i^* (y_i, z_i) \\ x_i \succsim_i^- y_i & \text{if and only if} & \forall w_i \in X_i,\ (w_i, y_i) \succsim_i^* (w_i, x_i). \end{array} \tag{16.43}$$

This means that $\succsim_i^+$ and the inverse of relation $\succsim_i^-$, $(\succsim_i^-)^{-1}$, can be interpreted as the marginal relations of relation $\succsim_i^*$ defined on $X_i \times X_i$: they play the same role with respect to $\succsim_i^*$ as that played by the marginal preferences $\succsim_i$ with respect to $\succsim$.

The following result can easily be proven using lemma 16.3(5–8).

Proposition 16.14. *For all $i \in N$, for all $x_i, y_i \in X_i$ we have:*

1) $x_i \succsim_i^+ y_i$ if and only if $x_i\,(\succsim_i^*)_1^+\,y_i$ if and only if $x_i\,(\succsim_i^*)_1^-\,y_i$ if and only if $x_i\,(\succsim_i^*)_1^\pm\,y_i$; and

2) $x_i \succsim_i^- y_i$ if and only if $y_i\,(\succsim_i^*)_2^+\,x_i$ if and only if $y_i\,(\succsim_i^*)_2^-\,x_i$ if and only if $y_i\,(\succsim_i^*)_2^\pm\,x_i$.

As a consequence, $\succsim_i^\pm = \succsim_i^+ \cap \succsim_i^-$ is the intersection of the (left-right trace) of $\succsim_i^*$ on the first dimension, $(\succsim_i^*)_1^\pm$, and the inverse of the (left-right) trace of $\succsim_i^*$ on the second dimension $(\succsim_i^*)_2^\pm$:

$$x_i \succsim_i^\pm y_i \text{ if and only if } x_i\,(\succsim_i^*)_1^\pm\,y_i \text{ and } y_i\,(\succsim_i^*)_2^\pm\,x_i. \tag{16.44}$$

Regarding $\succsim_i^{**}$, it is not difficult to see that its left-right trace on the first dimension is identical to $\succsim_i^\pm$, while its left-right trace on the second dimension is the inverse of $\succsim_i^\pm$, $(\succsim_i^\pm)^{-1}$.

We emphasize that these observations are true without making any hypothesis on traces; in particular, they are true even if traces are incomplete. In the case where $\succsim_i^*$ is a weak order (hence, when $\succsim$ satisfies axiom $RC1_i$), we may apply proposition 16.9 to $\succsim_i^*$. This relation therefore admits a numerical representation of the type

$$(x_i, y_i) \succsim_i^* (z_i, w_i) \text{ if and only if } \varphi_i(u_i(x_i), u_i(y_i)) \geq \varphi_i(u_i(z_i), u_i(w_i)),$$

where u_i is a function that separates the equivalence classes of the traces of $\succsim_i^*$. In view of equation (16.44) we can take a function that separates the equivalence classes of $\succsim_i^\pm$ for u_i. The fact that $\succsim_i^*$ is a weak order on the product set $X_i \times X_i$, i.e. a product of a set by itself, allows us to use the same function u_i on both dimensions.

Assume that $\succsim_i^*$ is weakly separable (since the product set on which $\succsim_i^*$ is defined has only two dimensions, 'weakly separable' is equivalent to 'separable' and 'weakly independent' is equivalent to 'independent'). Using the rest of proposition 16.9, we can build a numerical representation of $\succsim_i^*$ by a function $\psi_i(v_{i1}(x_i), v_{i2}(y_i))$, where v_{i1} is a numerical representation of the trace $(\succsim_i^*)_1^\pm$, v_{i2} is a numerical representation of the trace $(\succsim_i^*)_2^\pm$ and ψ_i is a function of two variables that is non-decreasing in both variables.

