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Additive and decomposable conjoint measurement with  
ordered categories

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## Abstract

Conjoint measurement studies binary relations defined on product sets and investigates the existence and uniqueness of numerical representations of such relations. It has proved to be quite a powerful tool to analyze and compare MCDM techniques designed to build a preference relation between multiattributed alternatives and has been an inspiring guide to many assessment protocols. These MCDM techniques lead to a relative evaluation model of the alternatives through a preference relation. Such models are not always appropriate to build meaningful recommendations. This has recently lead to the development of MCDM techniques aiming at building evaluation models having a more absolute character. In such techniques, the output of the analysis is, most often, a partition of the set of alternatives into several ordered categories defined with respect to outside norms, e.g., separating “Attractive” and “Unattractive” alternatives. In spite of their interest, the theoretical foundations of such MCDM techniques have not been much investigated. The purpose of this paper is to contribute to this analysis. More precisely, we show how to adapt classic conjoint measurement results to make them applicable for the study of such MCDM techniques. We concentrate on additive models. Our results may be seen as an attempt to provide an axiomatic basis to the well-known UTADIS technique that sorts alternatives using an additive value function model.

**Keywords:** Decision with multiple attributes, Sorting, Conjoint measurement, UTADIS.

# Représentations numériques additives et décomposables de catégories ordonnées

## Résumé

La théorie du mesurage conjoint étudie la question de la représentation numérique d'une relation binaire définie sur un produit cartésien. Cette théorie s'est révélée très utile pour comparer et analyser diverses techniques d'aide multicritère à la décision. Elle a également été la source de nombreux protocoles d'élicitation.

Les techniques d'aide à la décision utilisant une relation binaire comparant des actions évaluées sur plusieurs attributs conduisent, en général, à des modèles d'évaluation ayant un caractère relatif. Or, de tels modèles ne sont pas toujours adaptés pour bâtir une prescription pertinente.

Ceci a conduit au développement de techniques multicritères conduisant à des modèles d'évaluation ayant un caractère plus absolu. Dans ces techniques, le résultat se présente généralement sous la forme d'une affectation des actions à diverses catégories ordonnées, ces catégories étant définies par rapport à des normes indépendantes des actions à évaluer. On pourra, par exemple, séparer les actions satisfaisantes de celles étant insatisfaisantes. En dépit de leur intérêt, les fondements théoriques de telles méthodes ont été peu étudiés. L'objectif de cet article est de contribuer à cette étude. Plus précisément, on montre comment adapter les résultats classiques du mesurage conjoint pour couvrir le cas de catégories ordonnées. On étudie plus spécifiquement le cas de représentations additives. Ce travail peut alors être vu comme une tentative de donner à la méthode UTADIS une base théorique solide.

**Mots-clés:** Analyse multicritère, Tri, Mesurage conjoint, UTADIS.

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# 1 Introduction and motivation

The aim of most MCDM techniques (see Belton and Stewart, 2001, Bouyssou et al., 2006, for recent reviews) is to build a model allowing to compare alternatives evaluated on several attributes in terms of preference. Conjoint measurement (see Krantz et al., 1971, Ch. 6 & 7 or Fishburn, 1970, Ch. 4 & 5) is a branch of measurement theory studying binary relations defined on product sets and investigating the existence and uniqueness of numerical representations of such relations. It has proved to be a powerful tool to analyze and compare MCDM techniques. It has also been an inspiring guide to many assessment protocols (see, e.g., Keeney and Raiffa, 1976 or von Winterfeldt and Edwards, 1986).

In some instances, building a recommendation on the basis of a preference relation between alternatives does not seem to be fully adequate. Indeed, a preference relation between alternatives is an evaluation model that has only a *relative* character and, for instance, it may well happen that the best alternatives are not desirable at all. This calls for MCDM techniques building evaluation models having a more absolute character. Such models belong to what Roy (1996) called the “sorting problem statement”. Suppose for instance that an academic institution wants a model that would help the committee responsible for the admission of students in a given program. A model only aiming at building a relation comparing students in terms of “performance” is unlikely to be much useful. We expect such an institution to be primarily interested in a model that would isolate, within the set of all candidates, the applicants that are most likely to meet its standards defining what a “good” student is.

MCDM techniques designed to cope with such problems most often lead to build a partition of the set of alternatives into ordered categories, e.g., through the comparison of alternatives to “norms” or the analysis of assignment examples. This type of techniques has recently attracted much attention in the literature (see Greco et al., 1999, 2002a, 2005, Zopounidis and Doumpos, 2000a, 2002, for reviews). Several techniques have been designed to tackle such problems such as UTADIS (see Jacquet-Lagrèze, 1995, Zopounidis and Doumpos, 2000b), ELECTRE TRI (see Mousseau et al., 2000, Roy and Bouyssou, 1993, Wei, 1992), filtering methods (see Henriot, 2000, Perny, 1998), methods based on the Choquet integral (see Marichal et al., 2005, Marichal and Roubens, 2001, Meyer and Roubens, 2005), methods inspired by PROMETHEE (Doumpos and Zopounidis, 2002, 2004, Figueira et al., 2004), methods based on rough sets (Greco et al., 2001, 2002b, Słowiński et al., 2002) or the interactive approach introduced in Köksalan and Ulu (2003).

The aim of this paper is to contribute to a recent trend of research (see Bouyssou and Marchant, 2007a,b, Greco et al., 2001, Słowiński et al., 2002) aiming at providing sound theoretical foundations to such methods. Greco et al. (2001) and

Słowiński et al. (2002), extending Goldstein (1991), have concentrated on models admitting a “decomposable” numerical representation that have a simple interpretation in terms of “decision rules” and investigated some of its variants. Bouyssou and Marchant (2007a,b) have studied a model that is close to the one used in the ELECTRE TRI technique (see Mousseau et al., 2000, Roy and Bouyssou, 1993, Wei, 1992) that turns out to be a particular case of the decomposable models studied in Greco et al. (2001) and Słowiński et al. (2002). The aim of this paper is to pursue this line of research.

With UTADIS in mind, we concentrate in this paper on the question of exhibiting conditions allowing to build an additive numerical representation. It is important to note that this problem has already been tackled in depth by Vind (1991) (see also Vind, 2003, Ch. 5 & 9). Vind (1991) assumes that the set of alternatives has a “continuous structure” and that there are at least four attributes. Besides being rather complex, these results do not cover all cases that may be interesting for analyzing MCDM techniques designed to sort alternatives between ordered categories. This motivates the present paper.

Our main objective will be to show how to adapt the classical results of conjoint measurement characterizing the additive value function model (i.e., the ones presented in Krantz et al., 1971, Ch. 6 & 7) to the case of ordered categories (a related path was followed by Nakamura, 2004, for the case of decision making under risk). In performing such an adaptation, we will try to keep things as simple as possible. A companion paper (Bouyssou and Marchant, 2008c) is devoted to the more technical issues involved by such an adaptation.

The rest of this paper is organized as follows. We introduce our setting in Section 2. Section 3 contains our main results. They are discussed in a final section. The appendix contains several additional results that cover cases not dealt with in the main text or extending them.

## 2 Definitions and notation

### 2.1 The setting

Let  $n \geq 2$  be an integer and  $X = X_1 \times X_2 \times \cdots \times X_n$  be a set of objects. Elements  $x, y, z, \dots$  of  $X$  will be interpreted as alternatives evaluated on a set  $N = \{1, 2, \dots, n\}$  of attributes. For any nonempty subset  $J$  of the set of attributes  $N$ , we denote by  $X_J$  (resp.  $X_{-J}$ ) the set  $\prod_{i \in J} X_i$  (resp.  $\prod_{i \notin J} X_i$ ). With customary abuse of notation, when  $x, y \in X$ ,  $(x_J, y_{-J})$  will denote the element  $w \in X$  such that  $w_i = x_i$  if  $i \in J$  and  $w_i = y_i$  otherwise. We often omit braces around sets and write, e.g.,  $X_{-i}$ ,  $X_{-ij}$  or  $(x_i, x_j, y_{-ij})$ .

The traditional primitive of conjoint measurement is a binary relation  $\succsim$  defined

on  $X$  with  $x \succsim y$  interpreted as “ $x$  is at least as good as  $y$ ”. Our primitives will consist here in the assignment of each object  $x$  to a category that has an interpretation in terms of the intrinsic desirability of  $x$ . Throughout the main text, we concentrate on the case in which there are only two ordered categories. We consider more general cases in the appendix.

## 2.2 Primitives

The most natural primitive for our study seems to be a partition of the set  $X$  between ordered categories, i.e., a twofold partition  $\langle \mathcal{A}, \mathcal{U} \rangle$  with the convention that  $\mathcal{A}$  contains “Acceptable” alternatives and  $\mathcal{U}$  “Unacceptable” ones. It is useful to interpret  $\langle \mathcal{A}, \mathcal{U} \rangle$  as the result of a sorting model between ordered categories applied to the alternatives in  $X$ . As argued in Bouyssou and Marchant (2007a,b), the hypothesis that the ordering of categories is known beforehand is not really a restriction and remains in line with the type of data that is likely to be collected.

When the primitives consist of an ordered partition, each alternative  $x \in X$  is unambiguously assigned to one and only one of the categories  $\langle \mathcal{A}, \mathcal{U} \rangle$ . This is restrictive. Indeed, when asked to assigned an alternative  $x \in X$  to a category, a subject may well hesitate. Models tolerating such hesitations were considered in Greco et al. (2001) and Słowiński et al. (2002). We will not investigate them here. Another reason for enlarging the framework of ordered partitions is the following. Suppose that  $x \in \mathcal{A}$  and that  $y \in \mathcal{U}$ . In order to delineate categories  $\mathcal{A}$  and  $\mathcal{U}$ , one may try to find an alternative in  $\mathcal{A}$  that is slightly worse than  $x$  and an alternative in  $\mathcal{U}$  is slightly better than  $y$ . Iterating this process, we are likely to find alternatives that lie “at the frontier” between categories  $\mathcal{A}$  and  $\mathcal{U}$ . The consideration of alternatives at the frontier between consecutive categories was suggested in Goldstein (1991). It is central in Nakamura (2004) and in what follows.

Therefore, our primitives will consist here in a twofold covering  $\langle \mathcal{A}^*, \mathcal{U}^* \rangle$  of  $X$ , i.e., of two sets  $\mathcal{A}^*$  and  $\mathcal{U}^*$  such that  $\mathcal{A}^* \cup \mathcal{U}^* = X$ ,  $\mathcal{A}^* \neq \emptyset$  and  $\mathcal{U}^* \neq \emptyset$ . The alternatives in  $\mathcal{A}^* \cap \mathcal{U}^* = \mathcal{F}$  are supposed to lie at the frontier between the two categories. We note  $\mathcal{A} = \mathcal{A}^* \setminus \mathcal{F}$  and  $\mathcal{U} = \mathcal{U}^* \setminus \mathcal{F}$ . The alternatives in  $\mathcal{A}$  (resp. in  $\mathcal{U}$ ) are therefore interpreted as being unambiguously acceptable (resp. unacceptable). In all what follows, we alternatively view our primitives as consisting in a threefold partition  $\langle \mathcal{A}, \mathcal{F}, \mathcal{U} \rangle$  of  $X$  with the category  $\mathcal{F}$  playing the special role of a frontier between  $\mathcal{A}$  and  $\mathcal{U}$ . Abusing terminology, we will speak of  $\langle \mathcal{A}, \mathcal{F}, \mathcal{U} \rangle$  as an ordered covering of  $X$ . We sometimes write  $\mathcal{AF}$  instead  $\mathcal{A}^* = \mathcal{A} \cup \mathcal{F}$  and  $\mathcal{FU}$  instead of  $\mathcal{U}^* = \mathcal{F} \cup \mathcal{U}$ .

We say that an attribute  $i \in N$  is *influential* for  $\langle \mathcal{A}, \mathcal{F}, \mathcal{U} \rangle$  if there are  $x_i, y_i \in X_i$  and  $a_{-i} \in X_{-i}$  such that  $(x_i, a_{-i})$  and  $(y_i, a_{-i})$  do not belong to the same category in  $\langle \mathcal{A}, \mathcal{F}, \mathcal{U} \rangle$ .

We say that  $\langle \mathcal{A}, \mathcal{F}, \mathcal{U} \rangle$  is non-degenerate if both  $\mathcal{A}$  and  $\mathcal{U}$  are nonempty.

## 2.3 Models

Goldstein (1991) suggested the use of conjoint measurement techniques for the analysis of twofold coverings of a set of multiattributed alternatives through decomposable models of the type:

$$\begin{aligned} x \in \mathcal{A} &\Leftrightarrow F[v_1(x_1), v_2(x_2), \dots, v_n(x_n)] > 0, \\ x \in \mathcal{F} &\Leftrightarrow F[v_1(x_1), v_2(x_2), \dots, v_n(x_n)] = 0, \end{aligned}$$

where  $v_i$  is a real-valued function on  $X_i$  and  $F$  is a real-valued function on  $\prod_{i=1}^n v_i(X_i)$  that may have several additional properties, e.g., being one-to-one, nondecreasing or increasing in all its arguments. This analysis was extended in Greco et al. (2001) and Słowiński et al. (2002) to deal with an arbitrary number of categories, when there is no frontier. It is not difficult to extend this analysis to cope with frontiers (see Appendix F).

In the main text, we will concentrate on the case in which the above function  $F$  can be made additive. This special case is of direct interest to MCDM techniques like UTADIS sorting alternatives between ordered categories on the basis of an additive value function. On the theoretical side, it should be apparent that constraining  $F$  to be additive raises a measurement problem that is significantly more complex than the one dealt with decomposable models.

Hence our main task will be to find conditions on  $\langle \mathcal{A}, \mathcal{F}, \mathcal{U} \rangle$  that ensure the existence of real valued functions  $v_i$  on  $X_i$  such that, for all  $x \in X$ ,

$$\begin{aligned} x \in \mathcal{A} &\Leftrightarrow \sum_{i=1}^n v_i(x_i) > 0, \\ x \in \mathcal{F} &\Leftrightarrow \sum_{i=1}^n v_i(x_i) = 0. \end{aligned} \tag{A}$$

Such a problem was already tackled in Vind (1991) (see also Vind, 2003, Ch. 5 & 9). Although Vind's results are extremely useful, they are rather complex and do not cover all cases that may be interesting for analyzing MCDM techniques designed to sort alternatives between ordered categories. With the aim of obtaining simpler results that would cover a larger number of cases, we will show how to adapt the classical results of conjoint measurement characterizing the additive value function model (see Krantz et al., 1971, Ch. 6) to analyze model (A). The price to pay for this will be the use of solvability conditions that are quite strong. In a companion paper (Bouyssou and Marchant, 2008c), we consider less strong assumptions that lead to results that are more powerful but that are no more simple adaptations of classical conjoint measurement results.

Let us note that the study of model (A) involves many different cases. When  $n = 2$ , the analysis of model (A) belongs more to the field of ordinal measurement than to that of conjoint measurement. We briefly consider this particular case in Appendix A. This involves simple extensions of classical results on biorders to cope with our framework. In the same vein, the case in which  $X$  is finite involves the use of standard techniques. This is tackled in Appendix B. In the main text, we therefore concentrate on the case in which  $n \geq 3$  and the set of alternatives is not supposed to be finite. Appendix E deals with the case in which there are more than two ordered categories.

### 3 Additive representations

Our aim in this section is to present condition ensuring the existence of an additive representation of  $\langle \mathcal{A}, \mathcal{F}, \mathcal{U} \rangle$  when there are at least three attributes. Starting with  $\langle \mathcal{A}, \mathcal{F}, \mathcal{U} \rangle$  on  $X = X_1 \times \dots \times X_2 \times \dots \times X_n$ , our strategy will be to build a binary relation on a product set that leaves out one attribute, i.e., on a set  $\prod_{i \neq j} X_i$ . We will impose conditions on  $\langle \mathcal{A}, \mathcal{F}, \mathcal{U} \rangle$  ensuring that this binary relation satisfies the standard axioms of conjoint measurement as given in Krantz et al. (1971, Ch. 6). This ensures the existence of an additive representation of the binary relation. Bringing the attribute that was left out in the construction of the binary relation back into the picture again, we will show that the additive representation of the binary relation can be used to obtain an additive representation of the ordered covering  $\langle \mathcal{A}, \mathcal{F}, \mathcal{U} \rangle$ .

#### 3.1 Axioms

Our first condition generalizes to subsets a linearity condition that is central in the characterization of the decomposable model for ordered partitions.

**Definition 1** (Linearity)

Let  $\langle \mathcal{A}, \mathcal{F}, \mathcal{U} \rangle$  be an ordered covering of  $X$  and  $I \subseteq N$ . We say that  $\langle \mathcal{A}, \mathcal{F}, \mathcal{U} \rangle$  is

1.  $\mathcal{A}$ -linear on  $I \subseteq N$  (condition  $\mathcal{A}$ -linear<sub>I</sub>) if

$$\left. \begin{array}{l} (x_I, a_{-I}) \in \mathcal{A} \\ \text{and} \\ (y_I, b_{-I}) \in \mathcal{A} \end{array} \right\} \Rightarrow \left\{ \begin{array}{l} (y_I, a_{-I}) \in \mathcal{A} \\ \text{or} \\ (x_I, b_{-I}) \in \mathcal{A} \end{array} \right. \quad (\mathcal{A}\text{-linear}_I)$$

2.  $\mathcal{F}$ -linear on  $I \subseteq N$  if

$$\left. \begin{array}{l} (x_I, a_{-I}) \in \mathcal{F} \\ \text{and} \\ (y_I, b_{-I}) \in \mathcal{F} \end{array} \right\} \Rightarrow \left\{ \begin{array}{l} (y_I, a_{-I}) \in \mathcal{AF} \\ \text{or} \\ (x_I, b_{-I}) \in \mathcal{AF} \end{array} \right. \quad (\mathcal{F}\text{-linear}_I)$$

3.  $\mathcal{AF}$ -linear on  $I \subseteq N$  if

$$\left. \begin{array}{l} (x_I, a_{-I}) \in \mathcal{A} \\ \text{and} \\ (y_I, b_{-I}) \in \mathcal{F} \end{array} \right\} \Rightarrow \left\{ \begin{array}{l} (y_I, a_{-I}) \in \mathcal{A} \\ \text{or} \\ (x_I, b_{-I}) \in \mathcal{AF} \end{array} \right. \quad (\mathcal{AF}\text{-linear}_I)$$

for all  $x_I, y_I \in X_I$  and  $a_{-I}, b_{-I} \in X_{-I}$ . We say that  $\langle \mathcal{A}, \mathcal{F}, \mathcal{U} \rangle$  is  $\text{linear}_I$  if it is  $\mathcal{A}$ - $\text{linear}_I$ ,  $\mathcal{F}$ - $\text{linear}_I$  and  $\mathcal{AF}$ - $\text{linear}_I$ . We say that  $\langle \mathcal{A}, \mathcal{F}, \mathcal{U} \rangle$  is strongly linear if it satisfies  $\text{linear}_I$ , for all  $I \subseteq N$ .

It is easy to check that the existence of an additive representation implies strong linearity. The consequences of our linearity conditions can be clearly understood considering the trace that  $\langle \mathcal{A}, \mathcal{F}, \mathcal{U} \rangle$  generates on each  $X_I$ .

Let  $I \subseteq N$ . We define on  $X_I$  the binary relation  $\succsim_I$  letting, for all  $x_I, y_I \in X_I$ ,

$$x_I \succsim_I y_I \Leftrightarrow \text{for all } a_{-I} \in X_{-I}, \left\{ \begin{array}{l} (y_I, a_{-I}) \in \mathcal{A} \Rightarrow (x_I, a_{-I}) \in \mathcal{A}, \\ (y_I, a_{-I}) \in \mathcal{F} \Rightarrow (x_I, a_{-I}) \in \mathcal{AF}. \end{array} \right.$$

We say that  $\succsim_I$  is the trace on  $X_I$  generated by  $\langle \mathcal{A}, \mathcal{F}, \mathcal{U} \rangle$ . By construction,  $\succsim_I$  is always reflexive and transitive. We use  $\succ_I$  and  $\sim_I$  as is usual. We omit the obvious proof the following result.

**Lemma 2**

For all  $x, y \in X$  and all  $I \subseteq N$ , we have:

$$\begin{aligned} [y \in \mathcal{A} \text{ and } x_I \succsim_I y_I] &\Rightarrow (x_I, y_{-I}) \in \mathcal{A}, \\ [y \in \mathcal{F} \text{ and } x_I \succsim_I y_I] &\Rightarrow (x_I, y_{-I}) \in \mathcal{AF}, \\ [x_i \succsim_i y_i, \text{ for all } i \in I] &\Rightarrow [x_I \succsim_I y_I]. \end{aligned}$$

Furthermore, a covering  $\langle \mathcal{A}, \mathcal{F}, \mathcal{U} \rangle$  is  $\text{linear}_I$  iff  $\succsim_I$  is complete.

It is easy to build examples showing that, in general, conditions  $\mathcal{A}$ - $\text{linear}_I$ ,  $\mathcal{F}$ - $\text{linear}_I$  and  $\mathcal{AF}$ - $\text{linear}_I$  are independent. Condition  $\mathcal{A}$ - $\text{linear}_I$ , for all  $I \subseteq N$  is the main necessary condition used in Vind (1991) together with topological assumptions on  $X$  that make it possible to implicitly deal with the alternatives in  $\mathcal{F}$ . In our algebraic setting, we need to impose conditions dealing with the alternatives in  $\mathcal{F}$ . In view of our interpretation of  $\mathcal{F}$  as the frontier between two categories, such conditions are less intuitive than conditions that would only involve  $\mathcal{A}$  and  $\mathcal{U}$ . This seems unavoidable however.

Our next condition aims at capturing the special role played by category  $\mathcal{F}$ .

**Definition 3** (Thinness)

Let  $I \subseteq N$ . We say that the covering  $\langle \mathcal{A}, \mathcal{F}, \mathcal{U} \rangle$  is  $\text{thin}_I$  if,

$$\left. \begin{array}{l} (x_I, a_{-I}) \in \mathcal{F} \\ \text{and} \\ (y_I, a_{-I}) \in \mathcal{F} \end{array} \right\} \Rightarrow \left\{ \begin{array}{l} (x_I, b_{-I}) \in \mathcal{A} \Leftrightarrow (y_I, b_{-I}) \in \mathcal{A}, \\ (x_I, b_{-I}) \in \mathcal{U} \Leftrightarrow (y_I, b_{-I}) \in \mathcal{U}, \end{array} \right.$$

for all  $x_I, y_I \in X_I$  and  $a_{-I}, b_{-I} \in X_{-I}$ . We say that  $\langle \mathcal{A}, \mathcal{F}, \mathcal{U} \rangle$  is strongly thin if it is  $\text{thin}_I$ , for all  $I \subseteq N$ .

It is easy to check that the existence of an additive representation implies that the covering must be strongly thin. We omit the simple proof of the following lemma.

**Lemma 4**

Suppose that  $\langle \mathcal{A}, \mathcal{F}, \mathcal{U} \rangle$  is  $\text{linear}_I$  and  $\text{thin}_I$  for  $I \subseteq N$ . Then

$$\begin{aligned} [(x_I, a_{-I}) \in \mathcal{F} \text{ and } y_I \succ_I x_I] &\Rightarrow (y_I, a_{-I}) \in \mathcal{A}, \\ [(x_I, a_{-I}) \in \mathcal{F} \text{ and } x_I \succ_I z_I] &\Rightarrow (z_I, a_{-I}) \in \mathcal{U}, \end{aligned}$$

for all  $x_I, y_I, z_I \in X_I$  and  $a_{-I} \in X_{-I}$ .

