Chapter 4
Building Recommendations

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Abstract This chapter briefly presents a number of techniques that can be used to build recommendations in each of three classical problem statements (choosing, ranking, and sorting) on the basis of a preference model. We start with the simple case of a preference model based on a value function. We then turn to more complex cases.

4.1 Introduction

In Chap. 3, various preference models for alternatives evaluated on several attributes/criteria were presented. Two main types of preference models were analyzed:

- preference models based on *value functions* leading to a weak order on the set of alternatives,
- preference models in which incomparability and/or intransitivity may occur.

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Preference models are tools built by the analyst in the course of a decision aiding study, the main phases of which were described in Chap. 2. Having built one or several preference models does not mean that the work of the analyst is over. Going from a preference model to *recommendations* requires many different tasks. Some of them are rather informal, involving, e.g., a good communication strategy with the actors in the decision process, the need for transparency in this process, a sound management of multiple stakeholders, etc. We discuss here the formal tasks that are involved in the elaboration of a recommendation.

The nature of this recommendation that is looked for will be of crucial importance in this phase of the decision aiding study. The central element here is the *problem statement* that has been agreed upon at the problem formulation stage of the decision aiding process (see Chap. 2). We will restrict our attention here to the three main problem statements introduced in Roy (1996).

4.1.1 Choosing

The first problem statement, *choosing*, is quite familiar in Operational Research and in Economics. The task of the analyst is formulated in such a way that he either tries to isolate, in the set A of potential alternatives, a subset A' that is likely to contain the most desirable alternatives in A given the information available or to propose a procedure that will operate such a selection.

Examples in which such a problem statement seems appropriate are not difficult to find: a recruiter wants to select a unique applicant, an engineer wants to select the best possible technical device, a patient wants to choose the best possible treatment among those offered in a hospital, a manager wants to optimise the supply policy of a factory (for other examples, see Chaps. 7 and 14 in this volume). In all these examples, the selection is to be made on the sole basis of the comparison of potential alternatives. In other words, the "best" alternatives are not defined with respect to norms but with respect to the set of alternatives A; the evaluation is only *relative*. Therefore, it may occur that the subset A', while containing the most desirable alternatives within A, only contains poor ones.

4.1.2 Ranking

The second problem statement, *ranking*, is also familiar in Operational Research and Economics. The problem is formulated in such a way that the analyst tries to rank order the set of potential alternatives A according to their desirability or to propose a procedure that will operate such a ranking. The evaluation is performed, as in the preceding problem statement, on a *relative* basis: the top ranked alternatives are judged better than the others while nothing guarantees that they are "satisfactory".

It is not difficult to find examples in which this problem statement seems appropriate. A sports league wants to rank order the teams at the end of the season. An academic programme has to select a number of applicants: a competitive exam is organized which leads to rank ordering the applicants according to an "average grade" and applicants are then selected in the decreasing order of their average grades until the size of the programme is reached. An R&D department has to finance a number of research projects subject to a budget constraint: research projects are then rank ordered and financed till the budget constraint is binding. In practice, some authors tend to use a ranking problem statement whereas a choice problem statement would seem more natural (see Chap. 12 in this volume). This is often motivated by the fact that the ranking problem statement leads to richer information than the choice problem statement.

4.1.3 Sorting

The third problem statement, *sorting*, is designed to deal with absolute evaluation. The problem is here formulated in such a way that the analyst tries to partition the set of alternatives into several categories, the definition of these categories being intrinsic, or to propose a procedure that will generate such a partition. The essential distinctive characteristics of this problem statement therefore lie in the definition of the categories. Two main cases arise.

The definition of the categories may not refer to the desirability of the alternatives. Many problems that arise in pattern recognition, speech recognition or diagnosis are easily formulated in this way. We will only discuss here the case in which the definition of the categories refers to the *desirability* of the alternatives, e.g., a credit manager may want to isolate "good" risks and "bad" risks, an academic programme may wish to only enroll "good" students, etc. A crucial problem here will lie in the definition of the categories, i.e., of the *norms* defining what is a "good" risk, what is a "good" student. Several chapters in this volume (see Chaps. 15 and 19) adopt this problem statement.

4.1.4 Outline

In Sect. 4.2, we tackle the simple case in which the preference model takes the form of a value function. Section 4.3 is devoted to the case of making a recommendation on the basis of *several* value functions. Such a situation frequently arises when using Linear Programming-based assessment techniques of an additive value function. In Sect. 4.4 we deal with the more delicate case of deriving a recommendation on the basis of less well-structured preference models like the ones that are obtained with the so-called outranking methods, which includes the ELECTRE methods.

4.2 A Single Value Function

Many of the preference models envisaged in Chap. 3 are based on value functions. This means that the analyst has built a real-valued function V on the set of alternatives A that induces binary relation $\succsim$ on A, interpreted as an "at least as good" relation letting, for all $a, b \in A$:

$$a \succsim b \Leftrightarrow V(a) \geq V(b). \tag{4.1}$$

Such a relation $\succsim$ is a weak order (it is complete and transitive). It is therefore simple to use it to build a recommendation involving only a relative evaluation of the alternatives, the hard work involved in the assessment of a value function being rewarded at this stage of the decision aiding process.

In this section, we suppose that the value function V is only constrained by (4.1). This means that any increasing transformation of V would carry the same information as V.

4.2.1 Choosing

In a choosing problem statement, it is natural to look for alternatives that would be "at least as good" as all other alternatives, i.e., to identify the set $G(A, \succsim)$ of *greatest alternatives* in A given the binary relation $\succsim$ defined by:

$$G(A, \succsim) = \{a \in A : a \succsim b, \forall b \in A\}.$$

Since $\succsim$ is complete and transitive, $G(A, \succsim)$ will, in general,[1] be nonempty. Finding the alternatives in $G(A, \succsim)$ is equivalent to finding the solutions to the following optimisation problem:

$$\max_{a \in A} V(a).$$

Note that the set of solutions to this optimisation problem is unchanged if V is replaced by any value function satisfying (4.1), i.e., by any value function obtained from V applying to it an increasing transformation.

The set $G(A, \succsim)$ may contain more than one element. In this case, all alternatives in $G(A, \succsim)$ are indifferent and compare in the same way to all other alternatives. Therefore, the preference model defined by V offers no means of distinguishing

[1] This is true when A is finite. The general case may be more tricky: while the relation $\geq$ on $\mathbb{R}$ is complete and transitive, $G(\geq, \mathbb{R})$ is clearly empty. The same is true with $\geq$ on the open $]0, 1[$ interval.

between them. All alternatives in $G(A, \succsim)$ are strictly preferred to all alternatives outside $G(A, \succsim)$. The rejection of the latter therefore seems fully justified: all recommended alternatives are judged strictly better than all rejected alternatives.