Since $\succsim_i^+ = (\succsim_i^*)_1^\pm$ and $\succsim_i^- = ((\succsim_i^*)_2^\pm)^{-1}$, we can alternatively represent $\succsim_i^*$ by $\phi_i(u_{i1}(x_i), u_{i2}(y_i))$, where u_{i1} is a numerical representation of $\succsim_i^+$, u_{i2} is a numerical representation of $\succsim_i^-$ and ϕ_i is a function of two variables that is non-decreasing in its first variable and non-increasing in the second. (We can take, for instance, $u_{i1} = v_{i1}$, $u_{i2} = -v_{i2}$ and $\phi_i = \psi_i$.) The latter opens the door to a representation of $\succsim_i^*$ by a function $\varphi_i(u_i(x_i), u_i(y_i))$, with the same function u_i on both dimensions. Indeed, as soon as $\succsim_i^+$ and $\succsim_i^-$ are not incompatible, i.e. as soon as $\succsim_i^\pm$ is a weak order, we can use for u_i a numerical representation of the weak order $\succsim_i^\pm$.

The case of $\succsim_i^{**}$ is simpler. As above, its trace on the first dimension is $\succsim_i^\pm$ and on the second dimension is $(\succsim_i^\pm)^{-1}$. Hence, as soon as $\succsim_i^{**}$ is a weakly separable weak order and $\succsim_i^+$ a weak order, we can build a representation of $\succsim_i^{**}$ of the type $\varphi_i(u_i(x_i), u_i(y_i))$, where u_i is a numerical representation of the weak order $\succsim_i^\pm$ and φ_i is non-decreasing in its first argument and non-increasing in its second one.

In the framework of our models, it is on $\succsim$ that we have to determine conditions which guarantee the separability or the independence of $\succsim_i^*$ or $\succsim_i^{**}$. Separability conditions for $\succsim_i^*$ and $\succsim_i^{**}$ are stated in the following proposition. In contrast (and this may sound strange initially) the independence of $\succsim_i^{**}$ is a consequence of none of our models, even the more constrained model ($L2D11$). We shall discuss this issue after we prove proposition 16.15 below.

Proposition 16.15. *If X_i is denumerable and $\succsim$ verifies $AC123_i$ and $RC1_i$, then $\succsim_i^*$ is a separable weak order on X_i^2 and any numerical representation $p_i(x_i, y_i)$ of $\succsim_i^*$ factorizes into*

$$p_i(x_i, y_i) = \varphi_i(u_i(x_i), u_i(y_i)), \tag{16.45}$$

where u_i is a numerical representation of weak order $\succsim_i^\pm$ and φ_i is a function defined on $u_i((X_i)^2)$, non-decreasing in its first argument and non-increasing in its second one.

If, in addition, $\succsim$ satisfies $RC2_i$, the same can be said of relation $\succsim_i^{**}$ and of its numerical representations.

Proof. We know that $\succsim$ verifies $RC1_i$ if and only if $\succsim_i^*$ is a complete weak order on X_i^2. This weak order is separable if, for all x_i, y_i, z_i, w_i in X_i, neither of the following conjunctions occurs:

1) $(x_i, z_i) \succ_i^* (y_i, z_i)$ and $(y_i, w_i) \succ_i^* (x_i, w_i)$
2) $(z_i, x_i) \succ_i^* (z_i, y_i)$ and $(w_i, y_i) \succ_i^* (w_i, x_i)$.

Since $\succsim_i^*$ is a complete relation, forbidding conjunction (1) is equivalent to ensuring that:

$$\left.\begin{array}{r}(x_i, z_i) \succsim_i^* (y_i, z_i)\\ \text{and}\\ (y_i, w_i) \succsim_i^* (x_i, w_i)\end{array}\right\} \Rightarrow \left\{\begin{array}{l}(y_i, z_i) \succsim_i^* (x_i, z_i)\\ \text{or}\\ (x_i, w_i) \succsim_i^* (y_i, w_i).\end{array}\right.$$

We know that $\succsim_i^\pm$ is the intersection of the first trace $(\succsim_i^*)_1^\pm$ of $\succsim_i^*$ and of the inverse of its second trace $(\succsim_i^*)_2^\pm$. Since $\succsim$ verifies $AC123_i$, $\succsim_i^\pm$ is a weak order. As a consequence, either $x_i \succsim_i^\pm y_i$ or $y_i \succsim_i^\pm x_i$. In the former case, starting from $(y_i, w_i) \succsim_i^* (y_i, w_i)$ and using definition (16.43), we obtain $(x_i, w_i) \succsim_i^* (y_i, w_i)$. In the latter case, starting from $(x_i, z_i) \succsim_i^* (x_i, z_i)$, we obtain $(y_i, w_i) \succsim_i^* (x_i, w_i)$.