It is easy to build examples showing that condition  $\text{thin}_I$  is, in general, independent from  $\mathcal{A}$ - $\text{linear}_I$ ,  $\mathcal{F}$ - $\text{linear}_I$  and  $\mathcal{AF}$ - $\text{linear}_I$ .

The next condition will only come into play when  $n = 3$ .

**Definition 5** (Thomsen condition)

We say that  $\langle \mathcal{A}, \mathcal{F}, \mathcal{U} \rangle$  on  $X$  satisfies the Thomsen condition if

$$\left. \begin{array}{l} (x_i, x_j, a_{-ij}) \in \mathcal{F} \\ (x_i, x_j) \sim_{ij} (y_i, y_j) \\ (y_i, z_j, b_{-ij}) \in \mathcal{F} \\ (y_i, z_j) \sim_{ij} (z_i, x_j) \end{array} \right\} \Rightarrow (x_i, z_j) \sim_{ij} (z_i, y_j),$$

for all  $i, j \in N$  with  $i \neq j$ , all  $x_i, y_i, z_i \in X_i$ , all  $x_j, y_j, z_j \in X_j$  and all  $a_{ij}, b_{-ij} \in X_{-ij}$ .

Let us show that this condition is necessary for model (A). Indeed,  $(x_i, x_j, a_{-ij}) \in \mathcal{F}$  and  $(x_i, x_j) \sim_{ij} (y_i, y_j)$  imply that  $(y_i, y_j, a_{-ij}) \in \mathcal{F}$ . This implies  $v_i(x_i) + v_j(x_j) = v_i(y_i) + v_j(y_j)$ . Similarly,  $(y_i, z_j, b_{-ij}) \in \mathcal{F}$  and  $(y_i, z_j) \sim_{ij} (z_i, x_j)$  lead to  $v_i(y_i) + v_j(z_j) = v_i(z_i) + v_j(x_j)$ . Hence, we have  $v_i(z_i) + v_j(y_j) = v_i(x_i) + v_j(z_j)$ , so that  $(x_i, z_j) \sim_{ij} (z_i, y_j)$ .

Our next condition is a possible formalization of an Archimedean condition for ordered coverings.

**Definition 6** (Archimedean condition)

Let  $i, j \in N$  with  $i \neq j$ . Let  $\mathbb{K}$  be any set of consecutive integers (positive or negative, finite or infinite). We say that  $x_i^\kappa \in X_i$ ,  $\kappa \in \mathbb{K}$ , is a standard sequence for  $\langle \mathcal{A}, \mathcal{F}, \mathcal{U} \rangle$  on attribute  $i \in N$  (with respect to attribute  $j \in N$ ) if there are  $a_{-ij}, b_{-ij} \in X_{-ij}$  and  $x_j^\kappa \in X_j$ ,  $\kappa \in \mathbb{K}$ , such that  $\text{Not}[a_{-ij} \sim_{-ij} b_{-ij}]$  (i.e., there are  $a_i \in X_i$  and  $b_j \in X_j$  such that  $(a_i, a_j, a_{-ij})$  and  $(a_i, a_j, b_{-ij})$  do not belong to the same category) and

$$\begin{aligned} (x_i^\kappa, x_j^\kappa, a_{-ij}) &\in \mathcal{F}, \\ (x_i^{\kappa+1}, x_j^\kappa, b_{-ij}) &\in \mathcal{F}, \end{aligned} \tag{1}$$

for all  $\kappa \in \mathbb{K}$ .

The standard sequence is strictly bounded if there are  $\underline{x}_i, \bar{x}_i \in X_i$  such that  $\bar{x}_i \succ_i x_i^\kappa$  and  $x_i^\kappa \succ_i \underline{x}_i$  for all  $\kappa \in \mathbb{K}$ .

The covering  $\langle \mathcal{A}, \mathcal{F}, \mathcal{U} \rangle$  satisfies the Archimedean condition if every strictly bounded standard sequence is bounded.

Suppose that  $x_i^\kappa \in X_i$ ,  $\kappa \in \mathbb{K}$ , is a standard sequence on attribute  $i$  with respect to attribute  $j$ . Because we have supposed that  $\text{Not}[a_{-ij} \sim_{-ij} b_{-ij}]$ , in any additive representation  $\langle v_i \rangle_{i \in N}$  of  $\langle \mathcal{A}, \mathcal{F}, \mathcal{U} \rangle$ , we must have

$$\sum_{k \neq i, j} v_k(a_k) - \sum_{k \neq i, j} v_k(b_k) = \delta \neq 0.$$

Furthermore, (1) implies

$$\begin{aligned} v_i(x_i^\kappa) + v_j(x_j^\kappa) + \sum_{k \neq i, j} v_k(a_k) &= 0, \\ v_i(x_i^{\kappa+1}) + v_j(x_j^\kappa) + \sum_{k \neq i, j} v_k(b_k) &= 0. \end{aligned}$$

This implies that, for all  $\kappa \in \mathbb{K}$ , we have

$$v_i(x_i^{\kappa+1}) - v_i(x_i^\kappa) = \delta \neq 0.$$

Furthermore if the standard sequence is strictly bounded by  $\underline{x}_i$  and  $\bar{x}_i \in X_i$ , it is easy to check that we must have

$$v_i(\underline{x}_i) < v_i(x_i^\kappa) < v_i(\bar{x}_i).$$

This shows that the Archimedean condition is necessary for the existence of an additive representation.

Our main unnecessary assumption is a strong solvability assumption that says that category  $\mathcal{F}$  can always be reached by modifying an evaluation on a single attribute.

**Definition 7** (Unrestricted solvability)

We say that  $\langle \mathcal{A}, \mathcal{F}, \mathcal{U} \rangle$  satisfies unrestricted solvability if, for all  $i \in N$  and all  $x_{-i} \in X_{-i}$ ,  $(x_i, x_{-i}) \in \mathcal{F}$ , for some  $x_i \in X_i$ .

On top of unrestricted solvability, we will also suppose that  $\langle \mathcal{A}, \mathcal{F}, \mathcal{U} \rangle$  is a non-degenerate covering.

Let us note that if  $\langle \mathcal{A}, \mathcal{F}, \mathcal{U} \rangle$  is a non-degenerate covering satisfying unrestricted solvability and that is strongly linear and strongly thin, then, for all  $i \in N$ , there are  $z_i, w_i \in X_i$  such that  $z_i \succ_i w_i$ . Indeed using non-degeneracy, we know that  $x \in \mathcal{A}$ , for some  $x \in X$ . Using unrestricted solvability, we have  $(y_i, x_{-i}) \in \mathcal{F}$ , for some  $y_i \in X_i$ . This implies that  $x_i \sim_i y_i$  is impossible. Therefore, under the above conditions, all attributes are influent for  $\langle \mathcal{A}, \mathcal{F}, \mathcal{U} \rangle$ . Indeed, consider any  $x_i, y_i \in X_i$  such that  $x_i \succ_i y_i$ . Using unrestricted solvability, we find  $a_{-i} \in X_{-i}$  such that  $(x_i, a_{-i}) \in \mathcal{F}$ . Since  $x_i \succ_i y_i$ , Lemma 4 implies  $(y_i, a_{-i}) \in \mathcal{U}$ .

## 3.2 Results

Our main result is the following:

**Proposition 8**

Suppose that  $\langle \mathcal{A}, \mathcal{F}, \mathcal{U} \rangle$  is an ordered covering of a set  $X = X_1 \times X_2 \times \cdots \times X_n$  with  $n \geq 3$ . Suppose that  $\langle \mathcal{A}, \mathcal{F}, \mathcal{U} \rangle$  is non-degenerate and satisfies unrestricted solvability, strong linearity, strong thinness and the Archimedean condition. If  $n = 3$ , suppose furthermore that  $\langle \mathcal{A}, \mathcal{F}, \mathcal{U} \rangle$  satisfies the Thomsen condition. Then there is an additive representation of  $\langle \mathcal{A}, \mathcal{F}, \mathcal{U} \rangle$ .

The uniqueness of the representation is as follows.

**Proposition 9**

Under the conditions of Proposition 8,  $\langle u_i \rangle_{i \in N}$  and  $\langle v_i \rangle_{i \in N}$  are two additive representations, both using the threshold 0 for  $\mathcal{F}$ , of  $\langle \mathcal{A}, \mathcal{F}, \mathcal{U} \rangle$  iff there are real numbers  $\beta_1, \beta_2, \dots, \beta_n, \alpha$  with  $\alpha > 0$  and  $\sum_{i=1}^n \beta_i = 0$  such that for all  $i \in N$  and all  $x_i \in X_i$ ,  $v_i(x_i) = \alpha u_i(x_i) + \beta_i$ .

The proof of the above two propositions appears in the next section. Before that, a few remarks are in order.

1. In the above uniqueness result, we have supposed that the threshold 0 was fixed. This may give the impression that the uniqueness result is stronger than what it really is. If the threshold used for  $\mathcal{F}$  is taken to be variable from one representation to another, it is easy to see that one goes from an additive representation to another one simply by multiplying all functions  $u_i$  by the same positive constant and adding a constant  $\beta_i$  to each of them. If the first representation uses a null threshold, the second one will use a threshold equal to  $\sum_{i=1}^n \beta_i$ .

2. Proposition 8 uses strong linearity and strong thinness. Although this allows to simply grasp the conditions underlying the result, this involves some redundancy. For instance, it is clear that conditions  $\mathcal{A}$ -linear $_I$  and  $\mathcal{A}$ -linear $_{-I}$  are equivalent. We show in Appendix C how to weaken the set of conditions used above.
3. In Appendix E, we show how to extend this result to more than two ordered categories.

### 3.3 Proofs

Take any  $j \in N$ . Define on the set  $\prod_{i \neq j} X_i$  the binary relation  $\succsim^{(j)}$  letting, for all  $x_{-j}, y_{-j} \in X_{-j}$ ,

$$x_{-j} \succsim^{(j)} y_{-j} \Leftrightarrow (a_j, x_{-j}) \in \mathcal{AF} \text{ and } (a_j, y_{-j}) \in \mathcal{FU},$$

for some  $a_j \in X_j$ . We use  $\succ^{(j)}$  and  $\sim^{(j)}$  as is usual. Our proof rests on the following lemma showing that under the conditions of Proposition 8, the relation  $\succsim^{(j)}$  will satisfy the classical conditions ensuring the existence of an additive representation for this relation.

**Lemma 10**

*Let  $\langle \mathcal{A}, \mathcal{F}, \mathcal{U} \rangle$  be an ordered covering on a set  $X$ . Suppose that this covering is strongly linear and strongly thin. Suppose furthermore that unrestricted solvability holds. Let  $j \in N$ . We have:*

1. For all  $x_{-j}, y_{-j} \in X_{-j}$ ,

$$x_{-j} \succsim^{(j)} y_{-j} \Leftrightarrow x_{-j} \succsim_{-j} y_{-j}. \quad (2)$$

2. For all  $x_{-j}, y_{-j} \in X_{-j}$ ,

$$x_{-j} \sim^{(j)} y_{-j} \Leftrightarrow \left\{ \begin{array}{l} (a_j, x_{-j}) \in \mathcal{F} \\ (a_j, y_{-j}) \in \mathcal{F} \end{array} \right\} \text{ for some } a_j \in X_j. \quad (3)$$

3. For all  $x_{-j}, y_{-j} \in X_{-j}$ ,

$$x_{-j} \succ^{(j)} y_{-j} \Leftrightarrow \left\{ \begin{array}{l} (a_j, x_{-j}) \in \mathcal{A} \\ (a_j, y_{-j}) \in \mathcal{F} \end{array} \right\} \text{ for some } a_j \in X_j. \quad (4)$$

4. The binary relation  $\succsim^{(j)}$  is independent, i.e., for all  $i \in N \setminus \{j\}$ , all  $x_i, y_i \in X_i$  and all  $a_{-ij}, b_{-ij} \in X_{-ij}$ ,

$$(x_i, a_{-ij}) \succsim^{(j)} (x_i, b_{-ij}) \Leftrightarrow (y_i, a_{-ij}) \succsim^{(j)} (y_i, b_{-ij}).$$

5. The binary relation  $\succsim^{(j)}$  satisfies unrestricted solvability, i.e., for all  $y_{-j} \in X_{-j}$ , all  $i \in N \setminus \{j\}$  and all  $a_{-ij} \in X_{-ij}$ ,  $(x_i, a_{-ij}) \sim^{(j)} y_{-j}$ , for some  $x_i \in X_i$ .
6. If  $\langle \mathcal{A}, \mathcal{F}, \mathcal{U} \rangle$  is non-degenerate, there is at least one essential attribute for  $\succsim^{(j)}$ , i.e.,  $(x_i, a_{-ij}) \succ^{(j)} (y_i, a_{-ij})$ , for some  $i \in N \setminus \{j\}$ , some  $x_i, y_i \in X_i$  and some  $a_{-ij} \in X_{-ij}$ .
7. If  $\langle \mathcal{A}, \mathcal{F}, \mathcal{U} \rangle$  satisfies the Archimedean condition, then  $\succsim^{(j)}$  satisfies the Archimedean condition. More precisely, let  $\mathbb{K}$  be any set of consecutive integers (positive or negative, finite or infinite). We say that the set  $\{x_i^\kappa \in X_i : \kappa \in \mathbb{K}\}$  is a standard sequence for  $\succsim^{(j)}$  on attribute  $i \in N$  if there are  $a_{-ij}, b_{-ij} \in X_{-ij}$  such that  $(y_i, a_{-ij}) \succ^{(j)} (y_i, b_{-ij})$ , for some  $y_i \in X_i$  and  $(x_i^\kappa, a_{-ij}) \sim^{(j)} (x_i^{\kappa+1}, b_{-ij})$ , for all  $\kappa \in \mathbb{K}$ . This standard sequence is said to be strictly bounded if there are  $\underline{x}_i, \bar{x}_i \in X_i$  such that, for all  $\kappa \in \mathbb{K}$ ,  $(\bar{x}_i, a_{-ij}) \succ^{(j)} (x_i^\kappa, a_{-ij}) \succ^{(j)} (\underline{x}_i, a_{-ij})$ , for all  $a_{-ij} \in X_{-ij}$ . The relation  $\succsim^{(j)}$  is said to satisfy the Archimedean condition, if, for all  $i \in N \setminus \{j\}$ , any standard sequence on attribute  $i$  that is strictly bounded is finite.
8. Suppose that  $n = 3$  and let  $N = \{i, j, k\}$ . If  $\langle \mathcal{A}, \mathcal{F}, \mathcal{U} \rangle$  satisfies the Thomsen condition, then  $\succsim^{(j)}$  satisfies the Thomsen condition, i.e., for all  $i, k \in N \setminus \{j\}$  with  $i \neq k$ , all  $x_i, y_i, z_i \in X_i$  and all  $x_k, y_k, z_k \in X_k$ ,

$$\left. \begin{array}{l} (x_i, x_k) \sim^{(j)} (y_i, y_k) \\ \text{and} \\ (y_i, z_k) \sim^{(j)} (z_i, x_k) \end{array} \right\} \Rightarrow (x_i, z_k) \sim^{(j)} (z_i, y_k).$$

9. If there is an additive representation for  $\succsim^{(j)}$ , then there is an additive representation for  $\langle \mathcal{A}, \mathcal{F}, \mathcal{U} \rangle$ .

#### PROOF

Let us say that  $\langle \mathcal{A}, \mathcal{F}, \mathcal{U} \rangle$  is  $\nu$ - $\mathcal{A}$ -linear if it satisfies  $\mathcal{A}$ -linear $_I$  for all  $I \subseteq N$  such that  $|I| = \nu$ . We use a similar convention for  $\nu$ - $\mathcal{F}$ -linear,  $\nu$ - $\mathcal{AF}$ -linear,  $\nu$ -linear and  $\nu$ -thin.

*Part 1.* Suppose that  $x_{-j} \succsim_{-j} y_{-j}$ , so that, for all  $a_j \in X_j$ ,  $(a_j, y_{-j}) \in \mathcal{A} \Rightarrow (a_j, x_{-j}) \in \mathcal{A}$  and  $(a_j, y_{-j}) \in \mathcal{F} \Rightarrow (a_j, x_{-j}) \in \mathcal{AF}$ . Using unrestricted solvability, we know that  $(b_j, y_{-j}) \in \mathcal{F}$ , for some  $b_j \in X_j$ . Because  $x_{-j} \succsim_{-j} y_{-j}$ , this implies that  $(b_j, x_{-j}) \in \mathcal{AF}$ , so that  $x_{-j} \succsim^{(j)} y_{-j}$ .

Suppose now that  $x_{-j} \succsim^{(j)} y_{-j}$  so that  $(a_j, x_{-j}) \in \mathcal{AF}$  and  $(a_j, y_{-j}) \in \mathcal{FU}$ , for some  $a_j \in X_j$ . Suppose that  $\text{Not}[x_{-j} \succsim_{-j} y_{-j}]$ . Using  $(n-1)$ -linear, we know that  $\succsim_{-j}$  is complete so that we have  $y_{-j} \succ_{-j} x_{-j}$ . Using  $(n-1)$ -linear and  $(n-1)$ -thin,  $y_{-j} \succ_{-j} x_{-j}$  and  $(a_j, x_{-j}) \in \mathcal{AF}$  imply  $(a_j, y_{-j}) \in \mathcal{A}$ , a contradiction.

*Part 2.* The  $\Leftarrow$  part follows from the definition of  $\succsim^{(j)}$ . Let us prove the  $\Rightarrow$  part. Suppose that  $x_{-j} \sim^{(j)} y_{-j}$ , so that for some  $a_j, b_j \in X_j$ ,

$$\begin{aligned} (a_j, x_{-j}) &\in \mathcal{AF} \text{ and } (a_j, y_{-j}) \in \mathcal{FU}, \\ (b_j, y_{-j}) &\in \mathcal{AF} \text{ and } (b_j, x_{-j}) \in \mathcal{FU}. \end{aligned}$$

Using unrestricted solvability on attribute  $j$ , we know that there is a  $c_j \in X_j$  such that  $(c_j, x_{-j}) \in \mathcal{F}$ . If  $(c_j, y_{-j}) \in \mathcal{F}$ , there is nothing to prove. Suppose that  $(c_j, y_{-j}) \in \mathcal{U}$ . Using  $(n-1)$ -linear, this implies that  $x_{-j} \succ_{-j} y_{-j}$ . Using  $(n-1)$ -linear and  $(n-1)$ -thin,  $(b_j, y_{-j}) \in \mathcal{AF}$  and  $x_{-j} \succ_{-j} y_{-j}$  imply  $(b_j, x_{-j}) \in \mathcal{A}$ , a contradiction. Similarly if  $(c_j, y_{-j}) \in \mathcal{A}$ , we obtain  $y_{-j} \succ_{-j} x_{-j}$  so that  $(a_j, x_{-j}) \in \mathcal{AF}$  implies  $(a_j, y_{-j}) \in \mathcal{A}$ , a contradiction.

*Part 3.* Suppose that  $x_{-j} \succ^{(j)} y_{-j}$ . Using unrestricted solvability on attribute  $j$ , we know that  $(a_j, y_{-j}) \in \mathcal{F}$ , for some  $a_j \in X_j$ . We have either  $(a_j, x_{-j}) \in \mathcal{A}$  or  $(a_j, x_{-j}) \in \mathcal{FU}$ . The latter case implies  $y_{-j} \succ^{(j)} x_{-j}$  and is therefore impossible. Therefore, we have  $(a_j, y_{-j}) \in \mathcal{F}$  and  $(a_j, x_{-j}) \in \mathcal{A}$ .

Conversely, suppose that  $(a_j, x_{-j}) \in \mathcal{A}$  and  $(a_j, y_{-j}) \in \mathcal{F}$ , for some  $a_j \in X_j$ . This implies  $x_{-j} \succ^{(j)} y_{-j}$ . Suppose now that  $y_{-j} \succ^{(j)} x_{-j}$ , so that  $x_{-j} \sim^{(j)} y_{-j}$ . Using (3), we have  $(b_j, x_{-j}) \in \mathcal{F}$  and  $(b_j, y_{-j}) \in \mathcal{F}$ , for some  $b_j \in X_j$ . Using  $(n-1)$ -thin, this implies  $x_{-j} \sim_{-j} y_{-j}$ . This contradicts the fact that  $(a_j, x_{-j}) \in \mathcal{A}$  and  $(a_j, y_{-j}) \in \mathcal{F}$ .

*Part 4.* Suppose that, for some  $x_i, y_i \in X_i$  and some  $a_{-ij}, b_{-ij} \in X_{-ij}$ ,  $(x_i, a_{-ij}) \succ^{(j)} (x_i, b_{-ij})$  and  $(y_i, b_{-ij}) \succ^{(j)} (y_i, a_{-ij})$ . Using the definition of  $\succ^{(j)}$ ,  $(x_i, a_{-ij}) \succ^{(j)} (x_i, b_{-ij})$  implies  $(c_j, x_i, a_{-ij}) \in \mathcal{AF}$  and  $(c_j, x_i, b_{-ij}) \in \mathcal{FU}$ , for some  $c_j \in X_j$ . Using (4),  $(y_i, b_{-ij}) \succ^{(j)} (y_i, a_{-ij})$  implies  $(d_j, y_i, a_{-ij}) \in \mathcal{A}$  and  $(d_j, y_i, b_{-ij}) \in \mathcal{F}$ , for some  $d_j \in X_j$ . Using  $(n-2)$ -linear and  $(n-2)$ -thin, this implies  $b_{-ij} \succ_{-ij} a_{-ij}$ . But  $(c_j, x_i, a_{-ij}) \in \mathcal{AF}$  and  $b_{-ij} \succ_{-ij} a_{-ij}$  imply, using  $(n-2)$ -linear and  $(n-2)$ -thin,  $(c_j, x_i, b_{-ij}) \in \mathcal{A}$ , a contradiction.

*Part 5.* Let  $y_{-j} \in X_{-j}$  and  $a_{-ij} \in X_{-ij}$ . We must show that  $y_{-j} \sim^{(j)} (b_i, a_{-ij})$ , for some  $b_i \in X_i$ . Using unrestricted solvability on attribute  $j$ , we have  $(a_j, y_{-j}) \in \mathcal{F}$ , for some  $a_j \in X_j$ . Using unrestricted solvability on attribute  $i$ , we know that  $(a_j, b_i, a_{-ij}) \in \mathcal{F}$ , for some  $b_i \in X_i$ . The conclusion follows from (3).

*Part 6.* Because  $\langle \mathcal{A}, \mathcal{F}, \mathcal{U} \rangle$  is non-degenerate, we know that there are  $a_j \in X_j$ ,  $x_i, y_i \in X_i$  and  $a_{-ij} \in X_{-ij}$  such that  $(a_j, x_i, a_{-ij})$  and  $(a_j, y_i, a_{-ij})$  belong to two distinct categories, so that either

$$\begin{aligned} (a_j, x_i, a_{-ij}) &\in \mathcal{AF} \text{ and } (a_j, y_i, a_{-ij}) \in \mathcal{U} \text{ or} \\ (a_j, x_i, a_{-ij}) &\in \mathcal{A} \text{ and } (a_j, y_i, a_{-ij}) \in \mathcal{FU}. \end{aligned}$$

In either case, we obtain  $(x_i, a_{-ij}) \succ^{(j)} (y_i, a_{-ij})$  and, using 1-linear,  $x_i \succ_i y_i$ . Suppose that  $(x_i, a_{-ij}) \sim^{(j)} (y_i, a_{-ij})$ , so that, using (2),  $(b_j, x_i, a_{-ij}) \in \mathcal{F}$  and  $(b_j, y_i, a_{-ij}) \in \mathcal{F}$ , for some  $b_j \in X_j$ . Using 1-linear, 1-thin and Lemma 4,

$(b_j, y_i, a_{-ij}) \in \mathcal{F}$  and  $x_i \succ_i y_i$  imply  $(b_j, x_i, a_{-ij}) \in \mathcal{A}$ , a contradiction. Hence, we have  $(x_i, a_{-ij}) \succ^{(j)} (y_i, a_{-ij})$ , so that  $i$  is essential for  $\succ^{(j)}$ .