The set of maximal alternatives $M(A, \succsim)$ in A, given the binary relation $\succsim$, is defined by:

$$M(A, \succsim) = \{a \in A : \mathit{Not}[\, b \succ a\,], \forall b \in A\},$$

where $\succ$ is the asymmetric part of $\succsim$. It is often presented as the central notion in a choosing problem statement. When $\succsim$ is complete, we always have $G(A, \succsim) = M(A, \succsim)$. When A is finite, it is easy to show that $M(A, \succsim)$ is nonempty when $\succsim$ has no circuit in its asymmetric part $\succ$. For finite sets, the absence of any circuit in $\succ$ is, in fact, a necessary and sufficient condition for $M(B, \succsim)$ to be nonempty for all nonempty sets $B \subseteq A$.

4.2.2 Ranking

Let us now envisage the case of a ranking problem statement. The hard work of building a value function also pays off here since the binary relation $\succsim$ induced on A by the value function V (or by any increasing transformation of V) rank orders the alternatives from the best to the worst, which is precisely what is wanted. Apart from the necessity of conducting a robustness analysis, no additional work is required (on the notion of robustness analysis, see Roy, 1998, 2010).

4.2.3 Sorting

In both problem statements involving only a *relative evaluation* of alternatives, we have seen that the value function model provided an almost immediate way of deriving a recommendation. The situation is slightly more complex in a sorting problem statement, which calls for an *absolute evaluation*. It is indeed necessary to define the "norms" that will give sense to such an evaluation.

We will only envisage the case in which the absolute evaluation that is sought takes the form of a sorting of the alternatives between r ordered categories $C^1, C^2, \ldots, C^r$, with C^1 containing the least desirable alternatives. The definition of each category involves the definition of norms. These norms usually take two distinct forms. They may be modelled as *prototypes* of alternatives belonging to a category or as *limiting profiles* indicating the limit of each category. A "good" student may be defined using examples of past students in the programme: this would define the prototypes of the category of "good students". Alternatively, we could define, as is done in the French *baccalauréat*, an average grade above which, students are considered to be "good". This average grade implicitly defines a limiting profile for the category of "good students".

4.2.3.1 Limiting Profiles

When each category C^k is delimited by a limiting profile π^k, an alternative a should belong at least to the category C^k when it is preferred to π^k. It then becomes easy to use a value function to sort the alternatives: alternative $a \in A$ will belong to C^k if and only if $V(\pi^k) < V(a) < V(\pi^{k+1})$, where the unlikely cases of equality are dealt with conventionally, depending on the definition of the limiting profiles π^k. Note that the definition of a limiting profile implies that there is only one such profile per category. The main problem here lies in the definition of the limiting profiles π^k. We shall come back to this point in Sect. 4.3.3.

4.2.3.2 Prototypes

The situation is more delicate when categories are defined via prototypes. Suppose that category C^k has been defined by a set $P^{k,1}, P^{k,2}, \ldots$ of prototypes. A first step in the analysis consists in checking whether this information is consistent with the value function V, i.e., if the prototypes defining a category C^k are all preferred to the prototypes defining the category C^ℓ when $k > \ell$.

When this consistency test fails, the analyst may wish to reconsider the definition of V or of the various prototypes. When the prototypes are consistent, we may easily associate to each category C^k, its lowest prototype L^k and its highest prototype H^k in terms of the value function V. If $V(a) \in [V(L^k); V(H^k)]$, alternative a should be assigned to the category C^k. If this simple rule allows to assign each alternative to a well-defined category, no further analysis is required. When this is not the case, i.e., when there are alternatives $a \in A$ such that $V(a)$ falls between two intervals, we may either try to refine the information defining the categories, e.g., try to ask for new prototypes, or apply a simple rule e.g., replacing the intervals $[V(L^k); V(H^k)]$ by the interval $[(V(H^{k-1}) + V(L^k))/2; (V(H^k) + V(L^{k+1}))/2]$. Ideally we would need a similarity measure between alternatives and prototypes that would allow to classify a as a member of C^k if a is close to one or several of the prototypes defining C^k. The simple rule envisaged above amounts to using V as a very rough similarity measure since this amounts to saying that a is more similar to b than it is to c if $|V(a) - V(b)| < |V(a) - V(c)|$. It should however be noted that the assessment procedures of V do not necessarily guarantee that such a measure is appropriate. In general, this would call for the modelling of "preference differences" between alternatives, e.g., using a model in which:

$$a \succsim b \Leftrightarrow V(a) \geq V(b) \text{ and} \tag{4.2}$$

$$(a, b) \succsim^* (c, d) \Leftrightarrow V(a) - V(b) \geq V(c) - V(d), \tag{4.3}$$

where $\succsim^*$ is a binary relation on A^2 such that $(a, b) \succsim^* (c, d)$ is interpreted as "the preference difference between a and b is at least as large as the preference difference between c and d". A common mistake here is to use any V satisfying (4.2) as if it would automatically satisfy (4.3).

4.3 A Set of Value Functions

Motivated by the assessment of an additive value function via Linear Programming, this section studies techniques to build a recommendation on the basis of *several* value functions that cannot be deduced from one another using an increasing transformation. This is the case with techniques such as UTA (Jacquet-Lagrèze and Siskos, 1982). This method uses LP techniques to assess an additive value function, which, in general, leads to several possible value functions.

4.3.1 Choosing with a Set of Additive Value Functions

Suppose for example that, e.g., because we have assessed an additive value function with UTA, we have an entire set $\mathcal{V}$ of value functions compatible with the available information. Two main ways of exploiting this set $\mathcal{V}$ may be envisaged within a choosing problem statement.

The simplest way of using the set $\mathcal{V}$ is to consider that an alternative $a \in A$ should be included in the set $A' \subseteq A$ of recommended alternatives as soon as there is one additive value function in $\mathcal{V}$ such that using this function, a is at least as good as any other alternative in A.

When the set $\mathcal{V}$ comes from Linear Programming-based assessment techniques, such a test is easily performed using LP, since the elements in $\mathcal{V}$ correspond to the solution of a set of linear constraints. In fact, we only have to test whether the system of inequalities $V(a) \geq V(b)$, for all $b \in A$, is consistent for some $V \in \mathcal{V}$. This requires solving a linear programme for each alternative $a \in A$. This idea has been systematized in Greco et al. (2008, 2009).