We can show that conjunction (2) is also false in a similar way.

Let $p_i(x_i, y_i)$ and $u_i(x_i)$ be any numerical representation of the weak orders $\succsim_i^*$ and $\succsim_i^\pm$, respectively. Using the above conclusions, we verify directly that setting

$$\varphi_i(u_i(x_i), u_i(y_i)) = p_i(x_i, y_i)$$

defines unambiguously a function φ_i on $u_i(X_i)^2$ and that this function is non-decreasing in its first argument and non-increasing in its second one.

Regarding $\succsim_i^{**}$, the same considerations apply as soon as $\succsim_i^{**}$ is a weak order, which is ensured by $RC2_i$. $\diamond$

Let us now consider model ($L2D11$). It is straightforward that any preference $\succsim$ representable in this model is complete and satisfies $TAC12$ and TC. Hence, using lemmas 16.5(3), 16.5(4) and 16.10(2), we know that such a preference reacts in a strictly positive manner both to the traces on levels and to the traces on differences, i.e. if $(y_i, a_{-i}) \succsim (z_i, b_{-i})$, then

$$\begin{array}{rrcl} & x_i \succ_i^+ y_i & \Rightarrow & (x_i, a_{-i}) \succ (z_i, b_{-i}), \\ & z_i \succ_i^- w_i & \Rightarrow & (y_i, a_{-i}) \succ (w_i, b_{-i}) \\ \text{and} & (x_i, z_i) \succ_i^* (y_i, z_i) & \Rightarrow & (x_i, a_{-i}) \succ (z_i, b_{-i}). \end{array}$$

We cannot deduce from this, however, that $x_i \succ_i^+ y_i \Rightarrow (x_i, s_i) \succ_i^* (y_i, s_i)$ for all levels s_i or that $z_i \succ_i^- w_i \Rightarrow (t_i, w_i) \succ_i^* (t_i, z_i)$ for all levels t_i. In the former case (the other case being similar), for some levels s_i, it may indeed occur that comparing the difference (x_i, s_i) to the difference (y_i, s_i) does not reveal that x_i is at a higher level than y_i. One situation in which the higher level of x_i is certainly revealed is the following. If there exist $a_{-i}, b_{-i} \in X_{-i}$, such that $(y_i, a_{-i}) \sim (s_i, b_{-i})$ then, using the strict monotonicity of $\succsim$ with respect to $\succsim_i^+$, we have $(x_i, a_{-i}) \succ (s_i, b_{-i})$ hence $(x_i, s_i) \succ_i^* (y_i, s_i)$. If such a situation never occurs, it may happen that for all $a_{-i}, b_{-i} \in X_{-i}$ we always have either $(y_i, a_{-i}) \succ (s_i, b_{-i})$ and $(x_i, a_{-i}) \succ (s_i, b_{-i})$ or Not$[(y_i, a_{-i}) \succsim (s_i, b_{-i})]$ and Not$[(x_i, a_{-i}) \succsim (s_i, b_{-i})]$. In such a case, $(x_i, s_i) \sim_i^* (y_i, s_i)$ while this is not in contradiction with $x_i \succ_i^+ y_i$ [BOU 04a, example 17].

Condition $x_i \succ_i^\pm y_i \Rightarrow (x_i, w_i) \succ_i^* (y_i, w_i)$ is, however, necessary for the independence of $\succsim_i^{**}$. Indeed, $\succsim_i^{**}$ is independent if and only if for all x_i, y_i, z_i, w_i in X_i, $(x_i, z_i) \succsim_i^{**} (y_i, z_i) \Leftrightarrow (x_i, w_i) \succsim_i^{**} (y_i, w_i)$ and $(z_i, x_i) \succsim_i^{**} (z_i, y_i) \Leftrightarrow (w_i, x_i) \succsim_i^{**} (w_i, y_i)$. But $x_i \succ_i^\pm y_i$ implies $x_i \succ_i^+ y_i$ or $x_i \succ_i^- y_i$ (or both). In the former case, there exist levels a_{-i} and an alternative w such that $(x_i, a_{-i}) \succsim w$ and Not $(y_i, a_{-i}) \succsim w$. Hence, we have $(x_i, w_i) \succ_i^* (y_i, w_i)$. The latter case entails a similar conclusion. Hence, the independence of $\succsim_i^{**}$ implies that for all z_i, $(x_i, z_i) \succ_i^* (y_i, z_i)$.