*Part 7.* Let  $i \in N \setminus \{j\}$ . Consider a standard sequence  $\{x_i^\kappa \in X_i : \kappa \in \mathbb{K}\}$  for  $\succ^{(j)}$ . Hence, there are  $a_{-ij}, b_{-ij} \in X_{-ij}$  such that, for some  $c_i \in X_i$ ,  $\text{Not}[(c_i, a_{-ij}) \sim^{(j)} (c_i, b_{-ij})]$  and  $(x_i^\kappa, a_{-ij}) \sim^{(j)} (x_i^{\kappa+1}, b_{-ij})$ , for all  $\kappa \in \mathbb{K}$ . Using  $(n-1)$ -linear and  $(n-1)$ -thin, we know that that  $\text{Not}[a_{-ij} \sim_{-ij} b_{-ij}]$ . Furthermore, we have

$$\begin{aligned} (x_i^\kappa, x_j^\kappa, a_{-ij}) &\in \mathcal{F}, \\ (x_i^{\kappa+1}, x_j^\kappa, b_{-ij}) &\in \mathcal{F}, \end{aligned}$$

for some  $x_j^\kappa \in X_j$ ,  $\kappa \in \mathbb{K}$ . Hence,  $\{x_i^\kappa \in X_i : \kappa \in \mathbb{K}\}$  is a standard sequence for  $\langle \mathcal{A}, \mathcal{F}, \mathcal{U} \rangle$ .

Suppose that there are  $\underline{x}_i, \bar{x}_i \in X_i$  such that, for all  $\kappa \in \mathbb{K}$ ,  $\bar{x}_i \succ_i^{(j)} x_i^\kappa \succ_i^{(j)} \underline{x}_i$ , where  $\succ_i^{(j)}$  is the marginal relation induced by  $\succ^{(j)}$  on  $X_i$ . Using (2), this clearly implies  $\bar{x}_i \succ_i x_i^\kappa \succ_i \underline{x}_i$ , so that the standard sequence for  $\langle \mathcal{A}, \mathcal{F}, \mathcal{U} \rangle$  is bounded. Using the Archimedean condition for  $\langle \mathcal{A}, \mathcal{F}, \mathcal{U} \rangle$ , we know that this sequence must be finite.

*Part 8.* Suppose that  $n = 3$  and let  $N = \{i, j, k\}$ . Suppose that  $(x_i, x_k) \sim^{(j)} (y_i, y_k)$  and  $(y_i, z_k) \sim^{(j)} (z_i, x_k)$ . Using Part 4, we know that  $(x_i, x_k, a_j) \in \mathcal{F}$  and  $(y_i, z_k, b_j) \in \mathcal{F}$ , for some  $a_j, b_j \in X_j$ . Using Part 1, we have  $(x_i, x_k) \sim_{ik} (y_i, y_k)$  and  $(y_i, z_k) \sim_{ik} (z_i, x_k)$ . Using Thomsen, we therefore obtain  $(x_i, z_k) \sim_{ik} (z_i, y_k)$ . The conclusion follows from Part 1.

*Part 9.* Suppose that  $\langle u_i \rangle_{i \neq j}$  is an additive representation of  $\succ^{(j)}$ . Let  $x_j \in X_j$ . Using unrestricted solvability on any attribute  $i$  other than  $j$ , we can always find a  $a_{-j} \in X_{-j}$  such that  $(x_j, a_{-j}) \in \mathcal{F}$ . Now, define  $u_j$  letting, for all  $x_j \in X_j$ ,

$$u_j(x_j) = - \sum_{i \neq j} u_i(a_i) \text{ if } (x_j, a_{-j}) \in \mathcal{F}.$$

It is easy to see that  $u_j$  is well-defined. Indeed if  $(x_j, a_{-j}) \in \mathcal{F}$  and  $(x_j, b_{-j}) \in \mathcal{F}$ , (3) implies  $a_{-j} \sim^{(j)} b_{-j}$ , so that:

$$\sum_{i \neq j} u_i(a_i) = \sum_{i \neq j} u_i(b_i).$$

Let us now show that such a function  $u_j$  together with the functions  $\langle u_i \rangle_{i \neq j}$  give an additive representation for  $\langle \mathcal{A}, \mathcal{F}, \mathcal{U} \rangle$ .

If  $(x_j, x_{-j}) \in \mathcal{F}$ , then, by construction, we have  $u_j(x_j) + \sum_{i \neq j} u_i(x_i) = 0$ .

Suppose that  $(x_j, x_{-j}) \in \mathcal{A}$ . Using unrestricted solvability on any attribute other than  $j$ , we know that  $(x_j, a_{-j}) \in \mathcal{F}$ , for some  $a_{-j} \in X_{-j}$ . Hence, we have:

$$u_j(x_j) = - \sum_{i \neq j} u_i(a_i).$$

Using (4),  $(x_j, x_{-j}) \in \mathcal{A}$  and  $(x_j, a_{-j}) \in \mathcal{F}$  imply  $x_{-j} \succ^{(j)} a_{-j}$ , so that:

$$\sum_{i \neq j} u_i(x_i) > \sum_{i \neq j} u_i(a_i),$$

which implies

$$u_j(x_j) + \sum_{i \neq j} u_i(x_i) > 0.$$

That  $(x_j, x_{-j}) \in \mathcal{U}$  implies  $u_j(x_j) + \sum_{i \neq j} u_i(x_i) < 0$  is shown similarly. Hence we have built an additive representation of  $\langle \mathcal{A}, \mathcal{F}, \mathcal{U} \rangle$ .  $\square$

#### PROOF OF PROPOSITION 8

Using Parts 1–8 of Lemma 8, we know that  $\succ^{(j)}$  is an independent weak order satisfying unrestricted solvability and the Archimedean condition. Furthermore, we know that there is at least one essential attribute for  $\succ^{(j)}$  and that, if  $n = 3$ ,  $\succ^{(j)}$  satisfies the Thomsen condition. We can therefore use the classical theorems of conjoint measurement (see Krantz et al., 1971, Ch. 6)<sup>1</sup> to obtain an additive representation for  $\succ^{(j)}$ . The conclusion follows from Part 9 of Lemma 8.  $\square$

#### PROOF OF PROPOSITION 9

It is first clear that if  $\langle u_i \rangle_{i \in N}$  is an additive representation using the threshold 0 for  $\mathcal{F}$  of  $\langle \mathcal{A}, \mathcal{F}, \mathcal{U} \rangle$ , then  $\langle \alpha u_i + \beta_i \rangle_{i \in N}$  with  $\alpha > 0$  and  $\sum_{i=1}^n \beta_i = 0$  will also be an additive representation with the same threshold.

Let  $\langle u_i \rangle_{i \in N}$  be any additive representation of  $\langle \mathcal{A}, \mathcal{F}, \mathcal{U} \rangle$ . Let us show that  $\langle u_i \rangle_{i \neq j}$  must be an additive representation of  $\succ^{(j)}$ . Suppose that  $x_{-j} \sim^{(j)} y_{-j}$ . Using Part 2 of Lemma 10, we must have  $\sum_{i \neq j} u_i(x_i) = \sum_{i \neq j} u_i(y_i)$ . Similarly, using Part 3 of Lemma 10,  $x_{-j} \succ^{(j)} y_{-j}$  implies  $\sum_{i \neq j} u_i(x_i) > \sum_{i \neq j} u_i(y_i)$ . Hence, any additive representation of  $\langle \mathcal{A}, \mathcal{F}, \mathcal{U} \rangle$  must also be an additive representation  $\succ^{(j)}$ . Conversely, the proof of Part 9 of Lemma 8 has shown that, given any additive representation for  $\succ^{(j)}$ , we can obtain an additive representation for  $\langle \mathcal{A}, \mathcal{F}, \mathcal{U} \rangle$  that uses the same functions for  $i \neq j$ .

Because  $\succ^{(j)}$  satisfies all conditions the classical theorems of conjoint measurement (see Krantz et al., 1971, Ch. 6), we know that any two additive representations  $\langle u_i \rangle_{i \neq j}$  and  $\langle v_i \rangle_{i \neq j}$  must be such that

$$v_i(x_i) = \alpha u_i(x_i) + \beta_i.$$

with  $\alpha > 0$ .

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<sup>1</sup>More precisely, we make use of variants of Krantz et al. (1971, Theorem 6.2, page 257) (when  $n = 3$ ) and of Krantz et al. (1971, Theorem 6.13, page 302) (when  $n \geq 4$ ) in which restricted solvability is replaced by unrestricted solvability. In this case, if there is one essential attribute, then all attributes are essential.

Using unrestricted solvability on any attribute distinct from  $j$ , for all  $x_j \in X_j$ , we have  $(x_j, y_{-j}) \in \mathcal{F}$ , for some  $y_{-j} \in X_{-j}$ . This implies that if  $\langle u_i \rangle_{i \in N}$  and  $\langle v_i \rangle_{i \in N}$  are two representations of  $\langle \mathcal{A}, \mathcal{F}, \mathcal{U} \rangle$ , for all  $x_j \in X_j$ , we have

$$u_j(x_j) = - \sum_{i \neq j} u_i(y_i),$$

$$v_j(x_j) = - \sum_{i \neq j} v_i(y_i) = - \sum_{i \neq j} [\alpha u_i(y_i) + \beta_i],$$

where  $y_{-j} \in X_{-j}$  is such that  $(x_j, y_{-j}) \in \mathcal{F}$ . Therefore, we obtain  $v_j = \alpha u_j - \sum_{i \neq j} \beta_i$ . Hence, the two sets of functions will be such that, for all  $i \in N$ ,  $v_i = \alpha u_i + \beta_i$  with  $\alpha > 0$  and  $\sum_{i=1}^n \beta_i = 0$ .  $\square$

## 4 Discussion

This paper has shown how to adapt classical results of conjoint measurement giving conditions guaranteeing the existence of additive representations of binary relations to the case of ordered partitions. Our results are much simpler than the ones proposed in Vind (1991). This simplicity is mainly due to our use of unrestricted solvability. It cannot be overemphasized that this is a very strong hypothesis that forces all functions  $v_i$  used in a representation in model (A) to be unbounded. In spite of this very strong limitation, we have been able to deal with the  $n = 3$  case and our hypotheses do not exclude the case of equally-spaces structures that are clearly ruled out by the topological assumptions used in Vind (1991). Compared to the results in Vind (1991), we use no topological assumptions, which forces us to introduce conditions on the alternatives lying in  $\mathcal{F}$  at the frontier between categories.

The use of unrestricted solvability allows to keep things simple and to underline the logic of the construction thereby making our results simple corollaries of classical results. It is important to note that the approach taken here vitally depends on this hypothesis. Without it the binary relation  $\succsim^{(j)}$  would not be a weak order on the whole set  $\prod_{i \neq j} X_i$ , because it may well happen that  $Not[x_j \succsim^{(j)} y_j]$  and  $Not[y_j \succsim^{(j)} x_j]$ . This clearly invalidates the approach taken here. We investigate in Bouyssou and Marchant (2008c) another approach that uses the results in Chateauneuf and Wakker (1993) allowing to build additive representations of weak orders defined on subsets of product sets.

The analysis in this paper has underlined the importance of a small number of conditions (mainly linearity and thinness) for the existence of additive representations. This clearly calls now for empirical studies of their reasonableness.

# Appendices

## A Two attributes<sup>2</sup>

### A.1 Results

When there are only two attributes, it is well-known that the analysis of the additive value function model becomes more difficult than when there are more three attributes. However, in our setting, this case will turn out to be quite simple (in related contexts, this was already observed in Bouyssou, 1986, sect. 4 and Fishburn, 1991, sect. 5).

When  $X = X_1 \times X_2$  and  $\mathcal{F}$  is empty, necessary and sufficient conditions on  $\langle \mathcal{A}^*, \mathcal{U}^* \rangle$  to have a representation in model (A) can immediately be inferred from the results on biorders in Ducamp and Falmagne (1969), Doignon et al. (1984) and Doignon et al. (1987). In this case, condition  $\mathcal{A}$ -linear<sub>1</sub> (that is, when  $n = 2$ , clearly equivalent to  $\mathcal{A}$ -linear<sub>2</sub>) is necessary and sufficient for model (A) when  $X$  is finite or countably infinite (this was already noted, for finite sets, in Fishburn et al. (1991, Th. 3, p. 153) where  $\mathcal{A}$ -linear<sub>1</sub> is called the “no bad rectangle condition”). In the general case, it is straightforward to reformulate the order-density introduced in Doignon et al. (1984) to our setting. We leave details to the interested reader.

As shown below, these results are easily extended to cope with the possible existence of elements in  $\mathcal{F}$ . We have:

#### Proposition 11

Let  $\langle \mathcal{A}, \mathcal{F}, \mathcal{U} \rangle$  be an ordered covering on a finite or countably infinite set  $X = X_1 \times X_2$ . There are real-valued functions  $u_1$  on  $X_1$  and  $u_2$  on  $X_2$  such that, for all  $x \in X$ ,

$$\begin{aligned} x \in \mathcal{A} &\Leftrightarrow v_1(x_1) + v_2(x_2) > 0, \\ x \in \mathcal{F} &\Leftrightarrow v_1(x_1) + v_2(x_2) = 0, \end{aligned} \tag{5}$$

iff  $\langle \mathcal{A}, \mathcal{F}, \mathcal{U} \rangle$  satisfies  $\mathcal{A}$ -linear<sub>1</sub>,  $\mathcal{AF}$ -linear<sub>1</sub>,  $\mathcal{F}$ -linear<sub>1</sub> and  $\text{thin}_1$ . Furthermore, the functions  $v_1$  and  $v_2$  may always be chosen in such a way that for all  $x_1, y_1 \in X_1$  and  $x_2, y_2 \in X_2$ ,

$$\begin{aligned} x_1 \succsim_1 y_1 &\Leftrightarrow v_1(x_1) \geq v_1(y_1), \\ x_2 \succsim_2 y_2 &\Leftrightarrow v_2(x_2) \geq v_2(y_2). \end{aligned} \tag{6}$$

Necessity is clear. The proof of sufficiency is given in the next section.

#### Remark 12

It is clear that, when there only two attributes,  $\mathcal{A}$ -linear<sub>1</sub> is equivalent to  $\mathcal{A}$ -linear<sub>2</sub> and  $\mathcal{F}$ -linear<sub>1</sub> is equivalent to  $\mathcal{F}$ -linear<sub>2</sub>. Let us show that under the conditions

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<sup>2</sup> This section is a much abridged version of Bouyssou and Marchant (2008b).

of the above proposition, condition  $\mathcal{AF}$ -linear<sub>2</sub> holds, so that  $\succsim_2$  will be a weak order.

Suppose indeed that  $(x_1, x_2) \in \mathcal{A}$  and  $(y_1, y_2) \in \mathcal{F}$ . Suppose in violation with  $\mathcal{AF}$ -linear<sub>2</sub> that  $(x_1, y_2) \in \mathcal{FU}$  and  $(y_1, x_2) \in \mathcal{U}$ . Condition  $\mathcal{AF}$ -linear<sub>1</sub> implies either  $(y_1, x_2) \in \mathcal{A}$  or  $(x_1, y_2) \in \mathcal{AF}$ . Hence, we must have  $(x_1, y_2) \in \mathcal{F}$ . Since  $(y_1, y_2) \in \mathcal{F}$ ,  $(x_1, x_2) \in \mathcal{A}$  and  $(y_1, x_2) \notin \mathcal{A}$  violate thin<sub>1</sub>.

Similarly, let us show that under the conditions of the above proposition, condition thin<sub>2</sub> holds. Suppose that we have  $(x_1, x_2) \in \mathcal{F}$  and  $(x_1, y_2) \in \mathcal{F}$ , for some  $x_1 \in X_1$  and some  $x_2, y_2 \in X_2$ . We must show that  $(y_1, x_2)$  and  $(y_1, y_2)$  must belong to the same category, for all  $y_1 \in X_1$ . Suppose first that  $(y_1, x_2) \in \mathcal{A}$  and  $(y_1, y_2) \in \mathcal{F}$ . Since  $(x_1, y_2) \in \mathcal{F}$  and  $(y_1, y_2) \in \mathcal{F}$  the fact that  $(x_1, x_2) \in \mathcal{F}$  and  $(y_1, x_2) \in \mathcal{A}$  violates thin<sub>1</sub>. Suppose now that  $(y_1, x_2) \in \mathcal{F}$  and  $(y_1, y_2) \in \mathcal{U}$ . Since  $(x_1, x_2) \in \mathcal{F}$  and  $(y_1, x_2) \in \mathcal{F}$ ,  $(x_1, y_2) \in \mathcal{F}$  and  $(y_1, y_2) \in \mathcal{U}$  violates thin<sub>1</sub>. Suppose finally that  $(y_1, x_2) \in \mathcal{A}$  and  $(y_1, y_2) \in \mathcal{U}$ . Using  $\mathcal{AF}$ -linear<sub>1</sub>,  $(y_1, x_2) \in \mathcal{A}$  and  $(x_1, y_2) \in \mathcal{F}$  implies either  $(x_1, x_2) \in \mathcal{A}$  or  $(y_1, y_2) \in \mathcal{AF}$ , a contradiction.  $\bullet$

### Remark 13

It is not difficult to see that in Proposition 11, it is possible to replace the conjunction of  $\mathcal{AF}$ -linear<sub>1</sub>,  $\mathcal{F}$ -linear<sub>1</sub> and thin<sub>1</sub> by the requirement that

$$\left. \begin{array}{l} (x_1, a_{-1}) \in \mathcal{AF} \\ \text{and} \\ (y_1, b_{-1}) \in \mathcal{AF} \end{array} \right\} \Rightarrow \left\{ \begin{array}{l} (y_1, a_{-1}) \in \mathcal{AF} \\ \text{or} \\ (x_1, b_{-1}) \in \mathcal{AF} \end{array} \right.$$

for all  $x_1, y_1 \in X_1$  and  $a_{-1}, b_{-1} \in X_{-1}$ , together with thin<sub>1</sub> and thin<sub>2</sub>. This makes the result somewhat simpler. The present version of Proposition 11 nevertheless allows an easier comparison the conditions used in Proposition 8.  $\bullet$

The following examples show that the conditions used in Proposition 11 are independent. In all these examples, we have  $X = X_1 \times X_2$ .

### Example 14

Let  $X_1 = \{x_1, y_1\}$  and  $X_2 = \{x_2, y_2\}$ . Define  $\langle \mathcal{A}, \mathcal{F}, \mathcal{U} \rangle$  letting  $(x_1, x_2) \in \mathcal{A}$ ,  $(y_1, y_2) \in \mathcal{A}$ ,  $(x_1, y_2) \in \mathcal{U}$ ,  $(y_1, x_2) \in \mathcal{U}$ . It is clear that  $\mathcal{A}$ -linear<sub>1</sub> is violated. Conditions  $\mathcal{AF}$ -linear<sub>1</sub>,  $\mathcal{F}$ -linear<sub>1</sub> and thin<sub>1</sub> are trivially satisfied.  $\diamond$

### Example 15

Let  $X_1 = \{x_1, y_1, z_1\}$  and  $X_2 = \{x_2, y_2, z_2\}$ . Define  $\langle \mathcal{A}, \mathcal{F}, \mathcal{U} \rangle$  letting  $(z_1, z_2) \in \mathcal{A}$ ,  $(x_1, z_2) \in \mathcal{A}$ ,  $(y_1, z_2) \in \mathcal{A}$ ,  $(x_1, x_2) \in \mathcal{F}$ ,  $(y_1, y_2) \in \mathcal{F}$ ,  $(x_1, y_2) \in \mathcal{U}$ ,  $(y_1, x_2) \in \mathcal{U}$ ,  $(z_1, x_2) \in \mathcal{U}$ ,  $(z_1, y_2) \in \mathcal{U}$ . It is easy to check that conditions  $\mathcal{A}$ -linear<sub>1</sub>,  $\mathcal{AF}$ -linear<sub>1</sub> and thin<sub>1</sub> hold. Condition  $\mathcal{F}$ -linear<sub>1</sub> is violated since  $(x_1, x_2) \in \mathcal{F}$ ,  $(y_1, y_2) \in \mathcal{F}$ ,  $(x_1, y_2) \in \mathcal{U}$  and  $(y_1, x_2) \in \mathcal{U}$ .  $\diamond$

**Example 16**

Let  $X_1 = \{x_1, y_1\}$  and  $X_2 = \{x_2, y_2\}$ . Define  $\langle \mathcal{A}, \mathcal{F}, \mathcal{U} \rangle$  letting  $(x_1, y_2) \in \mathcal{F}$ ,  $(y_1, y_2) \in \mathcal{F}$ ,  $(x_1, x_2) \in \mathcal{A}$ ,  $(y_1, x_2) \in \mathcal{U}$ . It is clear that conditions  $\mathcal{A}$ -linear<sub>1</sub>,  $\mathcal{AF}$ -linear<sub>1</sub> and  $\mathcal{F}$ -linear<sub>1</sub> hold. Condition thin<sub>1</sub> is violated.  $\diamond$

**Example 17**

Let  $X_1 = \{x_1, y_1\}$  and  $X_2 = \{x_2, y_2\}$ . Define  $\langle \mathcal{A}, \mathcal{F}, \mathcal{U} \rangle$  letting  $(x_1, x_2) \in \mathcal{A}$ ,  $(y_1, y_2) \in \mathcal{F}$ ,  $(y_1, x_2) \in \mathcal{U}$ ,  $(x_1, y_2) \in \mathcal{U}$ . It is clear that conditions  $\mathcal{A}$ -linear<sub>1</sub>,  $\mathcal{F}$ -linear<sub>1</sub> and thin<sub>1</sub> hold. Condition  $\mathcal{AF}$ -linear<sub>1</sub> is violated.  $\diamond$

Extending Proposition 11 implies the introduction of an order-denseness condition. We say that the subset  $Y_1 \subseteq X_1$  is dense for the covering  $\langle \mathcal{A}, \mathcal{F}, \mathcal{U} \rangle$  if, for all  $(x_1, x_2) \in X$ ,

$$\begin{aligned} (x_1, x_2) \in \mathcal{A} &\Rightarrow [x_1 \succsim_1 x_1^* \text{ and } (x_1^*, x_2) \in \mathcal{A}], \\ &\text{and} \\ (x_1, x_2) \in \mathcal{U} &\Rightarrow [x_1^* \succsim_1 x_1 \text{ and } (x_1^*, x_2) \in \mathcal{U}], \end{aligned} \tag{7}$$

for some  $x_1^* \in Y_1$ . As detailed below, the asymmetry between  $X_1$  and  $X_2$  in the statement of the order-denseness condition is only due to simplicity considerations. We have:

**Proposition 18**

*Let  $\langle \mathcal{A}, \mathcal{F}, \mathcal{U} \rangle$  be an ordered covering on a set  $X = X_1 \times X_2$ . There are real-valued functions  $u_1$  on  $X_1$  and  $u_2$  on  $X_2$  such that (5) holds iff  $\langle \mathcal{A}, \mathcal{F}, \mathcal{U} \rangle$  satisfies  $\mathcal{A}$ -linear<sub>1</sub>,  $\mathcal{AF}$ -linear<sub>1</sub>,  $\mathcal{F}$ -linear<sub>1</sub>, thin<sub>1</sub> and there is a finite or countably infinite subset  $Y_1 \subseteq X_1$  that is dense for  $\langle \mathcal{A}, \mathcal{F}, \mathcal{U} \rangle$ . Furthermore, the functions  $u_1$  and  $u_2$  can always be chosen in such a way that (6) holds.*

The proof appears in the next section.