The above technique is very cautious and is likely to lead to quite large choice sets. A more refined analysis is based on the "proportion" of value functions $V \in \mathcal{V}$ for which an alternative is optimal. The "more functions" V in $\mathcal{V}$ give a as the optimal solution, the more confident we are in the fact that a can be recommended. In general, such an analysis unfortunately requires solving difficult computational problem (see Bana e Costa, 1986, 1988), even when $\mathcal{V}$ is defined by the solutions of a set of linear constraints. A possible solution would be to sample a few value functions within $\mathcal{V}$. Indeed, when $\mathcal{V}$ is defined by linear constraints, Jacquet-Lagrèze and Siskos (1982) suggested a simple way to build a finite subset $\mathcal{V}'$ of $\mathcal{V}$ that is "representative" of the whole set $\mathcal{V}$. An alternative approach is to use Monte-Carlo simulation (Charnetski and Soland, 1978; Lahdelma et al., 1998)

4.3.2 Ranking with a Set of Additive Value Functions

The crudest way of using the information contained in $\mathscr{V}$ is to build a partial preorder (i.e., a reflexive and transitive relation) T such that:

$$aTb \Leftrightarrow V(a) \geq V(b) \text{ for all } V \in \mathscr{V}, \tag{4.4}$$

i.e., letting a be ranked before b if it is so for every admissible function V in $\mathscr{V}$.

Testing if aTb can easily be done using LP when $\mathscr{V}$ is defined via linear constraints (this idea has been systematized in Greco et al. (2008) and Greco et al. (2009)). The use of such a technique is however limited since it implies solving $n(n-1)$ linear programmes when $|A| = n$. Furthermore, such a unanimity argument is likely to lead to a very poor recommendation: many alternatives will be incomparable when $\mathscr{V}$ is large.

When $|A|$ is too large to allow the use of the technique described above or when a richer result is sought, one may either try to restrict the domain $\mathscr{V}$ through emphasizing interaction with the decision maker during the assessment phase, or work with a representative set of value functions $\mathscr{V}'$, as mentioned above. Quite interesting examples of such techniques can be found in Siskos (1982).

4.3.3 Sorting with a Set of Additive Value Functions

In the techniques envisaged so far we did not consider the definition of the "norms" that are necessary to sort alternatives. A useful technique, in the spirit of UTA, consists in assessing the additive value function using examples of alternatives belonging to each of the ordered categories, that we called prototypes in Sect. 4.2.3.2. Such examples may come from past decisions or may be obtained from the decision maker as prototypical examples of each category. We may then try to infer limiting profiles and an additive value function on the basis of such information.

This amounts to assessing an additive value function V and thresholds s_k such that, for all prototypes $P^{k,j}$ of category C^k we have $V(P^{k,j}) \in [s^k, s^{k+1}[$. This is the basis of the UTADIS technique (see Jacquet-Lagrèze, 1995; Zopounidis and Doumpos, 2000b, 2001, 2002) and its variants (Zopounidis and Doumpos, 2000a).

Basically UTADIS uses a number of prototype alternatives for each ordered category whereas UTA uses a weak order on a subset of reference alternatives. Such a technique extends the traditional methods of discrimination used in Statistics considering the possibility of nonlinear value functions. As in Statistics, the assessment may use "cost of misclassification" which simply amounts to weighting the deviation variables in the LP used to assess the value function V appropriately. As in UTA, this leads to a whole set of possible additive value functions with associated limiting thresholds.

The way to make use of such information to build a recommendation has not been thoroughly studied in the literature. When the set $\mathscr{V}$ is defined via linear constraints, it is easy to use LP to compute for each alternative the subset of categories to which they may belong. This is computationally intensive. Another way to proceed is to consider a subset $\mathscr{V}'$ of representative additive value functions. For each alternative $a \in A$, it is easy to compute a set of possible assignments using $\mathscr{V}'$. One may then, for example, use the frequency with which each alternative is assigned to a category to devise a recommendation. For developments along this line, see Greco et al. (2010).

4.4 Other Preference Models

As argued in Chap. 3, the assessment of a value function is a demanding task. The analyst may then wish to use aggregation technique that have a more "ordinal" character. The price to pay for using such models is that the preference models to which they lead may be intransitive and/or incomplete. Using them to derive a recommendation is a difficult task. For space reasons, we restrict our attention to the case of crisp binary relations (the case of valued relations is dealt with in Bouyssou et al., 2006, , Chap. 7).

Suppose that you have built a preference relation on a set of alternatives using one of the techniques presented in Chap. 3 that does not guarantee the transitivity or the completeness of the result. This does not necessarily mean that *any* preference structure can be obtained with such a method. Let us first show, that for a number of well known techniques, this is unfortunately true.

Consider simple majority, i.e., the simplest "ordinal" technique for comparing alternatives. On each criterion, we suppose that alternatives can be compared using a weak order. Simple majority amounts to declaring that:

$$x \succsim y \Leftrightarrow |P(x,y)| \geq |P(y,x)|$$

where $P(x,y)$ denotes the set of criteria on which x is preferred to y. Clearly, a relation $\succsim$ obtained in such a way is always complete. Let T be any complete binary relation on a finite set of alternatives A. Besides completeness, no hypothesis is made on T; it may be the most intransitive relation you can think of, with circuits of any length in its asymmetric part. The surprising and disturbing fact, proved by McGarvey (1953), is that it is always possible to see T as the result of a simple majority aggregation. Extending this result, Bouyssou (1996) has shown that *any* reflexive relation on a finite set of alternatives may be obtained with ELECTRE I (Roy, 1968). Therefore, we have to tackle here quite a large class of preference models.

4.4.1 Motivating Examples

Many techniques for building recommendations on the basis of a non-necessarily transitive or complete binary relation have been proposed in the literature on Multiple Criteria Decision Making (MCDM). Most of them were justified on an ad hoc basis. But the intuition supporting these techniques might not work well in all cases. Let us illustrate this point with two examples.

Example 4.1 (Choice Procedures and Dominated Alternatives) Consider a set of alternatives $A = \{a, b, c, d\}$ evaluated on three criteria. Suppose that, on each criterion, alternatives are weakly ordered by a binary relation S_i. Suppose that the preference on each criterion are such that, using an obvious notation for weak orders:

$$\begin{aligned} &a P_1 b P_1 c P_1 d, \\ &c P_2 d P_2 a P_2 b, \\ &d P_3 a P_3 b P_3 c, \end{aligned}$$

where P_i denotes the asymmetric part of S_i.