Although we are unable to characterize the independence of $\succsim_i^{**}$ in terms of relation $\succsim$ and the previously introduced axioms (or the independence of $\succsim_i^*$), this will have no influence

Models E		Models $L0Dx$, $L1Dx$ and $L2Dx$	Conditions
$(D0)$	$\Leftrightarrow$	$(L0D0) \Leftrightarrow (L1D0) \Leftrightarrow (L2D0)$	$\varnothing$
$(D1)$	$\Leftrightarrow$	$(L0D1) \Leftrightarrow (L1D1) \Leftrightarrow (L2D1)$	
$\Updownarrow$		$\Updownarrow$	independent
$(D2)$	$\Leftrightarrow$	$(L0D2) \Leftrightarrow (L1D2) \Leftrightarrow (L2D2)$	
$(D3)$	$\Leftrightarrow$	$(L0D3) \Leftrightarrow (L1D3) \Leftrightarrow (L2D3)$	complete, independent

Table 16.5. *Models equivalent to ($D0$), ($D1$), ($D2$) and ($D3$)*

on the characterization of our models as we shall see. The only consequence is that we cannot guarantee the existence of *regular* representations for model ($L2D11$) (i.e. of representations in which u_i represents $\succsim_i^{\pm}$ and $\varphi_i(u_i(x_i), u_i(y_i))$ represents $\succsim_i^{**}$).

16.5.2. *Study of models ($L1D0$) to ($L1D11$) and ($L2D0$) to ($L2D11$)*

In this section, we assume that X is at most denumerable. The difficulties of the general case are mainly technical; they are fully dealt with in [BOU 04a]. Let us start with the study of the models where H is not supposed to be monotonic, i.e. models ($L1D0$) to ($L1D3$) and ($L2D0$) to ($L2D3$). It is easily understood that these models contribute nothing new with respect to the corresponding models on differences, that is models ($D0$), ($D1$) (which is equivalent to ($D2$)) and ($D3$). Indeed, the monotonicity of functions φ_i does not impose any additional constraint, since we do not require that function H reacts monotonically to the variations of functions φ_i. We can easily build the new representations on the basis of those of the models on differences by substituting $\varphi_i(u_i(x_i), u_i(y_i))$ to $p_i(x_i, y_i)$. The models equivalences are noted in Table 16.5; the equivalences with models ($L0D0$), ($L0D1$), ($L0D2$) and ($L0D3$) are also noted as well as the models characterizations.

As soon as we assume that H is non-decreasing, variations of φ_i are transmitted and additional constraints appear and impact on the characterization of the preference relations. Model ($L1D4$) is the first interesting one; it is equivalent to models ($L1D8$), ($L2D4$) and ($L2D8$). We verify immediately that a preference representable in model ($L1D4$) satisfies $AC123$ and these conditions, together with $RC1$, are necessary and sufficient for this model. To obtain a representation of a relation satisfying $RC1$ and $AC123$ in model ($L1D8$), let us start with the representation in model ($D8$) obtained through equation (16.36), i.e.

$$G([p_i(x_i, y_i)]) = \begin{cases} +\exp(\sum_{i=1}^n p_i(x_i, y_i)) & \text{if } x \succsim y, \\ -\exp(-\sum_{i=1}^n p_i(x_i, y_i)) & \text{otherwise} \end{cases}$$

where p_i is a numerical representation of $\succsim_i^*$ for all i.