**A.2 Proofs**

We prove Propositions 11 and 18 using a simple extension of biorders that may have an independent interest.

Let  $A = \{a, b, \dots\}$  and  $Z = \{p, q, \dots\}$  be two sets. We suppose throughout that  $A \cap Z = \emptyset$ . This is without loss of generality, since we can always build a disjoint duplication of  $A$  and  $Z$  (as in Doignon et al., 1984, Definition 4, p. 79). Following Doignon et al. (1984, 1987) define a binary relation between  $A$  and  $Z$  to be a subset of  $A \times Z$ . We often write  $a T p$  instead of  $(a, p) \in T$ . Define the trace of  $T$  on  $A$  as the binary relation  $T^A$  on  $A$  defined letting, for all  $a, b \in A$ ,

$$a T^A b \Leftrightarrow [b T p \Rightarrow a T p, \text{ for all } p \in Z].$$

Similarly, define the trace of  $T$  on  $Z$  as the binary relation  $T^Z$  on  $Z$  defined letting, for all  $p, q \in Z$ ,

$$p T^Z q \Leftrightarrow [a T p \Rightarrow a T q, \text{ for all } a \in A].$$

It is clear that the relations  $T^A$  and  $T^Z$  are always reflexive and transitive.

A binary relation  $T$  between  $A$  and  $Z$  is said to be a *biorder* if it has the Ferrers property, i.e., for all  $a, b \in A$  and all  $p, q \in Z$ , we have:

$$\left. \begin{array}{l} a T p \\ \text{and} \\ b T q \end{array} \right\} \Rightarrow \left\{ \begin{array}{l} a T q \\ \text{or} \\ b T p \end{array} \right.$$

It is not difficult to see that  $T$  has the Ferrers property iff  $T^A$  is complete iff  $T^Z$  is complete. When  $A$  and  $Z$  are at most countably infinite, Doignon et al. (1984) have shown that being a biorder is a necessary and sufficient condition for the existence of a real-valued function  $f$  on  $A$  and a real-valued function  $g$  on  $Z$  such that, for all  $a \in A$  and  $p \in Z$ ,

$$a T p \Leftrightarrow f(a) > g(p),$$

or, equivalently,

$$a T p \Leftrightarrow f(a) \geq g(p).$$

Consider now two disjoint relations  $T$  and  $I$  between the sets  $A$  and  $Z$ . We investigate below the conditions on  $T$  and  $I$  such that there are a real-valued function  $f$  on  $A$  and a real-valued function  $g$  on  $Z$  satisfying, for all  $a \in A$  and  $p \in Z$ ,

$$a T p \Leftrightarrow f(a) > g(p), \tag{8}$$

$$a I p \Leftrightarrow f(a) = g(p). \tag{9}$$

The above model constitutes a simple generalization of biorders. Apparently it has never been studied in the literature. The analysis below closely follows Doignon et al. (1984). We denote by  $R$  the relation between  $A$  and  $Z$  equal to  $T \cup I$  and  $U$  the relation between  $A$  and  $Z$  such that  $a U p \Leftrightarrow \text{Not}[a R p]$ . As above, let  $T^A$  (resp.  $T^Z$ ) be the trace of  $T$  on  $A$  (resp. on  $Z$ ). Similarly, let  $R^A$  (resp.  $R^Z$ ) be the trace of  $R$  on  $A$  (resp. on  $Z$ ). Define  $\succsim^A = T^A \cap R^A$  and  $\succsim^Z = T^Z \cap R^Z$ . It is clear that  $T^A$ ,  $R^A$ ,  $\succsim^A$ ,  $T^Z$ ,  $R^Z$ ,  $\succsim^Z$  are always reflexive and transitive. We know that  $T^A$  is complete iff  $T^Z$  is complete iff  $T$  is a biorder. Similarly,  $R^A$  is complete iff  $R^Z$  is complete iff  $R$  is a biorder.

It is easy to devise a number of necessary conditions on the disjoint relations  $T$  and  $I$  for the existence of our representation. In view of our analysis of biorders, it

is clear that both  $T$  and  $R$  must be biorders. Two other conditions are necessary for our representation: they capture the fact that the relation  $I$  is “thin”. Suppose that  $a I p$  and  $b I p$ . this implies  $f(a) = g(p)$  and  $f(b) = g(p)$ , so that  $f(a) = f(b)$ . Hence, for all  $q \in Z$ , we have  $a I q \Leftrightarrow b I q$  and  $a T q \Leftrightarrow b T q$ .

We say that thinness holds on  $A$  if

$$\left. \begin{array}{l} a I p \\ \text{and} \\ b I p \end{array} \right\} \Rightarrow \left\{ \begin{array}{l} a I q \Leftrightarrow b I q \\ \text{and} \\ a T q \Leftrightarrow b T q \end{array} \right.$$

for all  $a, b \in A$  and  $p, q \in Z$ . Similarly, we say thinness holds on  $Z$  if

$$\left. \begin{array}{l} a I p \\ \text{and} \\ a I q \end{array} \right\} \Rightarrow \left\{ \begin{array}{l} b I p \Leftrightarrow b I q \\ \text{and} \\ b T p \Leftrightarrow b T q \end{array} \right.$$

for all  $a, b \in A$  and  $p, q \in Z$ .

Some of the consequences of these conditions are collected below.

**Lemma 19**

1. *If two disjoint relations  $T$  and  $I$  between the sets  $A$  and  $Z$  have a representation (8–9), then  $T$  is a biorder,  $R$  is a biorder, and thinness holds on both  $A$  and  $Z$ .*
2. *For a pair of disjoint relations, the following four conditions are independent:  $T$  is a biorder,  $R$  is a biorder, thinness holds on  $A$ , thinness holds on  $Z$ .*
3. *If two disjoint relations  $T$  and  $I$  between  $A$  and  $Z$  are such that  $T$  and  $R$  are biorders and thinness holds on both  $A$  and  $Z$ , then the relations  $\succsim^A$  on  $A$  and  $\succsim^Z$  on  $Z$  are both complete.*
4. *Under the conditions of Part 3, we have:*

$$[a I p \text{ and } b \succ^A a] \Rightarrow b T p, \tag{10a}$$

$$[a I p \text{ and } p \succ^Z q] \Rightarrow a T q, \tag{10b}$$

$$[a I p \text{ and } a \succ^A c] \Rightarrow c U p, \tag{10c}$$

$$[a I p \text{ and } r \succ^Z p] \Rightarrow a U r, \tag{10d}$$

for all  $a, b, c \in A$  and  $p, q, r \in Z$

PROOF

Part 1 is obvious. The proof of Part 2 consists in exhibiting the required four examples.

**Example 20**

Let  $A = \{a, b\}$  and  $Z = \{p, q\}$ . Define  $T$  and  $I$  letting  $a T p$ ,  $b T q$ ,  $a I q$  and  $b I p$ . It is clear that  $T$  is not a biorder whereas  $R$  is. The two thinness conditions are trivially satisfied.  $\diamond$

**Example 21**

Let  $A = \{a, b, c\}$  and  $Z = \{p, q, r\}$ . Define  $T$  and  $I$  letting  $c T r$ ,  $a I p$  and  $b I q$ . It is clear that  $T$  is a biorder whereas  $R$  is not. The two thinness conditions are trivially satisfied.  $\diamond$

**Example 22**

Let  $A = \{a, b\}$  and  $Z = \{p, q\}$ . Define  $T$  and  $I$  letting  $b T p$ ,  $a I p$  and  $a I q$ . It is clear that both  $T$  and  $R$  are biorders. Thinness on  $A$  is trivially satisfied whereas thinness on  $Z$  is violated because  $a I p$ ,  $a I q$ ,  $b T p$  and  $\text{Not}[b T q]$ .  $\diamond$

**Example 23**

Let  $A = \{a, b\}$  and  $Z = \{p, q\}$ . Define  $T$  and  $I$  letting  $a T q$ ,  $a I p$  and  $b I p$ . It is clear that both  $T$  and  $R$  are biorders. Thinness on  $Z$  is trivially satisfied whereas thinness on  $A$  is violated because  $a I p$ ,  $b I p$ ,  $a T q$  and  $\text{Not}[b T q]$ .  $\diamond$

Part 3. Suppose that  $\succsim^A$  is not complete. Hence, for some  $a, b \in A$  and some  $p, q \in Z$ , we have

$$b T p \text{ and } \text{Not}[a T p], \text{ for some } p \in Z, \quad (11a)$$

or

$$b R p \text{ and } \text{Not}[a R p], \text{ for some } p \in Z, \quad (11b)$$

and

$$a T q \text{ and } \text{Not}[b T q], \text{ for some } q \in Z, \quad (11c)$$

or

$$a R q \text{ and } \text{Not}[b R q], \text{ for some } q \in Z, \quad (11d)$$

The combination of (11a) and (11c) violates the fact that  $T$  is a biorder. Similarly, the combination of (11b) and (11d) violates the fact that  $R$  is a biorder.

The combination of conditions (11a) and (11d) says that  $a R q$ ,  $b T p$ ,  $\text{Not}[a T p]$  and  $\text{Not}[b R q]$ . Notice that  $\text{Not}[a T p]$  implies either  $\text{Not}[a R p]$  or  $a I p$ . If  $\text{Not}[a R p]$ , since we know that  $a R q$ ,  $b R p$  and  $\text{Not}[b R q]$ , we have a violation of the fact that  $R$  is a biorder. Hence, we must have  $a I p$ . We know that  $a R q$  implies either  $a T q$  or  $a I q$ . Suppose that  $a T q$ . Since  $b T p$ , we obtain, using the fact that  $T$  is a biorder  $a T p$  or  $b T q$ , a contradiction. Therefore, we must have  $a I q$ . Using thinness on  $Z$ ,  $a I p$ ,  $a I q$  and  $b T p$  implies  $b T q$ , a contradiction. The proof for  $\succsim^Z$  is similar.

Part 4. Suppose that  $a I p$  and  $b \succ^A a$ . Since  $a I b$  implies  $a R p$  and  $b \succ^A a$  implies  $b \succsim^A a$ , we know that  $b R p$ . Suppose that  $b I p$ . Using thinness on  $A$ , it is easy to see that  $a I p$  and  $b I p$  imply  $b \sim^A a$ , a contradiction. Hence, we must have  $b T p$ , as required by (10a).

Suppose now that  $a I p$  and  $a \succ^A b$  and  $b R p$ . If  $b I p$ ,  $a I p$  and thinness on  $A$  imply  $a \sim^A b$ , a contradiction. Hence, we must have  $b T p$  and  $a \succsim^A b$  implies  $b T p$ , a contradiction. This shows that (10b) holds. The proofs of (10c) and (10d) with  $\succ^Z$  are similar.  $\square$

The above lemma gives all what is necessary to obtain the desired numerical representation on at most countable sets. We have:

**Proposition 24**

*Let  $A$  and  $Z$  be finite or countably infinite sets and let  $T$  and  $I$  be a pair of disjoint relations between  $A$  and  $Z$ . There are real valued functions  $f$  on  $A$  and  $g$  on  $Z$  such that (8) and (9) hold if and only if  $T$  is a biorder,  $R = T \cup I$  is a biorder and thinness holds on  $A$  and  $Z$ .*

*Furthermore, the functions  $f$  and  $g$  can always be chosen in such a way that, for all  $a, b \in A$  and  $p, q \in Z$ ,*

$$\begin{aligned} a \succsim^A b &\Leftrightarrow f(a) \geq f(b), \\ p \succsim^Z q &\Leftrightarrow g(p) \geq g(q). \end{aligned} \tag{12}$$

PROOF

Necessity results from Part 1 of Lemma 19. We show sufficiency. As explained above, we suppose w.l.o.g. that  $A$  and  $Z$  are disjoint.

We consider the relation  $Q$  on  $A \cup Z$  defined letting for all  $\alpha, \beta \in A \cup Z$ :

$$\alpha Q \beta \text{ iff } \begin{cases} \alpha, \beta \in A \text{ and } \alpha \succsim^A \beta, \\ \alpha, \beta \in Z \text{ and } \alpha \succsim^Z \beta, \\ \alpha \in A, \beta \in Z \text{ and } \alpha R \beta, \\ \alpha \in Z, \beta \in A \text{ and } \text{Not}[\beta T \alpha]. \end{cases} \tag{13}$$

The proof will be complete if we show our conditions imply that  $Q$  is a weak order. Indeed, because  $A$  and  $Z$  are both countable there will be a real-valued function  $h$  on  $A \cup Z$  such that, for all  $\alpha, \beta \in A \cup Z$ ,

$$\alpha Q \beta \Leftrightarrow h(\alpha) \geq h(\beta).$$

Suppose that, for some,  $a \in A$  and  $p \in Z$ , we have  $a T p$ . This implies  $a Q p$  and  $\text{Not}[p Q a]$  so that  $h(a) > h(p)$ . Similarly  $a I p$  implies both of  $a Q p$  and  $p Q a$ , so that  $h(a) = h(p)$ . If  $\text{Not}[a R b]$  we have  $\text{Not}[a Q p]$  and  $p Q a$ , so that  $h(a) < h(p)$ . Therefore defining  $f$  (resp.  $g$ ) to be the restriction of  $h$  on  $A$  (resp.

$Z$ ) leads to a representation satisfying (8) and (9). In view of the definition of  $Q$ , it is clear that (12) will hold.

Using Part 3 of Lemma 19, we know that  $\succsim^A$  on  $A$  is complete and that  $\succsim^Z$  on  $Z$  is complete. Therefore the only possible way to violate the completeness of  $Q$  is to suppose that, for some  $a \in A$  and  $p \in Z$ , we have  $\text{Not}[a Q p]$  and  $\text{Not}[p Q a]$ . This implies  $\text{Not}[a R p]$  and  $a T p$ , a contradiction. Therefore  $Q$  is complete.

It remains to show that  $Q$  is transitive, i.e., that, for all  $\alpha, \beta, \gamma \in A \cup Z$ ,  $\alpha Q \beta$  and  $\beta Q \gamma$  imply  $\alpha Q \gamma$ . Since each of  $\alpha, \beta, \gamma$  can belong either to  $A$  or to  $Z$ , there are 8 cases to examine.

1. If  $\alpha, \beta, \gamma \in A$ , the conclusion follows from the transitivity of  $\succsim^A$ .
2. If  $\alpha, \beta, \gamma \in Z$ , the conclusion follows from the transitivity of  $\succsim^Z$ .
3. Suppose that  $\alpha, \beta \in A$  and  $\gamma \in Z$ .  $\alpha Q \beta$  and  $\beta Q \gamma$  means that  $\alpha \succsim^A \beta$  and  $\beta R \gamma$ . Using the definition of  $\succsim^A$ , this implies  $\alpha R \gamma$ , so that  $\alpha Q \gamma$ .
4. Suppose that  $\alpha, \gamma \in A$  and  $\beta \in Z$ .  $\alpha Q \beta$  and  $\beta Q \gamma$  means that  $\alpha R \beta$  and  $\text{Not}[\gamma T \beta]$ . Suppose, in contradiction with the thesis, that  $\gamma \succ^A \alpha$ . If  $\alpha I \beta$ , (10a) implies  $\gamma T \beta$ , a contradiction. If  $\alpha T \beta$ ,  $\gamma \succ^A \alpha$  implies  $\gamma T \beta$ , a contradiction. Hence, we must have  $\alpha \succsim^A \gamma$ , so that  $\alpha Q \gamma$ .
5. Suppose that  $\beta, \gamma \in A$  and  $\alpha \in Z$ .  $\alpha Q \beta$  and  $\beta Q \gamma$  means that  $\text{Not}[\beta T \alpha]$  and  $\beta \succsim^A \gamma$ . Suppose that  $\gamma T \alpha$ . Using  $\beta \succsim^A \gamma$ , we obtain  $\beta T \alpha$ , a contradiction. Therefore, we must have  $\text{Not}[\gamma T \alpha]$  so that  $\alpha Q \gamma$ .
6. Suppose that  $\alpha, \beta \in Z$  and  $\gamma \in A$ .  $\alpha Q \beta$  and  $\beta Q \gamma$  means that  $\alpha \succsim^Z \beta$  and  $\text{Not}[\gamma T \beta]$ . Suppose that  $\gamma T \alpha$ . Using  $\alpha \succsim^Z \beta$ , we obtain  $\gamma T \beta$ , a contradiction. Hence, we must have  $\text{Not}[\gamma T \alpha]$ , so that  $\alpha Q \gamma$ .
7. Suppose that  $\alpha, \gamma \in Z$  and  $\beta \in A$ .  $\alpha Q \beta$  and  $\beta Q \gamma$  means that  $\text{Not}[\beta T \alpha]$  and  $\beta R \gamma$ . Suppose that  $\gamma \succ^Z \alpha$ . Using (10b), we obtain  $\beta T \alpha$ , a contradiction. Hence we must have  $\alpha \succsim^Z \gamma$ , so that  $\alpha Q \gamma$ .
8. Suppose that  $\beta, \gamma \in Z$  and  $\alpha \in A$ .  $\alpha Q \beta$  and  $\beta Q \gamma$  means that  $\alpha R \beta$  and  $\beta \succsim^Z \gamma$ . By definition, this implies  $\alpha R \gamma$ , so that  $\alpha Q \gamma$ .  $\square$

The sufficiency proof of Proposition 11 follows from Proposition 24. Indeed, consider the relations  $T$  and  $I$  between the sets  $X_1$  and  $X_2$  defined letting, for all  $x_1 \in X_1$  and all  $x_2 \in X_2$ ,

$$\begin{aligned} x_1 T x_2 &\Leftrightarrow (x_1, x_2) \in \mathcal{A}, \\ x_1 I x_2 &\Leftrightarrow (x_1, x_2) \in \mathcal{F}. \end{aligned} \tag{14}$$

It is routine to show that when  $\mathcal{A}$ -linear<sub>1</sub>,  $\mathcal{AF}$ -linear<sub>1</sub>,  $\mathcal{F}$ -linear<sub>1</sub> and thin<sub>1</sub> hold than the pair of disjoint relations  $T$  and  $I$  between the sets  $X_1$  and  $X_2$  satisfy the conditions of Proposition 24.

The extension of the preceding result to the general case calls for the introduction of an order-denseness condition. Let  $T$  and  $I$  be a pair of disjoint relations between  $A$  and  $Z$ . We say that a subset  $A^* \subseteq A$  is dense for the pair  $T$  and  $I$  if, for all  $a \in A$  and all  $p \in Z$ ,

$$a T p \Rightarrow [a \succ^A a^* \text{ and } a^* T p], \quad (15)$$

$$a U p \Rightarrow [a^* U p \text{ and } a^* \succ^A a], \quad (16)$$

for some  $a^* \in A^*$ .

**Remark 25**

The order-denseness used above is not symmetric between  $A$  and  $Z$ . We use it only to keep things simple. Following Doignon et al. (1984, Prop. 9, p. 84), it is not difficult to show that it is sufficient to require that there is finite or countably infinite subset of  $K \subseteq A \cup Z$  such that

$$a T p \Rightarrow \begin{cases} a \succ^A \alpha \text{ and } \alpha T p, \text{ for some } \alpha \in K \cap A, \\ \text{or} \\ a T \alpha \text{ and } \alpha \succ^Z p, \text{ for some } \alpha \in K \cap Z. \end{cases}$$

and

$$a U p \Rightarrow \begin{cases} a \succ^A \alpha \text{ and } \alpha U p, \text{ for some } \alpha \in K \cap A, \\ \text{or} \\ a U \alpha \text{ and } \alpha \succ^Z p, \text{ for some } \alpha \in K \cap Z. \end{cases}$$

A similar weakening of the order-denseness condition can be performed for the order-denseness condition used in Proposition 18. •

The existence of finite or countably infinite subset  $A^*$  that is dense for the pair  $T$  and  $I$  will guarantee the existence of numerical representation. We have:

**Proposition 26**

*Let  $A$  and  $Z$  be two sets and let  $T$  and  $I$  be a pair of disjoint relations between  $A$  and  $Z$ . There are real valued functions  $f$  on  $A$  and  $g$  on  $Z$  such that (8) and (9) hold if and only if  $T$  is a biorder,  $R = T \cup I$  is a biorder, thinness holds on  $A$  and  $Z$  and there is a finite or countably infinite subset  $A^* \subseteq A$  that is dense for the pair  $T$  and  $I$ . Furthermore, the functions  $f$  and  $g$  can always be chosen in such a way that (12) holds.*

**PROOF**

*Necessity.* Suppose that there are real valued functions  $f$  on  $A$  and  $g$  on  $Z$  such that (8) and (9) hold. Let us show that this implies the existence of a finite or

countably infinite subset  $A^* \subseteq A$  that is dense for the pair of disjoint relations  $T$  and  $I$ .

Let  $\lambda_j \in f(A)$  be such that

$$\mu_j < \lambda_j \text{ and } (\mu_j, \lambda_j) \cap f(A) = \emptyset, \quad (17)$$

for some  $\mu_j \in g(Z)$ . With each such  $\lambda_j \in f(A)$ , we associate a particular  $\mu_j \in g(Z)$  such that (17) holds. Suppose that  $\lambda_k < \lambda_j$ . The two intervals  $(\mu_k, \lambda_k)$  and  $(\mu_j, \lambda_j)$  are disjoint since  $\mu_j < \lambda_k$  would violate the fact that  $(\mu_j, \lambda_j) \cap f(A) = \emptyset$ . The collection of numbers  $\lambda_j$  must be countable because the intervals  $(\mu_j, \lambda_j)$  are nonempty and disjoint and, therefore, each contain a distinct rational number. Therefore, there is a finite or countably infinite set  $A_1^* \subseteq A$  such that  $f(A_1^*)$  contains all the  $\lambda_j$ .

Let  $\lambda_j \in f(A)$  be such that

$$\lambda_j < \mu_j \text{ and } (\lambda_j, \mu_j) \cap f(A) = \emptyset, \quad (18)$$

for some  $\mu_j \in g(Z)$ . With each such  $\lambda_j \in f(A)$ , we associate a particular  $\mu_j \in g(Z)$  such that (18) holds. Suppose that  $\lambda_j < \lambda_k$ . The two intervals  $(\lambda_k, \mu_k)$  and  $(\lambda_j, \mu_j)$  are disjoint since  $\lambda_j < \mu_k$  would violate the fact that  $(\lambda_k, \mu_k) \cap f(A) = \emptyset$ . The collection of numbers  $\lambda_j$  must be countable because the intervals  $(\lambda_j, \mu_j)$  are nonempty and disjoint and, therefore, each contain a distinct rational number. Therefore, there is a finite or countably infinite set  $A_2^* \subseteq A$  such that  $f(A_2^*)$  contains all the  $\lambda_j$ .