Alternative b is *strongly dominated* by alternative a (a is strictly preferred to b on all criteria). Intuitively, this gives a decisive argument not to include b in the set of recommended alternatives.

Suppose then that the above information is aggregated into a binary relation S using simple majority. It is not difficult to see that S is such that: $aPb, aPc, bPc, cPd, dPa, dPb$, where P denotes the asymmetric part of S. It is obvious that S is not well suited to select a subset of alternatives since its asymmetric part P contains a circuit involving all alternatives (aPb, bPc, cPd, dPa). The simplest way to get rid of such a circuit is to consider that all alternatives included in a circuit should be considered "equivalent". This can be done by considering the *transitive closure* of the relation, i.e., the smallest transitive relation containing it. But using the transitive closure of S would then lead to consider that all alternatives are equivalent and, hence, to propose the whole set A as the set of recommended alternatives. This is not sound since we have shown that b should *not* be recommended.

Example 4.2 (Ranking Procedures and Monotonicity) Let $A = \{a, b, c, d, e, f, g\}$. Using the result of Bouyssou (1996), we know that if we use simple majority or ELECTRE I, we might end up with a complete binary relation S such that: $aPb, aPf, bPc, bPd, bPe, bPf, cPa, cPe, cPf, cPg, dPa, dPc, dPe, dPf, dPg, ePa, ePf, ePg, fPg, gPa, gPb$, where P denotes the asymmetric part of S.

In order to obtain a ranking on the basis of such information, one may use a measure of the "desirability" of each alternative. A simple measure of the desirability of an alternative x consists in counting the number of alternatives y

such that xSy minus the number of alternatives z such that zSx. This measure is called the *Copeland score* of an alternative (Laslier, 1997).

A simple way of building a ranking on A goes as follows. Define the first equivalence class of the ranking as the alternatives that have obtained a maximal Copeland score. Remove these alternatives from the set of alternatives. Define the second equivalence class of the ranking as the alternatives with maximal Copeland scores in the reduced set. Repeat this procedure as long as there are unranked alternatives. Such a ranking procedure is intuitively appealing and leads to the following ranking, using obvious notations:

$$d \succ c \succ e \succ [a, g] \succ b \succ f,$$

which does not seem unreasonable.

Consider now a relation identical to the one above except that we now have aPd instead of dPa. Intuition suggests that the position of a has improved and we reasonably expect that this is reflected in the new ranking. But applying the same ranking method as before now leads to:

$$[b, c, d] \succ e \succ [a, f, g].$$

Such a result is quite disappointing since, before a was improved, a was ranked before b while, after the improvement of a, b is ranked before a.

These two examples show that the definition of sound procedures for deriving a recommendation on the basis of a non-necessarily transitive or complete binary relation is a difficult task. Intuitively appealing procedures sometimes produce disappointing results.

4.4.2 Choice Procedures

Let A be a set of alternatives. Suppose that you have built a preference relation S on A using an aggregation technique. Let us call $\mathscr{S}$ the set of all conceivable preference relations that can be obtained using such a technique. As shown above, $\mathscr{S}$ consists of all reflexive binary relations with ELECTRE I and all complete binary relations with simple majority. A choice procedure C is a function associating a nonempty subset $C(S)$ of A with each element S of $\mathscr{S}$. The choice procedure C should:

- be such that $C(S)$ is as small as possible given the available information,
- be such that there are clear arguments to justify the elimination of the alternatives in $A \setminus C(S)$, i.e., the alternatives that are not selected,
- be such that there is no built-in bias in favour of some alternatives, i.e., that the only arguments that can be taken into account in the determination of $C(S)$ are how these alternatives are related in terms of the relation S. Technically, this leads to requiring that C is *neutral*, i.e., that $C(S) = \sigma[C(S^\sigma)]$, where σ is any

one-to-one function on A and S^σ is the binary relation in $\mathscr{S}$ such that, for all $a, b \in A$, $S(a, b) = S^\sigma(\sigma(a), \sigma(b))$.
- react to the improvement of an alternative in the expected direction. Technically, the procedure should be *monotonic*, i.e., if $a \in C(S)$ and S' is identical to S except that $[aS'b$ and $Not[aSb]]$ or $[Not[bS'a]$ and $bSa]$, for some $b \in A$, then we should have $a \in C(S')$.

Let $S \in \mathscr{S}$. We shall always denote by P (resp. I) the asymmetric (resp. symmetric) part of S and J the associated incomparability relation, i.e., for all $a, b \in A$, aJb iff $[Not[aSb]$ and $Not[bSa]]$.

4.4.2.1 Procedures Based on Covering Relations

Suppose that there exists $a \in A$ such that aPb, for all $b \in A \setminus \{a\}$. Such an alternative is usually called a *Condorcet winner*. In this case, letting $C(S) = \{a\}$ seems to be the only reasonable choice. In fact, by construction:

- when there is a Condorcet winner, it is necessarily unique,
- there is *direct evidence* that a is better than all other alternatives.

Unfortunately, the existence of a Condorcet winner is an unlikely situation and we must agree on what to do in the absence of a Condorcet winner.

A simple extension of the notion of a Condorcet winner is that of *greatest* alternatives already introduced. Remember that an alternative $a \in A$ belongs to the set $G(A, S)$ of greatest alternatives in A given S if aSb, for all $b \in A$. If a belongs to $G(A, S)$, we have direct evidence that a is at least as good as any other alternative in A. Contrary to the case of Condorcet winners, there may be more than one greatest alternative. When the set of greatest alternatives is nonempty, it is tempting to put all alternatives on $G(A, S)$ in $C(S)$.

This seems a natural choice. Indeed, all greatest alternatives are indifferent, so there is no direct evidence that would allow to further refine the choice set $C(S)$. Contrary to the case in which S is a weak order, it should however be noted that there might be *indirect* evidence that allows to distinguish between greatest alternatives. As shown in the following example, indirect evidence may be usefully employed to narrow down the set of selected alternatives.

Example 4.3 Suppose that $A = \{a, b, c\}$ and S be such that aIb, bIc and aPc. Although both a and b belong to $G(A, S)$, we can use the way a and b compare to a third alternative, c, to distinguish between them. Here, since aPc while bIc, it is very tempting to use this indirect evidence to narrow $C(S)$ down to $\{a\}$.

Unfortunately, there is no clear-cut way of defining what should count as an *indirect evidence* that an alternative is better than another and to balance it with the direct evidence.