Using proposition 16.15, we can decompose $p_i(x_i, y_i)$, which is any numerical representation of $\succsim_i^*$, into $\varphi_i(u_i(x_i), u_i(y_i))$ in which u_i represents weak order $\succsim_i^{\pm}$ and φ_i is non-decreasing in its first argument and non-increasing in its second one. This shows that model

$(L1D8)$ is not more constrained than model $(L1D4)$. We can show, starting from the just constructed representation, that it is possible to change functions φ_i into functions that are increasing in their first argument and decreasing in their second one. This is possible without making any additional hypothesis on relation $\succsim$ [BOU 04a]. Note that this modified function will no longer, in general, be a numerical representation of $\succsim_i^*$. This proves that model $(L2D8)$ is not more constrained than model $(L1D4)$ and thus establishes the announced equivalence of the four models as well as their characterization.

Passing to model $(L1D5)$ and the equivalent models $(L1D9)$, $(L2D5)$ and $(L2D9)$, we first observe that independence of $\succsim$ is a necessary condition, in addition to $RC1$ and $AC123$. Assuming that these conditions are fulfilled, we then construct a representation of $\succsim$ in model $(L1D9)$ as in the previous paragraph. The only difference is that $p_i(x_i, y_i)$ is no longer any numerical representation of $\succsim_i^*$: the chosen representation satisfies an additional property, that is $p_i(x_i, x_i) = 0$. Using proposition 16.15, we decompose *this* numerical representation of $\succsim_i^*$ into $\varphi_i(u_i(x_i), u_i(y_i))$ where u_i represents weak order $\succsim_i^{\pm}$ and φ_i is non-decreasing in its first argument and non-increasing in its second one. We have in addition that $\varphi_i(u_i(x_i), u_i(x_i)) = 0$. As before, $\varphi_i(u_i(x_i), u_i(y_i))$ can be modified into a function that is increasing in its first argument and decreasing in its second one, while preserving the additional property $\phi_i(u_i(x_i), u_i(x_i)) = 0$. A representation of $\succsim$ in model $(L2D9)$ is therefore obtained.

Model $(L1D6)$ implies $RC12$ and $AC123$. The independence of $\succsim$ is a consequence of $RC12$ (as in model $(D6)$ of which it is a specialization). The procedure used with the previous models also applies here to characterize models $(L1D6)$, $(L1D10)$, $(L2D6)$ and $(L2D10)$ and show that they are equivalent. Let us start with equation (16.37). Here, function p_i is a representation of $\succsim_i^{**}$; it is antisymmetric. The antisymmetry of p_i is transferred to $\varphi_i(u_i(x_i), u_i(y_i))$ (as a consequence of proposition 16.15).

The last four models are not all equivalent. We distinguish three classes among them: $(L1D7)$ and $(L2D7)$ are equivalent; the last two are distinct models. Notice first that all these models correspond to complete relations. Models $(L1D7)$ and $(L2D7)$ correspond exactly to the complete relations $\succsim$ that fulfill conditions $RC12$ and $AC123$. A numerical representation can be constructed as before, starting from a representation in model $(D7)$.

For a preference $\succsim$ representable in model $(L1D11)$, it is clear that TC and AC123 are necessary since $(L1D11)$ is a special case of models $(L1D10)$ and $(D11)$. Under these hypotheses, the construction process used for model $(L1D7)$ leads to a representation in model $(L1D11)$.

Finally, for model $(L2D11)$, TC and $TAC12$ are necessary conditions. The construction of a representation starts as for model $(L1D7)$; we then transform function φ_i into a function non-decreasing in its first argument and non-increasing in its second one, which no longer is, in general, a numerical representation of $\succsim_i^{**}$.

Table 16.6 summarizes all characterization and equivalence results relative to models $(L1D4)$ to $(L1D11)$ and $(L2D4)$ to $(L2D11)$.

Models $L1Dx$		Models $L2Dx$	Conditions
$(L1D4) \Leftrightarrow (L1D8)$	$\Leftrightarrow$	$(L2D4) \Leftrightarrow (L2D8)$	$RC1$, $AC123$
$(L1D5) \Leftrightarrow (L1D9)$	$\Leftrightarrow$	$(L2D5) \Leftrightarrow (L2D9)$	independent, $RC1$, $AC123$
$(L1D6) \Leftrightarrow (L1D10)$	$\Leftrightarrow$	$(L2D6) \Leftrightarrow (L2D10)$	$RC12$, $AC123$
$(L1D7)$	$\Leftrightarrow$	$(L2D7)$	complete, $RC12$, $AC123$
$(L1D11)$			complete, TC, $AC123$
		$(L2D11)$	complete, TC, $TAC12$