Let us select a subset  $A_3^* \subseteq A$  such that for every pair of rational numbers  $p$  and  $q$  such that  $p < q$  the following condition holds:

$$(p, q) \cap f(A) \neq \emptyset \Rightarrow [p < f(a^*) < q, \text{ for some } a^* \in A_3^*]. \quad (19)$$

It is easy to see that the set  $A_3^* \subseteq A$  can always be taken to be finite or countably infinite.

Define  $A^* = A_1^* \cup A_2^* \cup A_3^*$ . By construction,  $A^* \subseteq A$  is finite or countably infinite. Let us show that  $A^*$  is dense for the pair  $T$  and  $I$ .

Suppose that  $a T p$ , so that  $f(a) > g(p)$ . If  $(g(p), f(a)) \cap f(A) = \emptyset$  then, by construction, we have  $f(a) = f(a^*)$ , for some  $a^* \in A_1^*$ . Because  $f(a) = f(a^*) > g(p)$ , we clearly have  $a \succ^A a^*$  and  $a^* T p$ . Otherwise we have  $(g(p), f(a)) \cap f(A) \neq \emptyset$  and let  $c$  be any element in  $A$  such that  $g(p) < f(c) < f(a)$ . Let  $r, r' \in \mathbb{Q}$  be such that  $g(p) < r < f(c) < r' < f(a)$ . By construction of the set  $A_3^*$ , we have  $g(p) < r < f(a^*) < r' < f(a)$ , for some  $a^* \in A_3^*$ . Because  $f(a^*) > g(p)$ , we have  $a^* T p$ . Because  $f(a^*) < f(a)$ , we have  $a \succ^A a^*$ .

Suppose now that  $a U p$ , so that  $f(a) < g(p)$ . If  $(f(a), g(p)) \cap f(A) = \emptyset$ , then, by construction, we have  $f(a) = f(a^*)$ , for some  $a^* \in A_2^*$ . Because  $f(a) = f(a^*) <$

$g(p)$ , we have  $a^* U p$  and  $a^* \succsim^A a$ . Otherwise we have  $(f(a), g(p)) \cap f(A) \neq \emptyset$  and let  $d$  be any element in  $A$  such that  $f(a) < f(d) < g(p)$ . Let  $r, r' \in \mathbb{Q}$  be such that  $f(a) < r < f(d) < r' < g(p)$ . By construction of the set  $A_3^*$ , we have  $f(a) < r < f(a^*) < r' < g(p)$ , for some  $a^* \in A_3^*$ . Because  $f(a^*) < g(p)$ , we have  $a^* U p$ . Because  $f(a) < f(a^*)$ , we have  $a^* \succsim^A a$ .

*Sufficiency.* The proof will be complete if we show that there is a countable subset  $B^*$  of  $A \cup Z$  that is dense for  $Q$ , i.e. that, for all  $\alpha, \beta \in A \cup Z$ ,

$$[\alpha Q \beta \text{ and } \text{Not}[\beta Q \alpha]] \Rightarrow [\alpha Q \gamma \text{ and } \gamma Q \beta, \text{ for some } \gamma \in B^*].$$

We show that the set  $A^*$ , as defined above, is dense for  $Q$ . There are four cases to consider.

1. Suppose that  $\alpha \in A$  and  $\beta \in Z$ . Then  $\alpha Q \beta$  and  $\text{Not}[\beta Q \alpha]$  implies  $\alpha T \beta$ . Using the fact that  $A^* \subseteq A$  is dense for the pair  $T$  and  $I$ , we obtain  $\alpha \succsim^A a^*$  and  $a^* T \beta$ , for some  $a^* \in A^*$ , so that  $\alpha Q a^*$  and  $a^* Q \beta$ .
2. Suppose that  $\alpha \in Z$  and  $\beta \in A$ . Then  $\alpha Q \beta$  and  $\text{Not}[\beta Q \alpha]$  implies  $\beta U \alpha$ . Using the fact that  $A^* \subseteq A$  is dense for the pair  $T$  and  $I$ , we obtain  $a^* \succsim^A \beta$  and  $a^* U \alpha$ , for some  $a^* \in A^*$ , so that  $\alpha Q a^*$  and  $a^* Q \beta$ .
3. Suppose that  $\alpha, \beta \in A$ , so that  $\alpha Q \beta$  and  $\text{Not}[\beta Q \alpha]$  implies  $\alpha \succ^A \beta$ . By definition, we have either

$$\alpha T p \text{ and } \text{Not}[\beta T p], \tag{20}$$

or

$$\alpha R p \text{ and } \beta U p, \tag{21}$$

for some  $p \in Z$ .

Suppose that (20) holds. Using the fact that  $A^* \subseteq A$  is dense for the pair  $T$  and  $I$ ,  $\alpha T p$  implies  $\alpha \succsim^A a^*$  and  $a^* T p$ , for some  $a^* \in A^*$ . Suppose that  $\beta \succ^A a^*$ . Using the definition of  $\succsim^A$ ,  $a^* T p$  and  $\beta \succ^A a^*$  imply  $\beta T p$ , a contradiction. Hence, we must have  $a^* \succsim^A \beta$ . Hence, we have  $\alpha \succsim^A a^*$  and  $a^* \succsim^A \beta$ , as required.

Suppose now that (21) holds. Using the fact that  $A^* \subseteq A$  is dense for the pair  $T$  and  $I$ ,  $\beta U p$  implies  $a^* U p$  and  $a^* \succsim^A \beta$ , for some  $a^* \in A^*$ . Suppose that  $a^* \succ^A \alpha$ . Using the definition of  $\succsim^A$  and (10a),  $\alpha R p$  and  $a^* \succ^A \alpha$  imply  $a^* T p$ , a contradiction. Hence, we have  $\alpha \succsim^A a^*$  and  $a^* \succsim^A \beta$ , as required.

4. Suppose that  $\alpha, \beta \in Z$ , so that  $\alpha Q \beta$  and  $\text{Not}[\beta Q \alpha]$  implies  $\alpha \succ^Z \beta$ . By definition, we have either

$$a T \beta \text{ and } \text{Not}[a T \alpha], \quad (22)$$

or

$$a R \beta \text{ and } a U \alpha, \quad (23)$$

for some  $a \in A$ .

Suppose that (22) holds. Using the fact that  $A^* \subseteq A$  is dense for the pair  $T$  and  $I$ ,  $a T \beta$  implies  $a \succ^A a^*$  and  $a^* T \beta$ , for some  $a^* \in A^*$ . If  $a^* T \alpha$ ,  $a \succ^A a^*$  implies  $a T \alpha$ , a contradiction. Therefore, we must have  $\text{Not}[a^* T \alpha]$ . Therefore, we have  $\text{Not}[a^* T \alpha]$  and  $a^* R \beta$ , so that  $\alpha Q a^*$  and  $a^* Q \beta$ , as required.

Suppose finally that (23) holds. Using the fact that  $A^* \subseteq A$  is dense for the pair  $T$  and  $I$ ,  $a U \alpha$  implies  $a^* U \alpha$  and  $a^* \succ^A a$ , for some  $a^* \in A^*$ . If  $a^* U \beta$ ,  $a^* \succ^A a$  implies  $a U \beta$ , a contradiction. Hence, we must have  $a^* R \beta$ . Therefore, we have  $a^* U \alpha$  and  $a^* R \beta$ , so that  $a Q a^*$  and  $a^* Q \beta$ , as required.  $\square$

It is easy to check that Proposition 18 follows from Proposition 26. Indeed suppose that  $\langle \mathcal{A}, \mathcal{F}, \mathcal{U} \rangle$  satisfies  $\mathcal{A}$ -linear<sub>1</sub>,  $\mathcal{AF}$ -linear<sub>1</sub>,  $\mathcal{F}$ -linear<sub>1</sub>, thin<sub>1</sub> and there is a finite or countably infinite subset  $Y_1 \subseteq X_1$  that is dense for  $\langle \mathcal{A}, \mathcal{F}, \mathcal{U} \rangle$ . Define the relations  $T$  and  $I$  between the sets  $X_1$  and  $X_2$  using (14). We have already observed that the pair of disjoint relations  $T$  and  $I$  satisfies the conditions of Proposition 24. Furthermore, it is clear that the existence of a finite or countably infinite subset  $Y_1 \subseteq X_1$  that is dense for  $\langle \mathcal{A}, \mathcal{F}, \mathcal{U} \rangle$  implies the existence of a finite or countably infinite subset  $A^* \subseteq A$  is dense for the pair  $T$  and  $I$ . The necessity of the density condition is shown similarly to what was done in the proof of necessity of Proposition 18.

### A.3 Extensions and comments

We have given above necessary and sufficient conditions for the existence of a numerical representation (5), using Proposition 26 as a building block. A different route is to suppose that some solvability assumptions hold. It is more direct but does not lead to necessary and sufficient conditions.

Let us say that the covering  $\langle \mathcal{A}, \mathcal{F}, \mathcal{U} \rangle$  satisfies restricted solvability on attribute 1 if  $(x_1, x_2) \in \mathcal{A}$  and  $(y_1, x_2) \in \mathcal{U}$  implies that  $(z_1, x_2) \in \mathcal{F}$ , for some  $z_1 \in X_1$ . A similar condition is defined on attribute 2.

These two solvability conditions are not necessary for the existence of a representation (5) (note, however, that they are considerably weaker than unrestricted

solvability used above). Nevertheless, they appear to be quite reasonable. If they are satisfied, it is possible to give a direct and simple proof of Proposition 18.

Suppose that  $\langle \mathcal{A}, \mathcal{F}, \mathcal{U} \rangle$  satisfies  $\mathcal{A}$ -linear<sub>1</sub>,  $\mathcal{AF}$ -linear<sub>1</sub>,  $\mathcal{F}$ -linear<sub>1</sub>, thin<sub>1</sub> and thin<sub>2</sub> and is such that restricted solvability holds on attribute 1. Suppose furthermore that there are real-valued function  $v_1$  on  $X_1$  on  $v_2$  on  $X_2$  such that:

$$\begin{aligned} x_1 \succsim_1 y_1 &\Leftrightarrow v_1(x_1) \geq v_1(y_1), \\ x_2 \succsim_2 y_2 &\Leftrightarrow v_2(x_2) \geq v_2(y_2), \end{aligned}$$

for all  $x_1, y_1 \in X_1$  and  $x_2, y_2 \in X_2$  (meaning that some order-denseness condition has been postulated on  $\succsim_1$  and  $\succsim_2$ ). We may suppose w.l.o.g. that the image of both  $v_1$  and  $v_2$  is included in  $(-1; 1)$ . Consider any such functions  $v_1$  and  $v_2$ .

We say that  $x_2 \in X_2$  is *maximal* if  $(x_1, x_2) \in \mathcal{A}$ , for all  $x_1 \in X_1$ . Similarly,  $x_2 \in X_2$  is *minimal* if  $(x_1, x_2) \in \mathcal{U}$ , for all  $x_1 \in X_1$ . It is clear that if both  $x_2$  and  $y_2$  are maximal (resp. minimal) then  $x_2 \sim_2 y_2$ .

Now consider any  $x_2 \in X_2$  that is neither minimal nor maximal. We claim that  $(x_1, x_2) \in \mathcal{F}$ , for some  $x_1 \in X_1$ . Indeed, we know that  $(z_1, x_2) \in \mathcal{AF}$  and  $(w_1, x_2) \in \mathcal{FU}$ , for some  $z_1, w_1 \in X_1$ . If either  $(z_1, x_2) \in \mathcal{F}$  or  $(w_1, x_2) \in \mathcal{F}$ , there is nothing to prove. If  $(z_1, x_2) \in \mathcal{A}$  and  $(w_1, x_2) \in \mathcal{U}$ , restricted solvability on attribute 1 leads to the desired conclusion.

Because of linearity and thinness, we know that

$$\begin{aligned} [(x_1, x_2) \in \mathcal{F} \text{ and } y_1 \succ_1 x_1] &\Rightarrow (y_1, x_2) \in \mathcal{A}, \\ [(x_1, x_2) \in \mathcal{F} \text{ and } x_1 \succ_1 z_1] &\Rightarrow (z_1, x_2) \in \mathcal{U}. \end{aligned}$$

Hence, for all  $x_2 \in X_2$ ,  $(x_1, x_2) \in \mathcal{F}$  and  $(x'_1, x_2) \in \mathcal{F}$  imply  $x'_1 \sim_1 x_1$ , so that  $v_1(x_1) = v_1(x'_1)$ . For all  $x_2 \in X_2$  that is neither maximal nor minimal, define the function  $u_2$  letting  $u_2(x_2) = -v_1(x_1(x_2))$  where  $x_1(x_2) \in X_1$  is such that  $(x_1(x_2), x_2) \in \mathcal{F}$ . The above observations have shown that  $u_2$  is well-defined. We extend  $u_2$ , letting  $u_2(x_2) = 1$  (resp.  $= -1$ ) for any maximal (resp. minimal)  $x_2 \in X_2$ .

It is easy to check that we have:

$$\begin{aligned} (x_1, x_2) \in \mathcal{A} &\Leftrightarrow v_1(x_1) + u_2(x_2) > 0, \\ (x_1, x_2) \in \mathcal{F} &\Leftrightarrow v_1(x_1) + u_2(x_2) = 0, \\ (x_1, x_2) \in \mathcal{U} &\Leftrightarrow v_1(x_1) + u_2(x_2) < 0. \end{aligned}$$

The proof is obvious if  $x_2$  is maximal or minimal. Suppose not, so that  $x_1(x_2)$  exists. If  $(x_1, x_2) \in \mathcal{F}$ , we have  $v_1(x_1) + u_2(x_2) = v_1(x_1) - v_1(x_1(x_2)) = 0$ , the first equality resulting from the definition of  $u_2$  and the second from the fact that it must be true that  $x_1 \sim_1 x_1(x_2)$ . Suppose that  $(x_1, x_2) \in \mathcal{A}$ . If, in contradiction with the thesis, we have  $v_1(x_1) + u_2(x_2) = v_1(x_1) - v_1(x_1(x_2)) \leq 0$ , we obtain

$v_1(x_1(x_2)) \geq v_1(x_1)$ , so that  $x_1(x_2) \succsim_1 x_1$ . By construction, we have  $(x_1(x_2), x_2) \in \mathcal{F}$  and  $x_1(x_2) \succsim_1 x_1$  implies that we cannot have  $(x_1, x_2) \in \mathcal{A}$ , a contradiction. The proof is similar if it is supposed that  $(x_1, x_2) \in \mathcal{U}$ .

A geometric interpretation of the above construction is the following. Consider the plane  $v_1(X_1) \times v_2(X_2)$ . It is easy to see that in this plane, the set of points corresponding to  $\mathcal{F}$  is a strictly decreasing curve. Geometrically, the re-scaling of  $v_2$  that is necessary to transform this curve into a line is easy to devise.

The general problem of studying families of curves that can be made parallel lines with such transformations have been studied in depth in Levine (1970) (see also Krantz et al., 1971, sec. 6.7, p. 283). Let us note that the results in Levine (1970) may also be used to tackle the case of three categories and two attributes. Under suitable generalizations of linearity, thinness and solvability, the curves corresponding to the two frontiers in the  $v_1(X_1) \times v_2(X_2)$  plane will not intersect and both be strictly decreasing. It is not difficult to see that the application of Theorem 3.B, p. 424 in Levine (1970) (or Th. 6.8, p. 286 in Krantz et al., 1971) allows to transform a decomposable transformation into an additive one. Because this case does not seem to have a particular interest, we do not develop.

## B The finite case

Let us consider a covering  $\langle \mathcal{A}, \mathcal{F}, \mathcal{U} \rangle$  on a *finite* set  $X$ . Since the case of two attributes was dealt with in the preceding section, we implicitly suppose here that  $n \geq 3$ . It turns out that results for this case are elementary adaptations of classical results of conjoint measurement introduced in the literature by Scott (1964).

Consider an ordered covering  $\langle \mathcal{A}, \mathcal{F}, \mathcal{U} \rangle$  on  $X$ . Define the binary relations on  $X$  letting, for all  $x, y \in X$ ,

$$x P y \Leftrightarrow \left\{ \begin{array}{c} y \in \mathcal{F} \text{ and } x \in \mathcal{A}, \\ \text{or} \\ y \in \mathcal{U} \text{ and } x \in \mathcal{AF}, \end{array} \right\}, \quad (24)$$

$$x I y \Leftrightarrow [x = y \text{ or } x, y \in \mathcal{F}] \quad (25)$$

We obviously have that  $P \cap I = \emptyset$ . We define  $R$  as  $P \cup I$ .

Consider now the possibility to represent the relations  $P$  and  $I$  in such a way that:

$$x P y \Rightarrow \sum_{i=1}^n f_i(x_i) > \sum_{i=1}^n f_i(y_i), \quad (26a)$$

$$x I y \Rightarrow \sum_{i=1}^n f_i(x_i) = \sum_{i=1}^n f_i(y_i). \quad (26b)$$

The connection between model (A) and the existence of a representation (26) is easily established.

**Lemma 27**

Let  $X$  be a finite set. An ordered covering  $\langle \mathcal{A}, \mathcal{F}, \mathcal{U} \rangle$  of  $X$  has a representation in model (A) if and only if there are real-valued function  $f_i$  on  $X_i$  such that (26) holds for the induced relations  $P$  and  $I$ .

PROOF

*Necessity.* Suppose that the ordered covering  $\langle \mathcal{A}, \mathcal{F}, \mathcal{U} \rangle$  of  $X$  has a representation in model (A). We have:

$$\begin{aligned} x I y &\Leftrightarrow [x = y \text{ or } x, y \in \mathcal{F}], \\ &\Rightarrow \sum_{i=1}^n v_i(x_i) = \sum_{i=1}^n v_i(y_i), \end{aligned}$$

and

$$\begin{aligned} x P y &\Leftrightarrow \left\{ \begin{array}{l} y \in \mathcal{F} \text{ and } x \in \mathcal{A} \\ \text{or} \\ y \in \mathcal{U} \text{ and } x \in \mathcal{AF} \end{array} \right\} \\ &\Rightarrow \sum_{i=1}^n v_i(x_i) > \sum_{i=1}^n v_i(y_i). \end{aligned}$$

*Sufficiency.* Suppose now that the relations  $P$  and  $I$  have a representation (26a–26b). Let

$$\begin{aligned} \gamma &= \min_{x \in \mathcal{A}} \sum_{i=1}^n f_i(x_i) & \delta &= \max_{x \in \mathcal{U}} \sum_{i=1}^n f_i(x_i) \\ \alpha &= \min_{x \in \mathcal{F}} \sum_{i=1}^n f_i(x_i) & \beta &= \max_{x \in \mathcal{F}} \sum_{i=1}^n f_i(x_i). \end{aligned}$$

It follows from (26b) that  $\alpha = \beta = \theta$ . Because  $x \in \mathcal{A}$  and  $y \in \mathcal{F}$  imply  $x P y$ , we have  $\gamma > \theta$ . Similarly, because  $x \in \mathcal{F}$  and  $y \in \mathcal{U}$  imply  $x P y$ , we have  $\theta > \beta$ .

Taking, for all  $i \in N$  and all  $x_i \in X_i$ ,  $v_i(x_i) = f_i(x_i) - \theta$  obviously leads to a representation of  $\langle \mathcal{A}, \mathcal{F}, \mathcal{U} \rangle$  in model (A).  $\square$

The condition ensuring that two binary relations  $P$  and  $I$  on a product set have a representation (26a–26b) are classical (see, e.g., Krantz et al., 1971, p. 428). We briefly recall them below.

Let  $\succsim$  be a reflexive relation on  $X$ . Let us first recall the conditions allowing to obtain a numerical representation of  $\succsim$  such that:

$$x \succ y \Rightarrow \sum_{i=1}^n f_i(x_i) > \sum_{i=1}^n f_i(y_i), \quad (27)$$

$$x \sim y \Rightarrow \sum_{i=1}^n f_i(x_i) = \sum_{i=1}^n f_i(y_i). \quad (28)$$

Consider  $2m$  elements (not necessarily distinct)  $x^1, x^2, \dots, x^m, y^1, y^2, \dots, y^m \in X$ . We say that  $(x^1, x^2, \dots, x^m) E^m (y^1, y^2, \dots, y^m)$  if, for all  $i \in N$ ,  $(x_i^1, x_i^2, \dots, x_i^m)$  is a permutation of  $(y_i^1, y_i^2, \dots, y_i^m)$ . We say that  $\succsim$  satisfies condition  $C^m$  if, for all  $x^1, x^2, \dots, x^m, y^1, y^2, \dots, y^m \in X$  such that  $(x^1, x^2, \dots, x^m) E^m (y^1, y^2, \dots, y^m)$ ,  $[x^j \succsim y^j, \text{ for } j = 1, 2, \dots, m-1] \Rightarrow \text{Not}[x^m \succ y^m]$ . We have:

**Proposition 28 (Krantz et al., 1971, Th. 9.1, page 430)**

*Let  $\succsim$  be a reflexive binary relation on a finite  $X$ . The relation  $\succsim$  has a representation in model (27–28) iff it satisfies condition  $C^m$  for  $m = 2, 3, \dots$*

Note that condition  $C^m$  is required to hold for  $m = 2, 3, \dots$ . This is in fact a denumerable scheme of conditions. It is well-known that there are binary relations defined on a finite set that satisfy  $C^m$  for  $m = 2, 3, \dots, t$  but that violates  $C^{t+1}$  (see Krantz et al., 1971, p. 427). Therefore (unless additional restrictions are imposed, e.g., on the cardinality of  $X$ , see Fishburn, 1997, 2001 or Wille, 2000 on the difficult combinatorial problems involved in the study of such restrictions), this denumerable scheme of conditions cannot be truncated. Using Lemma 27, this leads to:

**Proposition 29**

*Let  $\langle \mathcal{A}, \mathcal{F}, \mathcal{U} \rangle$  be an ordered covering on a finite set  $X$ . The covering  $\langle \mathcal{A}, \mathcal{F}, \mathcal{U} \rangle$  has a representation in model (A) iff the relation  $R = P \cup I$  satisfies condition  $C^m$ , for  $m = 2, 3, \dots$*

This very simple result prompts some remarks.

**Remark 30**

Proposition 29 raises even more difficult combinatorial questions than Proposition 28 does. Viewing the set  $X$  as being partially ordered by the relation  $P$ , it is clear that it will be impossible to find paths of arbitrary length in this poset (the maximal length of a path is 3, which is obtained by taking one alternative successively in each of  $\mathcal{A}$ ,  $\mathcal{F}$  and  $\mathcal{U}$ ). But this possibility is crucial in order to show that the denumerable scheme of conditions used in for Proposition 28 cannot be truncated. This raises the question of relating the possibility to truncate the denumerable scheme of conditions when there is a constraint on the length of the

maximal path in  $X$  for  $P$ . Unfortunately, we have no satisfactory answer at this time. •

**Remark 31**

We say that  $\succsim$  satisfies condition  $D^m$  if, for all  $x^1, x^2, \dots, x^m, y^1, y^2, \dots, y^m \in X$  such that  $(x^1, x^2, \dots, x^m) E^m (y^1, y^2, \dots, y^m)$ ,  $[x^j \succ y^j$  or  $x^j = y^j$  for  $j = 1, 2, \dots, m - 1] \Rightarrow \text{Not}[x^m \succ y^m]$ .