Suppose first that aPb so there is direct evidence that a is superior to b. If, for all $c \in A$, we have $cPa \Rightarrow cPb$, $cIa \Rightarrow cSb$, $bPc \Rightarrow aPc$ and $bIc \Rightarrow aSc$, there is no indirect evidence that b could be superior to a. In such a case, we say that a strongly covers b ($aSCb$) and it seems that the selection of b would be quite unwarranted. A cautious selection would then seem to be to select all alternatives that are not strongly covered by any other, i.e., the set $M(A, SC)$ of maximal alternatives in A for SC. When A is finite, $M(A, SC)$ is always nonempty since the strong covering relation is asymmetric and transitive and, thus, has no circuit. Therefore letting $C(S) = M(A, SC)$ defines a selection procedure. Note that the use of this selection procedure would allow to avoid selecting a strongly dominated alternative as was the case with the procedure envisaged in Example 4.1 since, in this example, a strongly covers b. With such a procedure, the rejection of the elements in $A \setminus C(S)$ would seem fully justified since for each $b \in A \setminus C(S)$, there would be an $a \in C(S)$ such that aPb. We leave to the reader the, easy, task of showing that this selection procedure is neutral and monotonic.

The relation SC is likely to be rather poor, so that the above procedure may result in large choice sets. In order to reject an alternative, it is necessary to have direct evidence against it and no indirect evidence in its favour. In Example 4.3, it would not allow to distinguish between the two greatest alternatives a and b since there is no direct evidence for a against b.

A less stringent procedure would consist in saying that the selection of b is unwarranted as soon as there is an alternative a such that there is direct evidence that a is at least as good as b while there is no indirect evidence that b is better than a. This would lead to the definition of a covering relation in which a weakly covers b ($aWCb$) as soon as aSb and for all $c \in A$, we have $cPa \Rightarrow cPb$, $cIa \Rightarrow cSb$, $bPc \Rightarrow aPc$ and $bIc \Rightarrow aSc$. Therefore, the weak covering relation WC is identical to the strict covering relation SC except that aIb is compatible with $aWCb$. Contrary to SC, the relation WC is not asymmetric. It is reflexive and transitive so its asymmetric part has no circuit. When A is finite, letting $C(S) = M(A, WC)$ therefore defines a selection procedure. For each non selected alternative b, there is a selected alternative a such that either aPb or aIb, while there is no indirect evidence that b might be superior to a. The theoretical properties of this choice procedure are quite distinct from the one relying on the strong covering relation (Dutta and Laslier, 1999; Peris and Subiza, 1999), while remaining neutral and monotonic.

A weakness of the procedure given above is that when a and b are incomparable, it is impossible to distinguish between them even when there is strong indirect evidence that one is better to the other. It is possible to modify the definition of the weak covering relation requiring only that there is no direct evidence against a, i.e., that aSb or aJb, while still requiring that there is no indirect evidence that b is superior to a. This very weak covering relation is still reflexive and transitive. Taking the maximal alternatives in A for the very weak covering relation therefore defines a selection procedure. It refines the above selection procedure based on the weak covering relation. This is however a price to pay. Using such a choice set does not prevent the existence of a non selected alternative b such that there is no alternative in the choice set for which there is direct evidence that it is at least as good as b. Therefore, the narrowing of the choice set, considering the very weak covering relation, may be judged unsatisfactory.

We refer to Dutta and Laslier (1999), Laslier (1997) and Peris and Subiza (1999) for a thorough study of the properties of choice sets that are based on some idea of "covering" i.e., mixing direct and indirect evidence to justify the selection of $C(S)$.

4.4.2.2 Procedures Based on Kernels

Quite a different path was taken by Roy (1968) and Roy and Skalka (1984) in the ELECTRE I and ELECTRE IS methods (a similar idea is already detailed in von Neumann and Morgenstern, 1947, , in the context of Game Theory). Note that the selection procedure is clear as soon as S is transitive. In fact, in such a case, the set of maximal elements in A, i.e., $M(A, S) = \{a \in A : Not[bPa] \text{ for all } b \in A\}$ is always nonempty and such that, for all $b \notin M(A, S)$, there is an alternative $a \in M(A, S)$ such that aSb. In fact, when S is transitive, the set $M(A, S)$ coincides with the set of maximal alternatives for the weak covering relation since, in this case, $S = WC$.

For $B \subseteq A$, we say that B is *dominating* if for all $c \notin B$ there is an alternative $b \in B$ such that bSc. Therefore the selection of the alternatives in a dominating subset always justifies the non selection of the other alternatives. By construction, the set A itself is dominating. When A is finite, there are therefore dominating subsets of minimal cardinality. If there is only one such dominating subset, it is a good candidate for the choice set $C(S)$. When S has circuits, there may be more than one dominating subset of minimal cardinality. Taking their union will generally result in quite an undiscriminating procedure. This is illustrated in the following example.

Example 4.4 Let $A = \{a, b, c, d, e\}$. Suppose that S is such that aPb, bPc, cPd, dPe and ePa. This relation has 5 dominating subsets of minimal cardinality, i.e., $\{a, c, e\}$, $\{a, b, d\}$, $\{a, c, d\}$, $\{b, c, e\}$ and $\{b, d, e\}$. The union of the minimal dominating subsets is A.

B. Roy therefore suggested to consider the relation S' obtained by reducing the circuits in S, i.e., to consider all alternatives that are involved in a circuit as a single alternative. Working with S' instead of S amounts to considering that all alternatives involved in a circuit compare similarly with alternatives outside the circuit. This is frequently a strong hypothesis implying the loss of a lot of information (this would be the case in Example 4.4). The following example illustrates the process of reducing the circuits of S.

Example 4.5 Let $A = \{a, b, c, d, e, f\}$ and consider the binary relation S such that: $aSb, aSc, aSd, aSe, aSf, bSc, bSf, cSa, cSe, dSe, eSf$. In order to build the relation S' obtained by reducing the circuits in S we need to find the maximal circuits in S (i.e., circuits that are not included in other circuits). There is only one circuit in S: aSb, bSc and cSa. Therefore the three alternatives a, b and c are replaced by a single one, say x, and there is an arc from x to another alternative if there is an arc in S going from either a, b or c to this alternative. Similarly there is an arc going from an alternative to x if there was an arc going from this alternative to either a, b or c in S. Therefore the binary relation S' is such that: $xS'd, xS'e, xS'f, dS'e, eS'f$.