Table 16.6. *Equivalences and characterization of models ($L1D4$) to ($L1D11$) and ($L2D4$) to ($L2D11$)*

16.5.3. *Examples*

Tversky's additive differences model (16.3) and the additive value function model (16.1) both use marginal traces on levels and on differences. They both verify, as we have seen in sections 16.3.8 and 16.4.5, the hypotheses of the more constrained models ($L8$) and ($D11$). As a result they belong to category ($L1D11$) in the models using both traces on levels and on differences.

The additive differences model can be viewed as a particular case of model (16.4); functions $p_i(x_i, y_i)$ that occur in the latter factorize into algebraic differences: $p_i(x_i, y_i) = \Phi_i(u_i(x_i) - u_i(y_i))$ where functions u_i represent the marginal traces $\succsim_i^{\pm}$ that are identical (in this case) to marginal preferences $\succsim_i$.

In the versions of outranking methods described in literature, differences of preference are generally expressed in terms of the levels. In the simple versions that we have presented, the majoritarian method without veto (condition (16.6)) or with veto (equations (16.41) and (16.42)), we have assumed that preference differences can be expressed directly in terms of the alternative description on the relevant attributes, i.e. as a difference between corresponding coordinates of vectors x and y. In other words, it has been assumed implicitly that $u_i(x_i) = x_i$. It is easy of course to adapt the descriptions of the outranking methods in order to show explicitly a coding of the descriptions (i.e. of the elements of X_i) by functions u_i. These transform the possibly unstructured sets X_i into subsets of the real numbers $u_i(X_i)$. To do this, we simply substitute x_i and y_i by $u_i(x_i)$ and $u_i(y_i)$, respectively, in expressions (16.6), (16.41) and (16.42). Through this, we obtain models on the levels and on the differences of type ($L1D10$) or, equivalently, of type ($L2D10$). Note that the representations in models as constrained as possible are not always the most natural or the most useful ones, as already observed with models on differences. (Compare equations (16.39) and (16.40) to (16.6)).

16.6. Conclusion

In this chapter, we have presented a general approach for describing binary relations on a product set. This approach is based on conjoint measurement models that do not exclude intransitive or incomplete preferences. The main tools for analyzing such preferences are simple: we use two types of marginal traces induced on each dimension by the global preference. These tools are powerful: they permit a complete analysis of a rather large variety of models as we have shown, limiting ourselves to the case where X is denumerable.

Our project was to discover how far it is possible to go, in terms of numerical representations of relations, by using only a small number of cancelation conditions and without imposing transitivity conditions to the relations or unnecessary structural properties on the set of objects X. Surprisingly, we can go rather far while remaining in the relatively poor setting that we have chosen. In addition, the cancelation conditions that we are using ($RC1$, $RC2$, independence, TC, $AC1$, $AC2$, $AC3$, $TAC1$, $TAC2$, $AC4$) are reasonably simple and remain close to the conditions used in traditional conjoint measurement models.

The framework that has been developed and the results obtained are promising in terms of applications and further developments. Some of them have been evoked above; let us emphasize the following in particular:

– The characterization of all relations compatible with a dominance relation: such a characterization has been obtained using the models based on the marginal traces on levels (see sections 16.3.5 and 16.3.6; see also [BOU 04b]).

– The characterization of preference relations that can be obtained by means of an 'ordinal aggregation model' using marginal traces on differences: such models can be used for analyzing majoritarian methods and outranking relations such as those obtained by methods of the ELECTRE type. We illustrate how this suggestion can be put into practice in section 16.4.5 (see also [BOU 01, BOU 05a, BOU 08]). This offers an alternative to the approach developed in [DUB 01, DUB 02a, DUB 03b, FAR 01].

– The characterization of 'ordinal' models for decision in the uncertain (Chapter 11). The models described in this chapter adapt to the decision in the uncertain; it is sufficient to suppose that all components X_i of the product set X are copies of a single set. The n components of the vector describing an alternative correspond to the evaluations of this alternative in the various 'states of Nature' [BOU 03a, BOU 03b, BOU 04c]. As for ordinal aggregation, models of the type studied in this chapter offer an alternative to the approach developed in [DUB 97, FAR 99, DUB 02b, DUB 03a].