As shown in Fishburn (1970, p. 44), requiring that condition  $D^m$  holds for  $m = 2, 3, \dots$  is a necessary and sufficient condition for the existence of real-valued functions  $u_i$  such that (27) holds. When the ordered covering is a partition, it is easy to see that we can replace  $C^m$  with  $D^m$  in the statement of Proposition 29.

This was already noted in Fishburn et al. (1991, Th. 2, p. 152) and Fishburn and Shepp (1999, Th. 2.2, p. 40) in the apparently different context of discrete tomography. It can also be inferred from the main result in Fishburn (1992) who studies binary relations  $\succsim$  on a product set  $X = X_1 \times X_2 \times \dots \times X_n$  having a numerical representation such that:

$$x \succsim y \Leftrightarrow \sum_{i=1}^n p_i(x_i, y_i) \geq 0.$$

Unless special properties are supposed for the functions  $p_i$  on  $X_i^2$ , the binary relation  $\succsim$  may be regarded as defining an ordered partition  $\langle \mathcal{A}, \mathcal{U} \rangle$  of the set  $Y = X_1^2 \times X_2^2 \times \dots \times X_n^2$ , taking  $((x_1, y_1), (x_2, y_2)) \in \mathcal{A}$  iff  $x \succsim y$ .

Fishburn et al. (1991, Th. 5, p. 155) give an example of an ordered partition with  $r = 2$  and  $n = 3$  showing that it is not possible to weaken condition  $D^m$  by requiring it to hold only with *distinct* elements  $x^1, x^2, \dots, x^m$  and *distinct* elements  $y^1, y^2, \dots, y^m$ . As shown in Fishburn et al. (1991), such a weakened condition characterizes, on finite sets, sets of uniqueness, i.e., sets that are uniquely determined by their projection count on each coordinate. •

It is clear that the above technique extends without difficulty to ordered partitions or coverings with more than two categories.

## C Refining conditions

Proposition 8 uses strong linearity and strong thinness. This involves some redundancy.

We say that  $\langle \mathcal{A}, \mathcal{F}, \mathcal{U} \rangle$  is  $\nu$ - $\mathcal{A}$ -linear if it satisfies  $\mathcal{A}$ -linear $_I$  for all  $I \subseteq N$  such that  $|I| = \nu$ . We use a similar convention for  $\nu$ - $\mathcal{F}$ -linear,  $\nu$ - $\mathcal{AF}$ -linear,  $\nu$ -linear and  $\nu$ -thin.

It is easy to check that the only consequences of strong linearity used in the proof of Lemma 10 are 1-linear,  $(n - 2)$ -linear and  $(n - 1)$ -linear. Similarly, the

only consequences of strong thinness that are used are 1-thin,  $(n - 2)$ -thin and  $(n - 1)$ -thin. Therefore these conditions can fully replace strong linearity and strong thinness in the statement of Proposition 8. Working with groups of  $n - 1$  or  $n - 2$  attributes is not particularly intuitive however. We show below how it is possible to work only with singletons and pairs.

The proof of the following lemma follows directly from the definition of linearity.

**Lemma 32**

1.  $\langle \mathcal{A}, \mathcal{F}, \mathcal{U} \rangle$  satisfies  $\mathcal{A}$ -linear<sub>I</sub> iff it satisfies  $\mathcal{A}$ -linear<sub>-I</sub>,
2.  $\langle \mathcal{A}, \mathcal{F}, \mathcal{U} \rangle$  satisfies  $\mathcal{F}$ -linear<sub>I</sub> iff it satisfies  $\mathcal{F}$ -linear<sub>-I</sub>.

We have:

**Lemma 33**

Let  $\langle \mathcal{A}, \mathcal{F}, \mathcal{U} \rangle$  be an ordered covering satisfying  $\mathcal{A}$ -linear<sub>I</sub>,  $\mathcal{F}$ -linear<sub>I</sub>, thin<sub>I</sub> and unrestricted solvability. Then  $\langle \mathcal{A}, \mathcal{F}, \mathcal{U} \rangle$  satisfies  $\mathcal{AF}$ -linear<sub>I</sub>.

PROOF

Suppose that  $\mathcal{AF}$ -linear<sub>I</sub> is violated, so that  $(x_I, a_{-I}) \in \mathcal{A}$ ,  $(y_I, b_{-I}) \in \mathcal{F}$ ,  $(y_I, a_{-I}) \notin \mathcal{A}$ , and  $(x_I, b_{-I}) \notin \mathcal{AF}$ .

Using unrestricted solvability, we can find a  $c_{-I} \in X_{-I}$  such that  $(x_I, c_{-I}) \in \mathcal{F}$ . Using  $\mathcal{F}$ -linear<sub>I</sub>,  $(x_I, c_{-I}) \in \mathcal{F}$ ,  $(y_I, b_{-I}) \in \mathcal{F}$ , and  $(x_I, b_{-I}) \notin \mathcal{AF}$  imply  $(y_I, c_{-I}) \in \mathcal{AF}$ . Suppose first that  $(y_I, c_{-I}) \in \mathcal{F}$ . Since  $(x_I, c_{-I}) \in \mathcal{F}$ , thin<sub>I</sub> implies that  $x_I \sim_I y_I$ , contradicting the fact that  $(x_I, a_{-I}) \in \mathcal{A}$  and  $(y_I, a_{-I}) \notin \mathcal{A}$ . Suppose now that  $(y_I, c_{-I}) \in \mathcal{A}$ . Since  $(x_I, a_{-I}) \in \mathcal{A}$ ,  $\mathcal{A}$ -linear<sub>I</sub> implies either  $(x_I, c_{-I}) \in \mathcal{A}$  or  $(y_I, a_{-I}) \in \mathcal{A}$ , a contradiction.  $\square$

**Lemma 34**

An ordered covering  $\langle \mathcal{A}, \mathcal{F}, \mathcal{U} \rangle$  satisfies  $\mathcal{AF}$ -linear<sub>I</sub> and thin<sub>I</sub> iff it satisfies  $\mathcal{AF}$ -linear<sub>-I</sub> and thin<sub>-I</sub>.

PROOF

Suppose that thin<sub>I</sub> is violated, so that  $(x_I, a_{-I}) \in \mathcal{F}$ ,  $(y_I, a_{-I}) \in \mathcal{F}$ ,  $(x_I, b_{-I}) \in \mathcal{A}$ ,  $(y_I, b_{-I}) \in \mathcal{FU}$ . If  $(y_I, b_{-I}) \in \mathcal{F}$ , thin<sub>-I</sub> and  $(y_I, a_{-I}) \in \mathcal{F}$  imply that  $a_{-I} \sim_{-I} b_{-I}$ , contradicting the fact that  $(x_I, a_{-I}) \in \mathcal{F}$  and  $(x_I, b_{-I}) \in \mathcal{A}$ . Suppose now that  $(y_I, b_{-I}) \in \mathcal{U}$ . Using  $\mathcal{AF}$ -linear<sub>-I</sub>,  $(x_I, b_{-I}) \in \mathcal{A}$  and  $(y_I, a_{-I}) \in \mathcal{F}$  imply either  $(x_I, a_{-I}) \in \mathcal{A}$  or  $(y_I, b_{-I}) \in \mathcal{AF}$ , a contradiction.

Suppose now that  $\mathcal{AF}$ -linear<sub>I</sub> is violated, so that we have  $(x_I, a_{-I}) \in \mathcal{A}$ ,  $(y_I, b_{-I}) \in \mathcal{F}$ ,  $(y_I, a_{-I}) \in \mathcal{FU}$  and  $(x_I, b_{-I}) \in \mathcal{U}$ . If  $(y_I, a_{-I}) \in \mathcal{F}$ , thin<sub>-I</sub> and  $(y_I, b_{-I}) \in \mathcal{F}$  imply that that  $a_{-I} \sim_{-I} b_{-I}$ , contradicting the fact that  $(x_I, a_{-I}) \in \mathcal{A}$  and  $(x_I, b_{-I}) \in \mathcal{U}$ . Suppose now that  $(y_I, a_{-I}) \in \mathcal{U}$ . Using  $\mathcal{AF}$ -linear<sub>-I</sub>,  $(x_I, a_{-I}) \in \mathcal{A}$  and  $(y_I, b_{-I}) \in \mathcal{F}$  imply either  $(x_I, b_{-I}) \in \mathcal{A}$  or  $(y_I, a_{-I}) \in \mathcal{AF}$ , a contradiction.  $\square$

When  $n = 3$ , we may strengthen Proposition 8 as follows.

**Proposition 35**

Let  $\langle \mathcal{A}, \mathcal{F}, \mathcal{U} \rangle$  be an ordered covering of a set  $X = X_1 \times X_2 \times X_3$ . If  $\langle \mathcal{A}, \mathcal{F}, \mathcal{U} \rangle$  is non-degenerate and satisfies unrestricted solvability, the Archimedean condition, 1- $\mathcal{A}$ -linear, 1- $\mathcal{F}$ -linear, 1-thin and the Thomsen condition, then there is an additive representation of  $\langle \mathcal{A}, \mathcal{F}, \mathcal{U} \rangle$ .

PROOF

Using Lemma 32, we know that 2- $\mathcal{A}$ -linear and 2- $\mathcal{F}$ -linear hold. Using Lemma 33, we know that 1- $\mathcal{AF}$ -linear holds. Using Lemma 34, we know that 2- $\mathcal{AF}$ -linear and 2-thin hold. The conclusion follows from Proposition 8.  $\square$

Let us now deal with the  $n \geq 4$  case. We have:

**Lemma 36**

If  $\langle \mathcal{A}, \mathcal{F}, \mathcal{U} \rangle$  satisfies unrestricted solvability, 1- $\mathcal{A}$ -linear, 1- $\mathcal{F}$ -linear, 1-thin and 2- $\mathcal{A}$ -linear, then it satisfies 2- $\mathcal{F}$ -linear.

PROOF

Suppose that 2- $\mathcal{F}$ -linear is violated, so that  $(x_i, x_j, a_{-ij}) \in \mathcal{F}$ ,  $(y_i, y_j, b_{-ij}) \in \mathcal{F}$ ,  $(y_i, y_j, a_{-ij}) \in \mathcal{U}$ ,  $(x_i, x_j, b_{-ij}) \in \mathcal{U}$ . Using Lemma 33, we know that 1- $\mathcal{AF}$ -linear holds, so that  $\succsim_i$  is complete, for all  $i \in N$ . It is clearly impossible that we have  $x_i \succsim_i y_i$  and  $x_j \succsim_j y_j$ . Indeed,  $(y_i, y_j, b_{-ij}) \in \mathcal{F}$ ,  $x_i \succsim_i y_i$  and  $x_j \succsim_j y_j$  would imply  $(x_i, x_j, b_{-ij}) \in \mathcal{AF}$ , a contradiction. Similarly, it is impossible that  $y_i \succsim_i x_i$  and  $y_j \succsim_j x_j$ .

Suppose henceforth that  $x_i \succ_i y_i$  and  $y_j \succ_j x_j$ , the other case being dealt with similarly. Because  $(y_i, y_j, b_{-ij}) \in \mathcal{F}$  and  $x_i \succ_i y_i$ , we obtain  $(x_i, y_j, b_{-ij}) \in \mathcal{A}$ . Using unrestricted solvability, we know that  $(x_i, w_j, b_{-ij}) \in \mathcal{F}$ , for some  $w_j \in X_j$ . Because  $(x_i, x_j, b_{-ij}) \in \mathcal{U}$ , we must have  $w_j \succ_j x_j$ . Hence, from  $(x_i, x_j, a_{-ij}) \in \mathcal{F}$ , we obtain  $(x_i, w_j, a_{-ij}) \in \mathcal{A}$ . Similarly, using unrestricted solvability, we can find a  $z_j \in X_j$  such that  $(y_i, z_j, a_{-ij}) \in \mathcal{F}$ . Because  $(y_i, y_j, a_{-ij}) \in \mathcal{U}$ , it must be true that  $z_j \succ_j y_j$ . Using  $(y_i, y_j, b_{-ij}) \in \mathcal{F}$ , we obtain  $(y_i, z_j, b_{-ij}) \in \mathcal{A}$ . Using 2- $\mathcal{A}$ -linear,  $(x_i, w_j, a_{-ij}) \in \mathcal{A}$  and  $(y_i, z_j, b_{-ij}) \in \mathcal{A}$  imply either  $(y_i, z_j, a_{-ij}) \in \mathcal{A}$  or  $(x_i, w_j, b_{-ij}) \in \mathcal{A}$ , a contradiction.  $\square$

When  $n \geq 4$ , we may use the following lemma to strengthen Proposition 8.

**Lemma 37**

Let  $n \geq 4$ . If  $\langle \mathcal{A}, \mathcal{F}, \mathcal{U} \rangle$  satisfies unrestricted solvability, 1- $\mathcal{A}$ -linear, 1- $\mathcal{F}$ -linear, 2- $\mathcal{A}$ -linear, 1-thin and 2-thin, then it satisfies  $(n-2)$ -linear,  $(n-1)$ -linear,  $(n-2)$ -thin and  $(n-1)$ -thin.

PROOF

Using Lemma 36, we know that 2- $\mathcal{F}$ -linear holds. Using Lemma 32, we know that  $(n-2)$ - $\mathcal{A}$ -linear,  $(n-1)$ - $\mathcal{A}$ -linear,  $(n-2)$ - $\mathcal{F}$ -linear and  $(n-1)$ - $\mathcal{F}$ -linear hold. Using Lemma 33, we know that 1- $\mathcal{AF}$ -linear and 2- $\mathcal{AF}$ -linear hold. Using Lemma 34, we know that  $(n-2)$ - $\mathcal{AF}$ -linear,  $(n-1)$ - $\mathcal{AF}$ -linear,  $(n-2)$ -thin and  $(n-1)$ -thin hold.  $\square$

It turns out that when the Archimedean condition is brought into the picture, condition 2-thin can also be omitted.

**Lemma 38**

Let  $n \geq 4$ . If  $\langle \mathcal{A}, \mathcal{F}, \mathcal{U} \rangle$  satisfies unrestricted solvability, 1- $\mathcal{A}$ -linear, 1- $\mathcal{F}$ -linear, 2- $\mathcal{A}$ -linear, 1-thin and the Archimedean condition, then it is 2-thin.

PROOF

Suppose that 2-thin is violated, so that  $(x_i, x_j, a_{-ij}) \in \mathcal{F}$ ,  $(y_i, y_j, a_{-ij}) \in \mathcal{F}$ ,  $(x_i, x_j, b_{-ij}) \in \mathcal{A}$ , and  $(y_i, y_j, b_{-ij}) \in \mathcal{FU}$ . Using Lemma 33, we know that 1- $\mathcal{AF}$ -linear holds. Using Lemma 36, we know that 2- $\mathcal{F}$ -linear holds. Note that it is clear impossible that  $a_{-ij} \sim_{-ij} b_{-ij}$ .

Suppose first that  $(y_i, y_j, b_{-ij}) \in \mathcal{U}$ . Using unrestricted solvability, we have  $(z_i, x_j, b_{-ij}) \in \mathcal{F}$ , for some  $z_i \in X_i$ . Because  $(x_i, x_j, b_{-ij}) \in \mathcal{A}$ , we must have that  $x_i \succ_i z_i$ . Now,  $(x_i, x_j, a_{-ij}) \in \mathcal{F}$  and  $x_i \succ_i z_i$  imply  $(z_i, x_j, a_{-ij}) \in \mathcal{U}$ . Using 2- $\mathcal{F}$ -linear,  $(z_i, x_j, b_{-ij}) \in \mathcal{F}$  and  $(y_i, y_j, a_{-ij}) \in \mathcal{F}$  imply either  $(z_i, x_j, a_{-ij}) \in \mathcal{AF}$  or  $(y_i, y_j, b_{-ij}) \in \mathcal{AF}$ , a contradiction.

Suppose now that  $(y_i, y_j, b_{-ij}) \in \mathcal{F}$ . It is easy to see that we cannot have  $x_i \sim_i y_i$  and  $x_j \sim_j y_j$ . Indeed, using 1-linear and 1-thin,  $(x_i, x_j, b_{-ij}) \in \mathcal{A}$  would imply  $(y_i, y_j, b_{-ij}) \in \mathcal{A}$ . Similarly, it is impossible to have  $[x_i \succsim_i y_i$  and  $x_j \succ_j y_j]$  or  $[y_i \succsim_i x_i$  and  $y_j \succ_j x_j]$ . Indeed, in the first case,  $(y_i, y_j, a_{-ij}) \in \mathcal{F}$  and  $x_i \succsim_i y_i$  would imply  $(x_i, y_j, a_{-ij}) \in \mathcal{AF}$ . Using  $x_j \succ_j y_j$ , we would obtain  $(x_i, x_j, a_{-ij}) \in \mathcal{A}$ .

Suppose henceforth that  $x_i \succ_i y_i$  and  $y_j \succ_j x_j$ , the other case being symmetric. Using  $(y_i, y_j, b_{-ij}) \in \mathcal{F}$  and  $y_j \succ_j x_j$ , we obtain  $(y_i, x_j, b_{-ij}) \in \mathcal{U}$ . Using unrestricted solvability, we can find a  $z_i \in X_i$  such that  $(z_i, x_j, b_{-ij}) \in \mathcal{F}$ . Because we know that  $(x_i, x_j, b_{-ij}) \in \mathcal{A}$ ,  $(z_i, x_j, b_{-ij}) \in \mathcal{F}$  and  $(y_i, x_j, b_{-ij}) \in \mathcal{U}$ , we must have  $x_i \succ_i z_i \succ_i y_i$ . Using  $(x_i, x_j, a_{-ij}) \in \mathcal{F}$  and  $x_i \succ_i z_i$ , we obtain  $(z_i, x_j, a_{-ij}) \in \mathcal{U}$ . Similarly,  $(y_i, y_j, a_{-ij}) \in \mathcal{F}$  and  $z_i \succ_i y_i$  imply  $(z_i, y_j, a_{-ij}) \in \mathcal{A}$ . Using unrestricted solvability, we can find a  $z_j \in X_j$  such that  $(z_i, z_j, a_{-ij}) \in \mathcal{F}$ . Because  $(z_i, y_j, a_{-ij}) \in \mathcal{A}$  and  $(z_i, x_j, a_{-ij}) \in \mathcal{U}$ , we must have  $y_j \succ_j z_j \succ_j x_j$ . Since  $(z_i, x_j, b_{-ij}) \in \mathcal{F}$  and  $z_j \succ_j x_j$ , we obtain  $(z_i, z_j, b_{-ij}) \in \mathcal{A}$ .

We have started with the hypothesis that  $x_i \succ_i y_i$ ,  $y_j \succ_j x_j$ ,  $(x_i, x_j, a_{-ij}) \in \mathcal{F}$ ,  $(y_i, y_j, a_{-ij}) \in \mathcal{F}$ ,  $(x_i, x_j, b_{-ij}) \in \mathcal{A}$ , and  $(y_i, y_j, b_{-ij}) \in \mathcal{F}$ . We have shown that we can find  $z_i \in X_i$  and  $z_j \in X_j$  such that  $x_i \succ_i z_i \succ_i y_i$ ,  $y_j \succ_j z_j \succ_j x_j$ ,

$(z_i, z_j, a_{-ij}) \in \mathcal{F}$  and  $(z_i, z_j, b_{-ij}) \in \mathcal{A}$ . Hence, we can iterate the above reasoning with  $z_i$  and  $z_j$  playing the roles of  $x_i$  and  $x_j$ . Rename  $z_i$  and  $z_j$  as  $z_i^1$  and  $z_j^1$ . This process will thus lead to find elements  $z_i^\kappa$  and  $z_j^\kappa$ ,  $\kappa = 1, 2, \dots$  such that  $x_i \succ_i \dots \succ_i z_i^\kappa \succ_i z_i^{\kappa-1} \succ_i \dots \succ_i z_i^1 \succ_i y_i$ ,  $y_j \succ_j \dots \succ_j z_j^\kappa \succ_j z_j^{\kappa-1} \succ_j \dots \succ_j z_j^1 \succ_j x_j$ ,  $(z_i^\kappa, z_j^\kappa, a_{-ij}) \in \mathcal{F}$  and  $(z_i^\kappa, z_j^\kappa, b_{-ij}) \in \mathcal{A}$ , for all  $\kappa$ . By construction, we have  $(z_i^{\kappa+1}, z_j^\kappa, b_{-ij}) \in \mathcal{F}$ , for  $\kappa = 1, 2, \dots$ . The sequence  $z_i^1, z_i^2, \dots$  is infinite. It is strictly bounded by  $x_i$  and  $y_i$ . We know that, for  $\kappa = 1, 2, \dots$ ,  $(z_i^\kappa, z_j^\kappa, a_{-ij}) \in \mathcal{F}$  and  $(z_i^{\kappa+1}, z_j^\kappa, b_{-ij}) \in \mathcal{F}$ . We have observed above that it is impossible that  $a_{-ij} \sim_{-ij} b_{-ij}$ . Hence, we have built an infinite standard sequence that is strictly bounded, violating the Archimedean condition.  $\square$

For  $n \geq 4$ , the conditions of Proposition 8 may therefore be weakened as follows.

**Proposition 39**

*Suppose that  $\langle \mathcal{A}, \mathcal{F}, \mathcal{U} \rangle$  is a non-degenerate ordered covering of a set  $X = X_1 \times X_2 \times \dots \times X_n$  with  $n \geq 4$ . Suppose that  $\langle \mathcal{A}, \mathcal{F}, \mathcal{U} \rangle$  satisfies unrestricted solvability, the Archimedean condition, 1- $\mathcal{A}$ -linear, 1- $\mathcal{F}$ -linear, 2- $\mathcal{A}$ -linear, 1-thin. Then there is an additive representation of  $\langle \mathcal{A}, \mathcal{F}, \mathcal{U} \rangle$ .*

PROOF

Using Lemma 38, we know that 2-thin holds. The proof therefore follows from Lemma 37, Proposition 8 and the observations at the beginning of this section.  $\square$

The next section discusses the independence of the conditions used in Propositions 35 and 39.

## D Examples

When  $n = 3$ , Proposition 35 uses, on top of non-degeneracy and unrestricted solvability, five necessary condition for an additive representation: 1- $\mathcal{A}$ -linear, 1- $\mathcal{F}$ -linear and 1-thin, the Thomsen condition and the Archimedean condition. Let us show that none of these five conditions can be dropped.