A famous result of Graph Theory (Berge, 1970; Roy, 1969–70) says that when a graph has no circuit, it has a unique *kernel*, defined as a dominating subset that is internally stable, i.e., such that there is no arc between any of its elements (this implies that the kernel is a minimal dominating subset). Reducing the circuits and taking the kernel of the relation is the selection procedure proposed in ELECTRE I. It is easy to verify that it is neutral and monotonic.

The procedure in ELECTRE IS (see Roy and Bouyssou, 1993; Roy and Skalka, 1984) amounts to a more sophisticated reduction of the circuits that takes the way the relation S has been defined into account. For recent developments along this line, including the extension of the notion of kernel to valued binary relation, see Bisdorff et al. (2008) (see also Chap. 5).

4.4.2.3 Other Procedures

The use of covering relations and of the notion of kernel are far from being the only possible choices to devise a selection procedure (Laslier, 1997; Peris and Subiza, 1999; Schwartz, 1986). Some of the possibilities that we do not investigate here are:

- selection procedures based on the consideration of relations close to S for which the choice is simple, e.g, orders or weak orders (see Barthélémy et al., 1989; Laslier, 1997; Charon and Hudry, 2007),
- selection procedures based on scores, e.g., Copeland scores (see van den Brink and Gilles, 2003; Henriet, 1985; Rubinstein, 1980),
- selection procedures that directly operate on the evaluations of the alternatives without building a relation S as an intermediate step (see Fishburn, 1977).

4.4.3 Ranking Procedures

Let A be a set of alternatives. Suppose that you have built a crisp relation S on A using some kind of aggregation technique. Let $\mathscr{S}$ be the set of all conceivable preference relations that can be obtained using such a technique. A ranking procedure $\succsim$ is a function associating a reflexive and transitive binary relation $\succsim(S)$ on A with each element S of $\mathscr{S}$. The task of building a transitive result on the basis of a binary relation, that might not be transitive or complete is not easy: we are in fact looking for a much richer result than that obtained using choice procedures. We expect such a ranking procedure to be:

- *neutral*, i.e., insensitive to the labelling of the alternatives,
- *faithful*, i.e., if S is a reflexive and transitive relation, we should have $\succsim(S) = S$,
- *monotonic*, i.e., the position of a in the ranking $\succsim(S)$ should not decrease if S is substituted by a relation S' in which the position of a has improved (see Example 4.2).

Clearly, this list is only partial, e.g., we would also expect the ranking $\succsim(S)$ to be linked to the covering relations defined above (see Vincke, 1992).

Several types of ranking procedures have been suggested in the literature:

1. Ranking procedures based on the transitive closure of S,
2. Ranking procedures based on scores, e.g., the Copeland score,
3. Ranking procedures based on the repeated use of a choice mechanism (as in Example 4.2),

We briefly illustrate each type of procedure below.

4.4.3.1 Procedures Based on the Transitive Closure

Let S be a reflexive binary relation on A. A simple way to obtain a reflexive and transitive relation $\succsim(S)$ on the basis of S is to take its transitive closure $\hat{S}$, i.e., the smallest transitive relation containing S. This defines a ranking procedure; it is easy to see that it is neutral, faithful and monotonic. In view of our discussion of choice procedures, the main defect of this ranking procedure should be apparent. All alternatives that are involved in a circuit of S will be equally ranked if we let $\succsim(S) = \hat{S}$. This often results in a huge loss of information.

A closely related ranking procedure is the one used in ELECTRE II (Roy and Bertier, 1973). It was originally designed to produce a reflexive and transitive relation on the basis of two nested reflexive relations. We present it below in the special case in which there is only one relation (the role of the second one being only to possibly refine the equivalence classes that are obtained).

Consider any reflexive relation S on A. The ranking procedure of ELECTRE II first consists, as with ELECTRE I, in reducing the circuits that may exist in S, replacing all alternatives involved in a circuit by a single vertex in the associated graph. Once this is done, we obtain, by construction, a relation with no circuit. We use this relation to build two weak orders. In the first one, T_1, the first equivalence class consists of the maximal elements (there is no element that is strictly preferred to them) of the relation with no circuit. These elements are then removed from the set of alternatives. The second equivalence class of T_1 consists of the maximal elements of the relation among those remaining and so forth.

The second weak order T_2 is obtained in a dual way, building the last equivalence class consisting of the minimal elements first (they are preferred to no other element) in the relation with no circuit, removing these elements from the set of alternatives and building the penultimate equivalence class of T_2 as the minimal elements among those remaining and so forth. In general, T_1 and T_2 are not identical. The reflexive and transitive relation $\succsim(S)$ is then taken to be the intersection of these two weak orders. Let us illustrate this process using a simple example.

Example 4.6 Let $A = \{a_1, a_2, \ldots, a_9\}$ and let S be such that: $a_1 S a_2, a_1 S a_4, a_1 S a_5$, $a_2 S a_3$, $a_3 S a_1, a_4 S a_6$, $a_6 S a_7, a_7 S a_9, a_8 S a_9$. The relation S has a circuit: $a_1 S a_2$, $a_2 S a_3$, $a_3 S a_1$. We therefore replace S on A with the relation S' on A' defined by: $bS'a_4, bS'a_5, a_4 S' a_6, a_6 S' a_7, a_7 S' a_9, a_8 S' a_9$, where a_1, a_2 and a_3 have been replaced by b. The relation S' has no circuit. Its set of maximal elements consists of $\{b, a_8\}$. Once these elements have been removed, the set of maximal elements is $\{a_4, a_5\}$. At the next iteration, we obtain $\{a_6\}$, then $\{a_7\}$ and $\{a_9\}$. Therefore the weak order T_1 is, using obvious notation:

$$[a_1, a_2, a_3, a_8] T_1 [a_4, a_5] T_1 a_6 T_1 a_7 T_1 a_9.$$

In a dual way, we obtain the weak order T_2:

$$[a_1, a_2, a_3] T_2 a_4 T_2 a_6 T_2 [a_7, a_8] T_2 [a_5, a_9].$$

The relations T_1 and T_2 are not identical. Taking their intersection leads to, abusing notation:

$$\begin{gathered}
[a_1, a_2, a_3] \succ a_4 \succ a_6 \succ a_7 \succ a_9, \\
[a_1, a_2, a_3] \succ a_8, \\
a_4 \succ a_5, \\
a_8 \succ a_7, a_8 \succ a_5, \\
a_5 \succ a_9.
\end{gathered}$$

What can be said of this result? First observe that the rationale for building two weak orders and for defining $\succsim(S)$ as their intersection is to introduce incomparability between alternatives that are difficult to compare using S. This is, for instance, the case between a_5 and all alternatives except a_1 or between a_8 and all alternatives except a_9. In this respect the success of the procedure is only limited since we finally conclude that $[a_1, a_2, a_3] \succ(S) a_8$, $a_8 \succ(S) a_7$, $a_4 \succ(S) a_5$ and $a_5 \succ(S) a_9$.