– The characterization of some particular functional forms for F, G or H [BOU 02a]: for instance, the cases where F, G or H are sums, the $\min$ operator, etc.

It is of course impossible to develop all these points here. The reader who will have followed us up to this point will not have any difficulty in imagining the spirit of these results.

Let us summarize in a few words the main message of this chapter:

– Faced with a non-transitive or incomplete relation, it is advisable to work with its marginal traces on levels and/or on differences.

– Conjoint measurement techniques can also be used to study non-transitive and incomplete relations.

– Setting aside the efficiency of elicitation procedures, we observe that substituting the additivity hypothesis by simple decomposability requirements often permits the fundamental features of a model to be captured in a simple way.

– Substituting additivity by a mere decomposability hypothesis amounts to using models that are intimately linked to rule-based modeling of preferences [GRE 99, GRE 01b, GRE 02]. In this way, one can consider the construction of elicitation procedures, using a machinery of rules induction issued from artificial intelligence.

The general framework and the results presented also contribute to a general theory of conjoint measurement. They allow us to outline a broad panorama of conjoint measurement models (Figure 16.1). The models are grouped according to whether:

– they use the traces on differences, in which case their functional form can be written in order to be non-decreasing in the functions $p_i(x_i, y_i)$;

– they use the traces on levels, in which case their functional form can be written in order to be non-decreasing in the functions $u_i(x_i)$ and non-increasing in the functions $u_i(y_i)$; or else

– they are transitive.

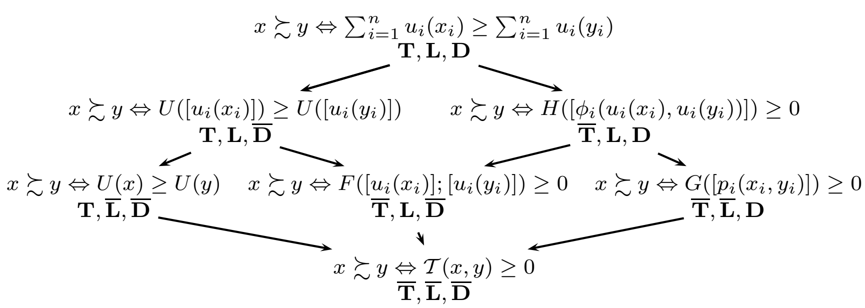

Figure 16.1. *Summary of preference models:* $\mathbf{T}$ *means 'transitive';* $\mathbf{L}$ *means 'uses marginal traces on levels';* $\mathbf{D}$ *means 'uses marginal traces on differences'; and for a property* $\mathbf{P}$, $\overline{\mathbf{P}}$ *means 'Not* $\mathbf{P}$*'*

In Figure 16.1, $\mathbf{T}$ denotes a transitive model, $\mathbf{L}$ a model that has complete marginal traces on levels and $\mathbf{D}$ a model that has complete marginal traces on differences.

In the family $\mathbf{L}$, all relations are weakly separable but it may happen that they are not weakly independent (and, *a fortiori*, not independent either). In contrast, family $\mathbf{D}$ contains only independent relations as soon as axiom $RC2$ is imposed. Marginal preference relations of preferences in family $\mathbf{L}$ tend to enjoy nice properties: they are complete and often semi-orders (as soon as axioms $AC3$ and either $AC1$ or $AC2$ are in force). The situation is quite different in family $\mathbf{D}$.

Note that all combinations of $\mathbf{T}$, $\mathbf{L}$ and $\mathbf{D}$ have been studied in literature except for the combination $\mathbf{T}, \overline{\mathbf{L}}, \mathbf{D}$. This is not surprising since, when $\mathbf{D}$ is in force, most models also use $RC2$; hence they are independent. When these properties are joined to transitivity and completeness of $\succsim$, $\succsim_i$ is a weak order, identical to $\succsim_i^{\pm}$. As a consequence, such models necessarily have complete marginal traces on levels.

16.7. Bibliography

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