**Example 40** (1-thin)

Take  $X = \mathbb{R}^3$  and let

$$\begin{aligned} x \in \mathcal{AF} &\Leftrightarrow x_1 + x_2 + x_3 \geq -1, \\ x \in \mathcal{FU} &\Leftrightarrow x_1 + x_2 + x_3 \leq 1. \end{aligned}$$

It is easy to see that this covering satisfies 1- $\mathcal{A}$ -linear, 1- $\mathcal{F}$ -linear, unrestricted solvability, Thomsen and the Archimedean condition. It clearly violates 1-thinness.  $\diamond$

**Example 41** (Thomsen condition)

Let  $X = \mathbb{R}^3$  and consider the ordered covering such that:

$$\begin{aligned} x \in \mathcal{A} &\Leftrightarrow x_1 + x_2 + \min(x_1, x_2) + x_3 > 0, \\ x \in \mathcal{F} &\Leftrightarrow x_1 + x_2 + \min(x_1, x_2) + x_3 = 0. \end{aligned}$$

It is easy to check that this covering is 1- $\mathcal{A}$ -linear, 1- $\mathcal{F}$ -linear and 1-thin. It satisfies unrestricted solvability as well as the Archimedean condition. The Thomsen condition is violated however since we have  $(18, 0, -18) \in \mathcal{F}$ ,  $(6, 6, -18) \in \mathcal{F}$ ,  $(30, 0, -30) \in \mathcal{F}$ ,  $(6, 18, -30) \in \mathcal{F}$  and  $(30, 6, -36) \in \mathcal{F}$  but  $(18, 18, -36) \in \mathcal{A}$ .  $\diamond$

**Example 42** (1- $\mathcal{A}$ -linear)

Take  $X = (0, 1)^3$  and let

$$\begin{aligned} x \in \mathcal{A} &\Leftrightarrow \begin{cases} x_1 + x_2 + x_3 > 1/2, \\ \text{and} \\ x_1 + x_2 + x_3 \neq 1, \end{cases} \\ x \in \mathcal{F} &\Leftrightarrow x_1 + x_2 + x_3 = 1. \end{aligned}$$

It is easy to check that this covering is 1- $\mathcal{F}$ -linear and 1-thin. It satisfies unrestricted solvability as well as the Archimedean condition and the Thomsen condition. Condition 1- $\mathcal{A}$ -linear is violated since, for instance,  $(1/2, 1/4, 0) \in \mathcal{A}$ ,  $(1/4, 1/2, 0) \in \mathcal{A}$ ,  $(1/4, 1/4, 0) \in \mathcal{F}\mathcal{U}$  and  $(1/2, 1/2, 0) \in \mathcal{F}\mathcal{U}$ .  $\diamond$

**Example 43** (1- $\mathcal{F}$ -linear)

Take  $X = (0, 1)^3$  and let

$$\begin{aligned} x \in \mathcal{A} &\Leftrightarrow x_1 + x_2 + x_3 > 3/2, \\ x \in \mathcal{F} &\Leftrightarrow x_1 + x_2 + x_3 = 1. \end{aligned}$$

It is easy to check that this covering is 1- $\mathcal{A}$ -linear and 1-thin. It satisfies unrestricted solvability as well as the Archimedean condition and the Thomsen condition. Condition 1- $\mathcal{F}$ -linear is violated since, for instance,  $(1/3, 1/3, 1/3) \in \mathcal{F}$ ,  $(1/2, 1/4, 1/4) \in \mathcal{F}$ ,  $(1/3, 1/4, 1/4) \in \mathcal{U}$  and  $(1/2, 1/3, 1/3) \in \mathcal{U}$ .  $\diamond$

**Example 44** (Archimedean condition)

Krantz et al. (1971, Example 4, p. 261) give a general technique to obtain a binary relation satisfying all conditions of classical results of conjoint measurement, except the Archimedean condition. Consider any such binary relation  $\succsim$  on a the set  $X$ .

Let  $\{\beta_i^\kappa \in X_i : \kappa \in \mathbb{K}\}$  be an infinite standard sequence for  $\succsim$  that is strictly bounded. Hence, there are  $a_{-i}, b_{-i} \in X_{-i}$  such that  $\text{Not}[a_{-i} \sim_{-i} b_{-i}]$  and  $(\beta_i^\kappa, a_{-i}) \sim (\beta_i^{\kappa+1}, b_{-i})$ , for all  $\kappa \in \mathbb{K}$ . Furthermore, there are  $\underline{\beta}_i, \bar{\beta}_i \in X_i$  such that, for all  $\kappa \in \mathbb{K}$ ,  $\bar{\beta}_i \succ_i \beta_i^\kappa \succ_i \underline{\beta}_i$ .

Using unrestricted solvability on any attribute other than  $i$ , we can find  $c_j, d_j \in X_j$  and  $\alpha_{-ij} \in X_{-ij}$  such that  $(c_j, \alpha_{-ij}) \sim_{-i} a_{-i}$  and  $(d_j, \alpha_{-ij}) \sim_{-i} b_{-i}$ . Hence, we have  $(\beta_i^\kappa, c_j, \alpha_{-ij}) \sim (\beta_i^{\kappa+1}, d_j, \alpha_{-ij})$ , for all  $\kappa \in \mathbb{K}$ .

Let us now build an ordered covering of  $X$  letting

$$\begin{aligned} x \in \mathcal{A} &\Leftrightarrow x \succ (\beta_i^1, c_j, \alpha_{-ij}), \\ x \in \mathcal{F} &\Leftrightarrow x \sim (\beta_i^1, c_j, \alpha_{-ij}). \end{aligned}$$

It is easy to check that this covering is strongly thin and strongly linear. It is clear that it satisfies unrestricted solvability as well as Thomsen.

By construction, we have  $(\beta_i^1, c_j, \alpha_{-ij}) \sim (\beta_i^2, d_j, \alpha_{-ij})$ , so that  $(\beta_i^1, c_j, \alpha_{-ij}^1) \in \mathcal{F}$  and  $(\beta_i^2, d_j, \alpha_{-ij}^1) \in \mathcal{F}$ , letting  $\alpha_{-ij}^1 = \alpha_{-ij}$ .

We know that  $(\beta_i^2, c_j, \alpha_{-ij}^1) \sim (\beta_i^3, d_j, \alpha_{-ij}^1)$ . Using unrestricted solvability on any attribute other than  $i$  and  $j$ , we can find  $\alpha_{-ij}^2 \in X_{-ij}$  such that  $(\beta_i^2, c_j, \alpha_{-ij}^2) \sim (\beta_i^1, c_j, \alpha_{-ij}^1)$ . This implies  $(\beta_i^2, c_j, \alpha_{-ij}^2) \in \mathcal{F}$ .

Let us show that we also have  $(\beta_i^3, d_j, \alpha_{-ij}^2) \in \mathcal{F}$ , i.e.,  $(\beta_i^3, d_j, \alpha_{-ij}^2) \sim (\beta_i^1, c_j, \alpha_{-ij}^1)$ . Suppose in contradiction with the thesis that  $(\beta_i^3, d_j, \alpha_{-ij}^2) \succ (\beta_i^1, c_j, \alpha_{-ij}^1)$ , the opposite case being dealt with similarly. Because  $(\beta_i^3, d_j, \alpha_{-ij}^2) \succ (\beta_i^1, c_j, \alpha_{-ij}^1)$  and  $(\beta_i^2, c_j, \alpha_{-ij}^2) \sim (\beta_i^1, c_j, \alpha_{-ij}^1)$ , we know that  $(\beta_i^3, d_j, \alpha_{-ij}^2) \succ (\beta_i^2, c_j, \alpha_{-ij}^2)$ . Using the independence of  $\succsim$ , this implies  $(\beta_i^3, d_j, \alpha_{-ij}^1) \succ (\beta_i^2, c_j, \alpha_{-ij}^1)$ , a contradiction.

Iterating the above reasoning shows that to the infinite standard sequence  $\beta_i^\kappa$  corresponds an infinite standard sequence for the ordered covering. It remains to show that this infinite standard sequence is bounded. Because we know that  $\bar{\beta}_i \succ_i \beta_i^\kappa \succ_i \underline{\beta}_i$  and  $(\beta_i^\kappa, c_j, \alpha_{-ij}^\kappa) \sim (\beta_i^1, c_j, \alpha_{-ij}^1)$ , we know that  $(\bar{\beta}_i, c_j, \alpha_{-ij}^\kappa) \succ (\beta_i^\kappa, c_j, \alpha_{-ij}^\kappa) \sim (\beta_i^1, c_j, \alpha_{-ij}^1)$  and  $(\beta_i^1, c_j, \alpha_{-ij}^1) \sim (\beta_i^\kappa, c_j, \alpha_{-ij}^\kappa) \succ (\bar{\beta}_i, c_j, \alpha_{-ij}^\kappa)$ , for all  $\kappa \in \mathbb{K}$ . This implies that  $\bar{\beta}_i \succ_i \beta_i^\kappa$  and  $\beta_i^\kappa \succ_i \underline{\beta}_i$ .  $\diamond$

When  $n \geq 4$ , Proposition 39 uses, on top of non-degeneracy and unrestricted solvability, five necessary conditions: the Archimedean condition, 1- $\mathcal{A}$ -linear, 1- $\mathcal{F}$ -linear, 2- $\mathcal{A}$ -linear, 1-thin. It is not difficult to adapt the above examples to show that none of the Archimedean condition, 1- $\mathcal{F}$ -linear and 1-thin can be omitted from this proposition. The following example show that 2- $\mathcal{A}$ -linear cannot be omitted either.

**Example 45** (2- $\mathcal{A}$ -linear)

Take  $X = \mathbb{R}^4$  and let

$$\begin{aligned} x \in \mathcal{AF} &\Leftrightarrow x_1 + x_2 + \min(x_1, x_2) + x_3 + x_4 + \min(x_3, x_4) \geq 0, \\ x \in \mathcal{FU} &\Leftrightarrow x_1 + x_2 + \min(x_1, x_2) + x_3 + x_4 + \min(x_3, x_4) \leq 0. \end{aligned}$$

It is easy to check that this covering satisfies unrestricted solvability, Archimedean condition as well as 1- $\mathcal{A}$ -linear, 1- $\mathcal{F}$ -linear and 1-thin. 2- $\mathcal{A}$ -linear is violated since, e.g.,  $(5, 5, -4, -4) \in \mathcal{A}$ ,  $(-4, -4, 5, 5) \in \mathcal{A}$  but  $(5, -4, 5, -4) \in \mathcal{U}$  and  $(-4, 5, -4, 5) \in \mathcal{U}$ .  $\diamond$

Unfortunately, we have been unable to show that condition 1- $\mathcal{A}$ -linear cannot be omitted (for more details on this point, see Bouyssou and Marchant, 2008c).

## E More than two categories

### E.1 Setting and model

Let  $r \geq 2$  be an integer. Let  $R = \{1, 2, \dots, r\}$  and  $R^* = \{1, 2, \dots, r-1\}$ . An  $r$ -fold ordered covering of the set  $X$  is a collection of nonempty sets  $\langle C^1, C^2, \dots, C^r \rangle$  such that  $C^1 \cup C^2 \cup \dots \cup C^r = X$  and  $C^k \cap C^\ell = \emptyset$ , for all  $k, \ell \in R$  such that  $|k - \ell| > 1$ .

The alternatives in  $\Upsilon^k = C^k \cap C^{k+1}$  are interpreted as lying at the frontier between categories  $C^k$  and  $C^{k+1}$ . For  $k \in R$ , we define  $\Delta^k = C^k \setminus [C^{k-1} \cup C^{k+1}]$ , with the convention that  $C^0 = C^{r+1} = \emptyset$ . The alternatives in  $\Delta^k$  are therefore the alternatives that belong to  $C^k$  and do not lie at the frontier between  $C^k$  and one of its adjacent categories. We define  $C^{\geq k} = \bigcup_{j=k}^r C^j$ ,  $C^{\leq k} = \bigcup_{j=1}^k C^j$  and  $C^{<k} = \bigcup_{j=1}^{k-1} C^j$ . Furthermore, we let  $\Delta^{\geq k} = \Delta^k \cup C^{\geq k+1}$  and  $\Delta^{\leq k} = \Delta^k \cup C^{<k}$ .

We consider an ordered covering  $\langle C^k \rangle_{k \in R}$ , possibly with  $r \geq 3$ . We are interested in a representation of  $\langle C^k \rangle_{k \in R}$  such that, for all  $x \in X$ , and all  $k \in R$ ,

$$x \in C^k \Leftrightarrow \sigma^{k-1} \leq \sum_{i=1}^n v_i(x_i) \leq \sigma^k, \quad (A^*)$$

with the convention that  $\sigma^0 = -\infty$ ,  $\sigma^r = +\infty$  and where  $\sigma^1, \sigma^2, \dots, \sigma^{r-1}$  are real numbers such that  $\sigma^1 < \sigma^2 < \dots < \sigma^{r-1}$  and  $v_i$  is a real-valued function on  $X_i$ . The basis of our analysis will be the results obtained in Section 3 for two categories.

### E.2 Axioms and result

Our first additional condition strengthens thinness so that if two alternatives differing in only one attribute are caught in a frontier, changing the  $n - 1$  common values of these two alternatives will never allow to distinguish them.

**Definition 46** (Generalized 1-thinness)

We say that  $\langle C^k \rangle_{k \in R}$  satisfies generalized 1-thinness if for all  $h \in R^*$ , all  $i \in N$ , all  $a_i, b_i \in X_i$  and all  $c_{-i}, d_{-i} \in X_{-i}$ ,  $[(a_i, c_{-i}) \in \Upsilon^h \text{ and } (b_i, c_{-i}) \in \Upsilon^h] \Rightarrow [(a_i, d_{-i}) \in C^k \Leftrightarrow (b_i, d_{-i}) \in C^k, \text{ for all } k \in R]$ .

It is easy to see that generalized 1-thinness is necessary for model (A\*). It clearly implies that all the twofold coverings  $\langle C^{\leq k}, C^{\geq k+1} \rangle$  induced by  $\langle C^k \rangle_{k \in R}$  are 1-thin.

Our second additional condition roughly says that two distinct frontiers between categories must have the same shape. It is simple to check that it is necessary for model  $(A^*)$ .

**Definition 47** (Parallelism)

An ordered covering  $\langle C^k \rangle_{k \in R}$  satisfies parallelism if

$$\left. \begin{array}{l} (a_i, a_j, c_{-ij}) \in \Upsilon^h \\ (b_i, b_j, c_{-ij}) \in \Upsilon^h \\ (a_i, a_j, d_{-ij}) \in \Upsilon^k \end{array} \right\} \Rightarrow (b_i, b_j, d_{-ij}) \in \Upsilon^k.$$

for all  $h, k \in R^*$ , all  $a_i, b_i \in X_i$ , all  $a_j, b_j \in X_j$  and all  $c_{-ij}, d_{-ij} \in X_{-ij}$ .

The next condition ensures a minimal consistency between all the twofold coverings that are induced from  $\langle C^k \rangle_{k \in R}$ . It is necessary for model  $(A^*)$ .

**Definition 48** (Mixed-1-linearity)

We say that  $\langle C^k \rangle_{k \in R}$  satisfies mixed-1-linearity if

$$\left. \begin{array}{l} (x_i, a_{-i}) \in \Delta^k \\ \text{and} \\ (y_i, b_{-i}) \in \Delta^h \end{array} \right\} \Rightarrow \left\{ \begin{array}{l} (y_i, a_{-i}) \in \Delta^{\geq k} \\ \text{or} \\ (x_i, b_{-i}) \in \Delta^{\geq h} \end{array} \right.$$

for all  $k, \ell \in R$ , all  $i \in N$ , all  $x_i, y_i \in X_i$  and all  $a_{-ij}, b_{-ij} \in X_{-ij}$ .

Our final additional condition is a strengthening of unrestricted solvability saying that starting with any alternative one can reach any frontier by modifying this alternative on a single attribute. This is a strong condition that is not necessary for model  $(A^*)$ .

**Definition 49** (Unrestricted solvability w.r.t. all frontiers)

An ordered covering  $\langle C^k \rangle_{k \in R}$  satisfies unrestricted solvability w.r.t. all frontiers if, for all  $h \in R^*$ , all  $i \in N$  and all  $a_{-i} \in X_{-i}$ ,  $(a_i, a_{-i}) \in \Upsilon^h$ , for some  $a_i \in X_i$ .

Our main result in this section is the following:

**Proposition 50**

Let  $\langle C^k \rangle_{k \in R}$  be an ordered covering of  $X$  such that:

1.  $\langle C^k \rangle_{k \in R}$  satisfies generalized 1-thinness, mixed-1-linearity, parallelism and unrestricted solvability w.r.t. all frontiers,
2. for some  $k \in R^*$ , the twofold covering  $\langle C^{\leq k}, C^{\geq k+1} \rangle$  is non-degenerate and satisfies [the Archimedean condition, 1- $\mathcal{A}$ -linear, 1- $\mathcal{F}$ -linear and the Thomsen condition (if  $n = 3$ )] or [the Archimedean condition, 1- $\mathcal{A}$ -linear, 1- $\mathcal{F}$ -linear and 2- $\mathcal{A}$ -linear (if  $n \geq 4$ )].

Then, there is an additive representation of  $\langle C^k \rangle_{k \in R}$  in model  $(A^*)$ .

PROOF

Let  $k \in R^*$  be such that  $\langle C^{\leq k}, C^{\geq k+1} \rangle$  satisfies the conditions mentioned in Part 2. Because generalized 1-thinness and unrestricted solvability w.r.t. all frontiers hold, we know using Proposition 35 or 39 that there is an additive representation of  $\langle C^{\leq k}, C^{\geq k+1} \rangle$  such that

$$x \in \Delta^{\geq k+1} \Leftrightarrow \sum_{i=1}^n v_i^{(k)}(x_i) > 0,$$

$$x \in \Upsilon^k \Leftrightarrow \sum_{i=1}^n v_i^{(k)}(x_i) = 0,$$

for all  $x \in X$ .

Let us show that this additive representation also gives an additive representation in model  $(A^*)$ .

Let  $h \in R^*$  with  $h \neq k$ . Consider first any two alternatives  $x, y \in \Upsilon^h$ .

Using unrestricted solvability w.r.t.  $\Upsilon^h$ , we can find  $z_1^2, z_1^3, \dots, z_1^n \in X_1$  such that

$$(z_1^2, y_2, x_3, x_4, x_5, \dots, x_n) \in \Upsilon^h,$$

$$(z_1^3, y_2, y_3, x_4, x_5, \dots, x_n) \in \Upsilon^h,$$

$$(z_1^4, y_2, y_3, y_4, x_5, \dots, x_n) \in \Upsilon^h,$$

$$\vdots$$

$$(z_1^n, y_2, y_3, y_4, y_5, \dots, y_n) \in \Upsilon^h.$$

Using unrestricted solvability w.r.t.  $\Upsilon^k$ , we know that we can find  $w_{-12} \in X_{-12}$  such that  $(x_1, x_2, w_{-12}) \in \Upsilon^k$ . Because  $(x_1, x_2, x_{-12}) \in \Upsilon^h$ ,  $(z_1^2, y_2, x_{-12}) \in \Upsilon^h$  and  $(x_1, x_2, w_{-12}) \in \Upsilon^k$ , parallelism implies that  $(z_1^2, y_2, w_{-12}) \in \Upsilon^k$ . Using the additive representation for  $\langle C^{\leq k}, C^{\geq k+1} \rangle$ , we therefore know that:

$$v_1^{(k)}(x_1) + v_2^{(k)}(x_2) = v_1^{(k)}(z_1^2) + v_2^{(k)}(y_2).$$

Let us now iterate the above reasoning. Using unrestricted solvability w.r.t.  $\Upsilon^k$ , we know that we can find  $w_{-13} \in X_{-13}$  such that  $(z_1^2, x_3, w_{-13}) \in \Upsilon^k$ . Because  $(z_1^2, y_2, x_3, x_{-123}) \in \Upsilon^h$ ,  $(z_1^3, y_2, y_3, x_{-123}) \in \Upsilon^h$  and  $(z_1^2, x_3, w_{-13}) \in \Upsilon^k$ , parallelism implies that  $(z_1^3, y_3, w_{-13}) \in \Upsilon^k$ . Using the additive representation for  $\langle C^{\leq k}, C^{\geq k+1} \rangle$ , we know that:

$$v_1^{(k)}(z_1^2) + v_3^{(k)}(x_3) = v_1^{(k)}(z_1^3) + v_3^{(k)}(y_3).$$

Combining what has been obtained so far shows that:

$$v_1^{(k)}(x_1) + v_2^{(k)}(x_2) + v_3^{(k)}(x_3) = v_1^{(k)}(z_1^3) + v_2^{(k)}(y_2) + v_3^{(k)}(y_3).$$

Iterating the above reasoning easily shows that we must have:

$$\begin{aligned} v_1^{(k)}(x_1) + v_2^{(k)}(x_2) + \cdots + v_{n-1}^{(k)}(x_{n-1}) + v_n^{(k)}(x_n) = \\ v_1^{(k)}(z_1^n) + v_2^{(k)}(y_2) + \cdots + v_{n-1}^{(k)}(y_{n-1}) + v_n^{(k)}(y_n). \end{aligned}$$

Because we have  $(y_1, y_{-1}) \in \Upsilon^h$  and  $(z_1^n, y_{-1}) \in \Upsilon^h$ , generalized thinness implies that  $y_1$  and  $z_1^n$  cannot be distinguished. In view of the proof of Proposition 8, this is easily seen to imply that  $v_1^{(k)}(z_1^n) = v_1^{(k)}(y_1)$ . Hence,  $x, y \in \Upsilon^h$  imply that

$$\begin{aligned} v_1^{(k)}(x_1) + v_2^{(k)}(x_2) + \cdots + v_{n-1}^{(k)}(x_{n-1}) + v_n^{(k)}(x_n) = \\ v_1^{(k)}(y_1) + v_2^{(k)}(y_2) + \cdots + v_{n-1}^{(k)}(y_{n-1}) + v_n^{(k)}(y_n), \end{aligned}$$

so that  $\langle v_1^{(k)}, v_2^{(k)}, \dots, v_n^{(k)} \rangle$  is not only an additive representation of  $\langle C^{\leq k}, C^{\geq k} \rangle$  but is also an additive representation of  $\Upsilon^h$ .

Suppose now that  $x \in \Delta^{\geq h+1}$ . Suppose for definiteness that we have  $x \in \Delta^\ell$  with  $\ell \geq h+1$ . Let us show that if  $w \in \Upsilon^h$ , we must have:

$$\begin{aligned} v_1^{(k)}(x_1) + v_2^{(k)}(x_2) + \cdots + v_{n-1}^{(k)}(x_{n-1}) + v_n^{(k)}(x_n) > \\ v_1^{(k)}(w_1) + v_2^{(k)}(w_2) + \cdots + v_{n-1}^{(k)}(w_{n-1}) + v_n^{(k)}(w_n). \end{aligned}$$

Using unrestricted solvability w.r.t.  $\Upsilon^h$ , we can find  $z_1 \in X_n$  such that  $(z_1, x_{-1}) \in \Upsilon^h$ . In view of what was shown above, it is sufficient to prove that we have:

$$\begin{aligned} v_1^{(k)}(x_1) + v_2^{(k)}(x_2) + \cdots + v_n^{(k)}(x_n) > \\ v_1^{(k)}(z_1) + v_2^{(k)}(x_2) + \cdots + v_n^{(k)}(x_n). \end{aligned}$$

Using unrestricted solvability w.r.t.  $\Upsilon^k$ , we know that we have  $(x_1, z_2, x_{-12}) \in \Upsilon^k$ , for some  $z_2 \in X_1$ . If  $(z_1, z_2, x_{-12}) \in \Upsilon^k$ , we have a clear violation of generalized thinness since  $(z_1, x_2, x_{-12}) \in \Upsilon^h$  and  $(x_1, x_2, x_{-12}) \in \Delta^{\geq h+1}$ . Suppose that we have  $(z_1, z_2, x_{-12}) \in \Delta^{\geq k+1}$ . Suppose for definiteness that  $(z_1, z_2, x_{-12}) \in \Delta^m$  with  $m \geq k+1$ . Since  $(x_1, x_2, x_{-12}) \in \Delta^\ell$  and  $(z_1, z_2, x_{-12}) \in \Delta^m$ , mixed-1-linearity implies either  $(z_1, x_2, x_{-12}) \in \Delta^\ell$  or  $(x_1, z_2, x_{-12}) \in \Delta^m$ . This is contradictory since we know that  $(z_1, x_2, x_{-12}) \in \Upsilon^h$  and  $(x_1, z_2, x_{-12}) \in \Upsilon^k$  with  $\ell \geq h+1$  and  $m \geq k+1$ . Hence, we must have  $(z_1, z_2, x_{-12}) \in \Delta^{\leq k}$ . Since  $(x_1, z_2, x_{-12}) \in \Upsilon^k$ , it follows from the proof of Proposition 8 that

$$\begin{aligned} v_1^{(k)}(x_1) + v_2^{(k)}(z_2) + v_3^{(k)}(x_3) + \cdots + v_n^{(k)}(x_n) > \\ v_1^{(k)}(z_1) + v_2^{(k)}(z_2) + v_3^{(k)}(x_3) + \cdots + v_n^{(k)}(x_n), \end{aligned}$$

which leads to the desired conclusion.