Let us also note that we would have obtained a similar result starting with the transitive closure $\hat{S}$ of S instead of S. Observe that, simply taking $\succsim(S) = \hat{S}$, would have probably been a better choice in this example.

The final result of the ranking procedure is obtained by taking the intersection of two weak orders. Since it is well-known that there are reflexive and transitive relations that cannot be obtained in such a way (Dushnik and Miller, 1941), this procedure is not faithful. We leave the proof that this procedure is indeed neutral and monotonic to the reader (this is detailed in Vincke, 1992).

4.4.3.2 Copeland Scores

We have seen that the procedure suggested in ELECTRE II does not satisfy all the requirements we intuitively would like to see satisfied. A simpler ranking procedure consists in rank ordering the elements in A according to their Copeland scores, i.e., the number of alternatives that they beat minus the number of alternatives that beat them. With Example 4.6, this would, abusing notation, give the weak order:

$$a_1 \succ a_8 \succ [a_2, a_3, a_4, a_6, a_7] \succ a_5 \succ a_9.$$

We cannot expect faithfulness with such a procedure, since the result of the procedure is obviously complete (note that the procedure treats indifference and incomparability similarly). On the other hand, such a procedure is neutral and monotonic.

The ranking procedure based on Copeland scores was characterized by Rubinstein (1980) (for the case of tournaments, i.e., complete and antisymmetric relations) and Henriet (1985) (for the case of complete relations). It is not difficult to extend Henriet's result to cover the case of an arbitrary reflexive relation (see Bouyssou, 1992). The main distinctive characteristic of this ranking procedure is that it is insensitive to the presence of circuits in S since the contribution of any circuit to the Copeland scores of the alternatives in the circuit is always zero.

Ranking procedures based on scores are quite common as soon as one deals with valued binary relations (a topic that is outside the scope of the present text). Let us simply mention here that the "net flow" score used in the PROMETHEE method (Brans and Vincke, 1985) can be seen as an extension of the Copeland score to the valued case (Bouyssou, 1992) (see Chap. 19). Other scores, e.g., scores that do not make use of the cardinal properties of the valuations can be envisaged (Bouyssou and Pirlot, 1997). Other ways of using scores are considered in Dias and Lamboray (2010).

4.4.3.3 Ranking by Repeated Choice

A possible way of combining the simplicity of such a ranking procedure with a move towards faithfulness consists in using the Copeland scores iteratively to build two weak orders T_1 and T_2. This would consist here in building the first equivalence class of a weak order T_1 with the alternatives having the highest Copeland scores, and iterating the procedures after having removed the already-ranked alternatives. For the relation in Example 4.6, we would obtain:

$$a_1 T_1 [a_2, a_4, a_8] T_1 a_6 T_1 a_7 T_1 [a_3, a_5, a_9].$$

Using a dual principle, we could also build a weak order T_2 the last equivalence class of which consists of alternatives having minimal Copeland scores and reiterate the process on the set of unranked alternatives. This would yield:

$$[a_1, a_2, a_3, a_8] T_2 a_4 T_2 a_6 T_2 [a_5, a_7] T_2 a_9.$$

Taking the intersection of these two weak orders is a much simplified version of the ranking procedure implemented in ELECTRE III (Roy, 1978). This leads to:

$$\begin{gathered} a_1 \succ [a_2, a_8] \succ a_4 \succ a_6 \succ a_7 \succ a_5 \succ a_9, \\ [a_2, a_8] \succ a_3 \succ a_5. \end{gathered}$$

Such a result does not seem to lead us closer to an adequate restitution of the uncertain positions of a_8 and a_5 within S. Furthermore, as observed in Example 4.2, such a ranking procedure is not monotonic, which seems to be a serious shortcoming.

4.4.4 *Sorting Procedures*

We have seen that the lack of transitivity and/or completeness raised serious difficulties when it comes to devising choosing and ranking procedures. These difficulties are somewhat less serious here. This is because, with sorting procedures, the assignment of an alternative only depends on its comparison to carefully selected reference actions defining the categories. The use of such reference points implies that, contrary to the case of choice and ranking procedures, the distinction between the phase of building a relation S and then using this relation in order to reach conclusions is blurred with the sorting problem statement. Reference points are used from the beginning and the relation S is mainly used to compare the alternatives in A to these reference points.

Early attempts to propose sorting procedures are Massaglia and Ostanello (1991), Moscarola and Roy (1977) and Roy (1981). A more general approach to the problem was suggested in Roy and Bouyssou (1993) and Yu (1992) with the so-called ELECTRE TRI approach that we present below.

4.4.4.1 An Overview of ELECTRE TRI

We consider the case of r ordered categories $C^1, C^2, \ldots, C^r$, with C^r containing the most desirable alternatives. We suppose, for the moment, that each category C^k is delimited by a limiting profile π^k. It is not restrictive to suppose that π^{k+1} strictly dominates[2] π^k, for all k. Furthermore, we can always find an alternative π^{r+1} that

[2]That is, π^{k+1} is at least as good as π^k on all criteria and strictly better on some criterion.

strongly dominates[3] all other alternatives in A and, conversely, an alternative π^1 that is strongly dominated by all other alternatives.

How can we use a preference relation between the alternatives in A and the set of limiting profiles to define a sorting procedure? Intuitively, since π^k is the lower limit of category C^k, we can apply the following two rules:

- if an alternative a is preferred to π^k, it should at least belong to category C^k,
- if π^k is preferred to a, a should at most belong to category C^{k-1},

the case in which a is indifferent to π^k is dealt with conventionally depending on the definition of the limiting profiles π^k.

When the relation S is complete and transitive, these two rules lead to unambiguously assign each alternative to a single category.

The situation is somewhat more complex when S is intransitive or incomplete. When S is compatible with the dominance relation (which is not a very restrictive hypothesis), as we have supposed that π^k strictly dominates π^{k-1}, it is possible to show (see Roy and Bouyssou, 1993, , Chap. 5) that when an alternative a is compared to the set of limiting profiles $\pi^1, \pi^2 \dots, \pi^{r+1}$, three distinct situations can arise:

1. $\pi^{r+1} P a, \pi^r P a, \dots, \pi^{k+1} P a, a P \pi^k, a P \pi^{k-1}, \dots, a P \pi^1$. In such a case, there is little doubt on how to assign a to one of $C^1, C^2, \dots, C^r$. Since $a P \pi^k$, a should be assigned at least to category C^k. But since $\pi^{k+1} P a$, a should be assigned at most to C^k. Hence, a should belong to C^k.
2. $\pi^{r+1} P a$, $\pi^{r+2} P a, \dots,$ $\pi^{\ell+1} P a$, $a I \pi^\ell$, $a I \pi^{\ell-1}, \dots,$ $a I \pi^{k+1}$, $a P \pi^k, \dots,$ $a P \pi^1$. The situation is here more complex. Since $\pi^{\ell+1} P a$, alternative a must be assigned at most to category C^ℓ. Similarly since $a P \pi^k$, a must be assigned at least to category C^k.

   The fact that a is indifferent to several consecutive limiting profiles is probably a sign that the definition of the categories is too precise with respect to the binary relation that is used by the sorting procedure: the profiles are too close to one another. This would probably call for a redefinition of the categories and/or for a different choice for S. In such a situation, an optimistic attitude consists in assigning a to the highest possible category, i.e., C^ℓ. A pessimistic attitude would assign a to C^k.
3. $\pi^{r+1} P a, \pi^r P a, \dots, \pi^{\ell+1} P a, a J \pi^\ell, a J \pi^{\ell-1}, \dots, a J \pi^{k+1}, a P \pi^k, \dots, a P \pi^1$. In this situation, a is incomparable to several consecutive profiles. This is a sign that, although we are sure that a must be assigned at most to category C^ℓ and at least to category C^k, the relation S does not provide enough information to opt for a category within this interval. Again, an optimistic attitude in such a situation consists in assigning a to the highest possible category, i.e., C^ℓ. A pessimistic attitude would be to assign a to C^k.

[3]That is, it is strictly better on all criteria.

The assignment procedure described above is the one introduced in ELECTRE TRI (Roy and Bouyssou, 1993; Yu, 1992) in which a is assigned to one of $C^1, C^2, \ldots, C^r$ using an optimistic procedure and a pessimistic procedure. Alternative a is always assigned to a higher category when using the optimistic procedure than when using the pessimistic procedure.

When S is identical to a dominance relation, the optimistic procedure suggested above coincides with a disjunctive sorting procedure. In fact a will be assigned to C^ℓ as soon as $\pi^{\ell+1} P a$ and $Not[\pi^\ell P a]$, which means that ℓ is the highest category such that, on some criterion $i \in N$, a is better than π^ℓ. Conversely, the pessimistic procedure coincides with a conjunctive assignment strategy: a will be assigned to C^k as soon as $Not[a P \pi^{k+1}]$ and $a P \pi^k$, which amounts to saying that k is the lowest category such that a dominates π^k.

It is worth noting that although the authors of this method have coupled this procedure with a particular definition of S (a crisp relation based on a concordance discordance principle), it can be applied to any relation that is compatible with a dominance relation.

An axiomatic analysis of ELECTRE TRI was recently proposed in Bouyssou and Marchant (2007a,b). For applications of ELECTRE TRI, see for instance Chaps. 9 and 7

4.4.4.2 Implementation of ELECTRE TRI

The ELECTRE TRI procedure described above supposes that the analyst has defined:

- the limiting profile π^k for each category C^k,
- the parameters involved in the definition of S: weights, indifference and preference thresholds, veto thresholds.

This is overly demanding in most applications involving the use of a sorting procedure. In many cases however, it is possible to obtain examples of alternatives that should be assigned to a given category. Like in the UTADIS method described earlier (see Sect. 4.3.3), one may use a "learning by examples" strategy to assign a value to these parameters. Several strategies for doing this were investigated in Dias and Mousseau (2006), Dias et al. (2002), Mousseau et al. (2001), Mousseau and Słowiński (1998), and Ngo The and Mousseau (2002). In Dias et al. (2002) and Dias and Mousseau (2003) a way to derive robust conclusions with ELECTRE TRI on the basis of several relations S is suggested. This shows that the analysis in Sect. 4.3 can be applied to preference models that are not based on a value function. An approach to the derivation of robust conclusions with ELECTRE TRI based on Monte-Carlo simulation is presented in Tervonen et al. (2009).

4.5 Conclusion

The difficulties presented in Sect. 4.4 raise the question of how to analyse and compare procedures designed to build recommendations. We would like to conclude with some thoughts on this point. Two main routes can be followed. The first one (see, e.g., Bouyssou and Vincke, 1997; Vincke, 1992) consists in defining a list of properties that seem "desirable" for such a technique (for example, never select a dominated alternative or respond to the improvement of an alternative in the expected way). Given such a list of properties one may then try:

- to analyse whether or not they are satisfied by a number of techniques,
- to establish "impossibility theorems", i.e., subsets of properties that cannot be simultaneously fulfilled,
- to determine, given the above-mentioned impossibility theorems, the techniques that satisfy most properties.

The second one (see, e.g., Bouyssou, 1992; Bouyssou and Perny, 1992; Bouyssou and Pirlot, 1997; Pirlot, 1995) consists in trying to find a list of properties that would "characterize" a given technique, i.e., a list of properties that this technique would be the only one to satisfy. This allows to emphasize the specific features of an exploitation technique and, thus, to compare it more easily with others.

These two types of analysis are not unrelated: ideally they should merge at the end, the characterizing properties exhibited by the second type of analysis being parts of the list of "desirable" properties used in the first type of analysis. Both types of analysis have their own problems. In the first, the main problem consists in defining the list of "desirable" properties. These properties should indeed cover every aspect of what seems to be constitutive of an "appropriate" technique. In the second, the characterizing properties will only be useful if they have a clear and simple interpretation, which may not always be the case when analysing a complex technique. We do hope that such analyses will continue to develop.

Let us finally mention that we have restricted our attention to procedures that only operate on the basis of the relation S. In particular, this excludes the use of some "reference points", i.e., of alternatives playing a particular role, as advocated by Dubois et al. (2003). When such reference points are taken into account, the separation between the phases of building a relation S and exploiting it in order to build a choice set is blurred. Indeed, it is then tempting to compare alternatives only to the reference points and not amongst themselves. Such approaches may offer an interesting alternative to the procedures presented above. They have not been worked out in much detail to date. In particular, the selection in practice of appropriate reference points does not seem to be an obvious task.

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