This shows that  $\langle v_1^{(k)}, v_2^{(k)}, \dots, v_n^{(k)} \rangle$  is not only an additive representation of  $\langle C^{\leq k}, C^{\geq k+1} \rangle$  but is also an additive representation of  $\langle C^k \rangle_{k \in R}$  in model  $(A^*)$ .  $\square$

The uniqueness of the representation built in Proposition 50 is as follows.

**Proposition 51**

*Suppose that  $r \geq 3$ . Under the conditions of Proposition 50,  $\langle u_i \rangle_{i \in N}$  and  $\langle v_i \rangle_{i \in N}$  are two additive representations of  $\langle C^k \rangle_{k \in R}$  using the same thresholds  $\sigma^1 < \sigma^2 < \dots < \sigma^{r-1}$  iff there are real numbers  $\beta_1, \beta_2, \dots, \beta_n$  with  $\sum_{i=1}^n \beta_i = 0$  such that for all  $i \in N$  and all  $x_i \in X_i$ ,  $v_i(x_i) = u_i(x_i) + \beta_i$ .*

PROOF

It is obvious that if  $\langle u_i \rangle_{i \in N}$  is an additive representation of  $\langle C^k \rangle_{k \in R}$  using the thresholds  $\sigma^1 < \sigma^2 < \dots < \sigma^{r-1}$  then then  $\langle u_i + \beta_i \rangle_{i \in N}$  with  $\sum_{i=1}^n \beta_i = 0$  is another additive representation of  $\langle C^k \rangle_{k \in R}$  using the same thresholds.

Let  $k$  be the index of a twofold covering satisfying the conditions of Part 2 of Proposition 50. It is clear that any additive representation  $\langle u_i \rangle_{i \in N}$  of  $\langle C^k \rangle_{k \in R}$  is also a representation of  $\langle C^{\leq k}, C^{\geq k+1} \rangle$  using the threshold  $\sigma_k$  for  $\Upsilon^k$ . We know from Proposition 9 the uniqueness of this representation. We may add a constant  $\beta_i$  to each  $u_i$  provided that  $\sum_{i=1}^n \beta_i = 0$ . We may also multiply each of the  $u_i$  by the same positive constant  $\alpha$ . Because we have here fixed the value of each  $x \in \Upsilon^\ell$  to the constant  $\sigma_\ell$ , such a dilatation is now incompatible with keeping the same thresholds. Hence, we must take  $\alpha = 1$ .  $\square$

**Remark 52**

As above, the uniqueness result given above relies on keeping the thresholds  $\sigma^1 < \sigma^2 < \dots < \sigma^{r-1}$  fixed. When they are not, one may choose arbitrarily two of them, via the multiplication of all  $u_i$  by the same positive constant and the addition of a constant  $\beta_i$  to each  $u_i$ . The value of the remaining thresholds is then determined by these choices.  $\bullet$

**Remark 53**

We give below examples showing that none of the additional necessary conditions used above can be dispensed with.

**Example 54**

Let  $X = \mathbb{R}^3$ . Let  $r = 3$ . Consider the ordered covering such that:

$$\begin{aligned} x \in \Delta^1 &\Leftrightarrow x_1 + x_2 + x_3 < 0, \\ x \in \Upsilon^1 &\Leftrightarrow 0 \leq x_1 + x_2 + x_3 \leq 1, \\ x \in \Delta^2 &\Leftrightarrow 1 < x_1 + x_2 + x_3 < 3, \\ x \in \Upsilon^2 &\Leftrightarrow 3 \leq x_1 + x_2 + x_3 \leq 4, \\ x \in \Delta^3 &\Leftrightarrow x_1 + x_2 + x_3 > 4. \end{aligned}$$

Generalized thinness is clearly violated. It is clear that mixed-1-linearity and parallelism hold.  $\diamond$

**Example 55**

Let  $X = \mathbb{R}^3$ . Let  $r = 3$ . Consider the ordered covering such that:

$$\begin{aligned} x \in \Delta^1 &\Leftrightarrow x_1 + x_2 + x_3 + \pi/2 < 0, \\ x \in \Upsilon^1 &\Leftrightarrow x_1 + x_2 + x_3 + \pi/2 = 0, \\ x \in \Delta^2 &\Leftrightarrow \begin{cases} x_1 + x_2 + x_3 + \pi/2 > 0, \\ \text{and} \\ x_1 + x_2 + x_3 + \arctan(x_1 + x_2 + x_3) < 0, \end{cases} \\ x \in \Upsilon^2 &\Leftrightarrow x_1 + x_2 + x_3 + \arctan(x_1 + x_2 + x_3) = 0, \\ x \in \Delta^3 &\Leftrightarrow x_1 + x_2 + x_3 + \arctan(x_1 + x_2 + x_3) > 0, \end{aligned}$$

Generalized thinness and mixed-1-linearity clearly hold. Parallelism is violated.  $\diamond$

**Example 56**

Let  $X = \mathbb{R}^3$ . Let  $r = 3$ . Consider the ordered covering such that:

$$\begin{aligned} x \in \Delta^1 &\Leftrightarrow x_1 + x_2 + x_3 < 0, \\ x \in \Upsilon^1 &\Leftrightarrow x_1 + x_2 + x_3 = 0, \\ x \in \Delta^3 &\Leftrightarrow 0 < x_1 + x_2 + x_3 < 1, \\ x \in \Upsilon^2 &\Leftrightarrow x_1 + x_2 + x_3 = 1, \\ x \in \Delta^2 &\Leftrightarrow 1 < x_1 + x_2 + x_3. \end{aligned}$$

Generalized thinness and parallelism clearly hold. Mixed-1-linearity is violated.  $\diamond$

## F Decomposable representations with more than two categories

### F.1 The models

Goldstein (1991) was the first to suggest the use of conjoint measurement techniques for the analysis of ordered partitions and ordered covering of a set of multiattributed alternatives through decomposable models when  $r = 2$ . His analysis was later generalized in Greco et al. (2001) to an arbitrary number of categories for ordered partitions. The case of ordered coverings with an arbitrary number of categories has not been analyzed in the literature. This section briefly tackles this case.

Let  $\langle C^k \rangle_{k \in R}$  be an ordered covering of  $X$ . We will be interested in conditions allowing to represent this ordered covering in such a way that, for all  $x \in X$  and all  $k \in R$ ,

$$x \in C^k \Leftrightarrow \sigma^{k-1} \leq F(u_1(x_1), u_2(x_2), \dots, u_n(x_n)) \leq \sigma^k, \quad (\text{M0})$$

with the convention that  $\sigma^0 = -\infty, \sigma^r = +\infty$  and where  $\sigma^1, \sigma^2, \dots, \sigma^{r-1}$  are real numbers such that  $\sigma^1 < \sigma^2 < \dots < \sigma^{r-1}$ ,  $u_i$  is a real-valued function on  $X_i$  and  $F$  is a real-valued function on  $\prod_{i=1}^n u_i(X_i)$  that is nondecreasing in all its arguments.

The special case of model (M0) in which the function  $F$  is supposed to be increasing in all its arguments is called model (M1).

In model (M0), we clearly have, for all  $k \in R$ ,

$$\begin{aligned} x \in \Upsilon^k &\Leftrightarrow F(u_1(x_1), u_2(x_2), \dots, u_n(x_n)) = \sigma^k, \\ x \in \Delta^k &\Leftrightarrow \sigma^{k-1} < F(u_1(x_1), u_2(x_2), \dots, u_n(x_n)) < \sigma^k, \end{aligned}$$

(using the convention adopted above, we have  $\Upsilon^r = \emptyset$ ). When  $F$  is only supposed to be nondecreasing in each of its arguments, the fact that alternatives lying on the frontier between two categories have a precise value for  $F$  will not be much of a constraint. Clearly, the situation changes in model (M1). The increasingness of  $F$  will then imply that the frontiers have to be “thin”.

## F.2 Model (M0)

Define on each  $X_i$  the binary relation  $\succsim_i^R$  letting, for all  $x_i, y_i \in X_i$ ,

$$\begin{aligned} x_i \succsim_i^R y_i &\Leftrightarrow \\ &\text{for all } a_{-i} \in X_{-i} \text{ and all } k \in R^*, \left\{ \begin{array}{l} (y_i, a_{-i}) \in C^k \Rightarrow (x_i, a_{-i}) \in C^{\geq k}, \\ (y_i, a_{-i}) \in \Delta^k \Rightarrow (x_i, a_{-i}) \in \Delta^{\geq k}. \end{array} \right\} \end{aligned}$$

We use  $\succ_i^R$  and  $\sim_i^R$  as is usual. It is not difficult to see that  $\succsim_i^R$  is always reflexive and transitive. When  $x_i \succsim_i^R y_i$ , any alternative  $(x_i, a_{-i})$  must belong to a category that is at least as good as the category containing the alternative  $(y_i, a_{-i})$ . Furthermore, if  $(y_i, a_{-i})$  does not lie on the frontier with the category below, the same will be true with  $(x_i, a_{-i})$ . Hence,  $\succsim_i^R$  may be interpreted as an “at least as good as” relation induced on  $X_i$  by the partition  $\langle C^k \rangle_{k \in R}$ . We have:

### Lemma 57

For all  $k \in R^*$  and all  $x, y \in X$ ,

1.  $[y \in C^k \text{ and } x_i \succsim_i^R y_i] \Rightarrow (x_i, y_{-i}) \in C^{\geq k},$
2.  $[y \in \Delta^k \text{ and } x_i \succsim_i^R y_i] \Rightarrow (x_i, y_{-i}) \in \Delta^{\geq k},$

3.  $[x_i \sim_i^R y_i, \text{ for all } i \in N] \Rightarrow [x \in C^k \Leftrightarrow y \in C^k]$ .

PROOF

Parts 1 and 2 are clear from the definition of  $\sim_i^R$ . Part 3 follows.  $\square$

**Definition 58**

Let  $\langle C^k \rangle_{k \in R}$  be an ordered covering of  $X$  and  $i \in N$ . We say that  $\langle C^k \rangle_{k \in R}$  is

1.  $\Delta$ -linear <sub>$i$</sub>  if

$$\left. \begin{array}{l} (x_i, a_{-i}) \in \Delta^k \\ \text{and} \\ (y_i, b_{-i}) \in \Delta^\ell \end{array} \right\} \Rightarrow \left\{ \begin{array}{l} (y_i, a_{-i}) \in \Delta^{\geq k} \\ \text{or} \\ (x_i, b_{-i}) \in \Delta^{\geq \ell} \end{array} \right. \quad (\Delta\text{-linear}_i)$$

2.  $C$ -linear <sub>$i$</sub>  if

$$\left. \begin{array}{l} (x_i, a_{-i}) \in C^k \\ \text{and} \\ (y_i, b_{-i}) \in C^\ell \end{array} \right\} \Rightarrow \left\{ \begin{array}{l} (y_i, a_{-i}) \in C^{\geq k} \\ \text{or} \\ (x_i, b_{-i}) \in C^{\geq \ell} \end{array} \right. \quad (C\text{-linear}_i)$$

3.  $\Delta C$ -linear <sub>$i$</sub>  if

$$\left. \begin{array}{l} (x_i, a_{-i}) \in \Delta^k \\ \text{and} \\ (y_i, b_{-i}) \in C^\ell \end{array} \right\} \Rightarrow \left\{ \begin{array}{l} (y_i, a_{-i}) \in \Delta^{\geq k} \\ \text{or} \\ (x_i, b_{-i}) \in C^{\geq \ell} \end{array} \right. \quad (\Delta C\text{-linear}_i)$$

for all  $x_i, y_i \in X_i$ ,  $k, \ell \in R$  and  $a_{-i}, b_{-i} \in X_{-i}$ . We say that  $\langle C^k \rangle_{k \in R}$  is  $\Delta$ -linear (resp.  $C$ -linear,  $\Delta C$ -linear) if it is  $\Delta$ -linear <sub>$i$</sub>  (resp.  $C$ -linear <sub>$i$</sub> ,  $\Delta C$ -linear <sub>$i$</sub> ) for all  $i \in N$ . We say that a covering is linear <sub>$i$</sub>  <sup>$R$</sup>  if it is  $\Delta$ -linear <sub>$i$</sub> ,  $C$ -linear <sub>$i$</sub>  and  $\Delta C$ -linear <sub>$i$</sub> . We say that a covering is linear <sup>$R$</sup>  if it is  $\Delta$ -linear,  $C$ -linear and  $\Delta C$ -linear.

Note that condition  $\Delta$ -linear <sub>$i$</sub>  is identical to the ‘‘mixed-1-linearity’’ condition used in Appendix E.

**Remark 59**

In all what follows, it is easy to check that, presence of  $\Delta$ -linear <sub>$i$</sub> , conditions  $C$ -linear <sub>$i$</sub>  and  $\Delta C$ -linear <sub>$i$</sub>  can be replaced by the conjunction of the following two weaker conditions:

$$\left. \begin{array}{l} (x_i, a_{-i}) \in \Upsilon^k \\ \text{and} \\ (y_i, b_{-i}) \in \Upsilon^\ell \end{array} \right\} \Rightarrow \left\{ \begin{array}{l} (y_i, a_{-i}) \in C^{\geq k} \\ \text{or} \\ (x_i, b_{-i}) \in C^{\geq \ell}, \end{array} \right.$$

$$\left. \begin{array}{l} (x_i, a_{-i}) \in \Delta^k \\ \text{and} \\ (y_i, b_{-i}) \in \Upsilon^\ell \end{array} \right\} \Rightarrow \left\{ \begin{array}{l} (y_i, a_{-i}) \in \Delta^{\geq k} \\ \text{or} \\ (x_i, b_{-i}) \in C^{\geq \ell}, \end{array} \right.$$

for all  $x_i, y_i \in X_i$ ,  $k, \ell \in R$  and  $a_{-i}, b_{-i} \in X_{-i}$  •

Some consequences of these linearity conditions are noted below.

**Lemma 60**

1. A partition  $\langle C^k \rangle_{k \in R}$  satisfies  $\Delta$ -linear<sub>*i*</sub>,  $C$ -linear<sub>*i*</sub> and  $\Delta C$ -linear<sub>*i*</sub> iff  $\succsim_i^R$  is complete.
2. Conditions  $\Delta$ -linear<sub>*i*</sub>,  $C$ -linear<sub>*i*</sub> and  $\Delta C$ -linear<sub>*i*</sub> are independent.

PROOF

Part 1. Suppose that  $\Delta$ -linear<sub>*i*</sub> is violated, so that  $(x_i, a_{-i}) \in \Delta^k$ ,  $(y_i, b_{-i}) \in \Delta^\ell$ ,  $(y_i, a_{-i}) \notin \Delta^{\geq k}$  and  $(x_i, b_{-i}) \notin \Delta^{\geq \ell}$ , for some  $x_i, y_i \in X_i$ ,  $k, \ell \in R$  and  $a_{-i}, b_{-i} \in X_{-i}$ . By definition of  $\succsim_i^R$ ,  $(x_i, a_{-i}) \in \Delta^k$  and  $(y_i, a_{-i}) \notin \Delta^{\geq k}$  imply  $\text{Not}[y_i \succsim_i^R x_i]$ . Similarly,  $(y_i, b_{-i}) \in \Delta^\ell$  and  $(x_i, b_{-i}) \notin \Delta^{\geq \ell}$  imply  $\text{Not}[x_i \succsim_i^R y_i]$ . A similar reasoning shows that a violation of  $C$ -linear<sub>*i*</sub> or  $\Delta C$ -linear<sub>*i*</sub> leads to violating the completeness of  $\succsim_i^R$ . Hence, the completeness of  $\succsim_i^R$  implies conditions  $\Delta$ -linear<sub>*i*</sub>,  $C$ -linear<sub>*i*</sub> and  $\Delta C$ -linear<sub>*i*</sub>.

Conversely, suppose that  $\succsim_i^R$  is not complete, so that, for some  $x_i, y_i \in X[i]$ , we have  $\text{Not}[x_i \succsim_i^R y_i]$  and  $\text{Not}[y_i \succsim_i^R x_i]$ . From the definition of  $\succsim_i^R$ , we know that  $\text{Not}[x_i \succsim_i^R y_i]$  implies either

$$(y_i, a_{-i}) \in C^k \text{ and } (x_i, a_{-i}) \notin C^{\geq k}, \quad (29a)$$

or

$$(y_i, a_{-i}) \in \Delta^k \text{ and } (x_i, a_{-i}) \notin \Delta^{\geq k}, \quad (29b)$$

for some  $a_{-i} \in X_{-i}$ . Similarly,  $\text{Not}[y_i \succsim_i^R x_i]$  implies either

$$(x_i, b_{-i}) \in C^k \text{ and } (y_i, b_{-i}) \notin C^{\geq k}, \quad (29c)$$

or

$$(x_i, b_{-i}) \in \Delta^k \text{ and } (y_i, b_{-i}) \notin \Delta^{\geq k}, \quad (29d)$$

for some  $b_{-i} \in X_{-i}$ . It is easy to see that the conjunction of (29a) and (29c) violates  $C$ -linear<sub>*i*</sub>. Similarly, the conjunction of (29b) and (29d) violates  $\Delta$ -linear<sub>*i*</sub>. Finally the conjunction of either (29a) and (29d) or (29b) and (29c) leads to a violation of  $\Delta C$ -linear<sub>*i*</sub>.

Part 2. We provide below the required three examples. In these three examples, we have  $X = \{x_1, y_1\} \times \{x_2, y_2\} \times \{x_3, y_3\}$  and  $r = 2$ . For notational convenience, define  $\mathcal{A} = \Delta^2$ ,  $\mathcal{U} = \Delta^1$ ,  $\mathcal{F} = \Upsilon^1$  and  $C^2 = \mathcal{A} \cup \mathcal{F} = \mathcal{AF}$ .

In this context, let

$$x_i \succsim_i^{\mathcal{A}} y_i \Leftrightarrow [\text{for all } a_{-i} \in X_{-i}, (y_i, a_{-i}) \in \mathcal{A} \Rightarrow (x_i, a_{-i}) \in \mathcal{A}],$$

$$x_i \succsim_i^{\mathcal{AF}} y_i \Leftrightarrow [\text{for all } a_{-i} \in X_{-i}, (y_i, a_{-i}) \in \mathcal{AF} \Rightarrow (x_i, a_{-i}) \in \mathcal{AF}],$$

for all  $x_i, y_i \in X_i$ ,

It is easy to see that the binary relations  $\succsim_i^{\mathcal{A}}$ ,  $\succsim_i^{\mathcal{AF}}$  are always reflexive and transitive. It is not difficult to check that:

- $\Delta$ -linear<sub>*i*</sub> holds iff  $\succsim_i^{\mathcal{A}}$  is complete,
- $C$ -linear<sub>*i*</sub> holds iff  $\succsim_i^{\mathcal{AF}}$  is complete,
- $\Delta C$ -linear<sub>*i*</sub> holds iff  $[Not[x_i \succsim_i^{\mathcal{A}} y_i] \Rightarrow y_i \succsim_i^{\mathcal{AF}} x_i]$ .

### Example 61

Consider the ordered covering such that:

$$\mathcal{A} = \{(x_1, x_2, x_3), (y_1, x_2, x_3), (y_1, y_2, x_3), (x_1, x_2, y_3)\},$$

$$\mathcal{F} = \{(x_1, y_2, x_3), (y_1, x_2, y_3)\},$$

$$\mathcal{U} = X \setminus [\mathcal{A} \cup \mathcal{F}].$$

It is easy to check that  $x_2 \succsim_2^{\mathcal{A}} y_2$ ,  $x_2 \succsim_2^{\mathcal{AF}} y_2$ ,  $x_3 \succsim_3^{\mathcal{A}} y_3$  and  $x_3 \succsim_3^{\mathcal{AF}} y_3$ . This shows that the covering is linear on attributes 2 and 3.

We have  $x_1 \succsim_1^{\mathcal{AF}} y_1$ . However,  $(x_1, x_2, y_3) \in \mathcal{A}$ ,  $(y_1, y_2, x_3) \in \mathcal{A}$ ,  $(y_1, x_2, y_3) \in \mathcal{F}$  and  $(x_1, y_2, x_3) \in \mathcal{F}$  implies that  $Not[x_1 \succsim_1^{\mathcal{A}} y_1]$  and  $Not[y_1 \succsim_1^{\mathcal{A}} x_1]$ . Hence, this ordered covering satisfies  $C$ -linear<sub>1</sub> and  $\Delta C$ -linear<sub>1</sub> but violates  $\Delta$ -linear<sub>1</sub>.  $\diamond$

### Example 62

Consider the ordered covering such that:

$$\mathcal{A} = \{(x_1, x_2, y_3), (y_1, x_2, y_3)\},$$

$$\mathcal{F} = \{(x_1, x_2, x_3), (y_1, y_2, y_3)\},$$

$$\mathcal{U} = X \setminus [\mathcal{A} \cup \mathcal{F}].$$

It is easy to check that  $x_2 \succsim_2^{\mathcal{A}} y_2$ ,  $x_2 \succsim_2^{\mathcal{AF}} y_2$ ,  $y_3 \succsim_3^{\mathcal{A}} x_3$  and  $y_3 \succsim_3^{\mathcal{AF}} x_3$ . This shows that the covering is linear on attributes 2 and 3.

We have  $x_1 \succsim_1^{\mathcal{A}} y_1$ . However,  $(x_1, x_2, x_3) \in \mathcal{F}$ ,  $(y_1, y_2, y_3) \in \mathcal{F}$ ,  $(x_1, y_2, y_3) \in \mathcal{U}$  and  $(y_1, x_2, x_3) \in \mathcal{U}$  implies that  $Not[x_1 \succsim_1^{\mathcal{AF}} y_1]$  and  $Not[y_1 \succsim_1^{\mathcal{AF}} x_1]$ . Hence, this ordered covering satisfies  $\Delta$ -linear<sub>1</sub> and  $\Delta C$ -linear<sub>1</sub> but violates  $C$ -linear<sub>1</sub>.  $\diamond$

**Example 63**

Consider the ordered covering such that:

$$\begin{aligned}\mathcal{A} &= \{(x_1, x_2, y_3), (y_1, x_2, y_3), (y_1, x_2, x_3)\}, \\ \mathcal{F} &= \{(x_1, x_2, x_3), (x_1, y_2, y_3)\}, \\ \mathcal{U} &= X \setminus [\mathcal{A} \cup \mathcal{F}].\end{aligned}$$

It is easy to check that  $x_2 \succ_2^{\mathcal{A}} y_2$ ,  $x_2 \succ_2^{\mathcal{F}} y_2$ ,  $y_3 \succ_3^{\mathcal{A}} x_3$  and  $y_3 \succ_3^{\mathcal{F}} x_3$ . This shows that the covering is linear on attributes 2 and 3.

We have  $y_1 \succ_1^{\mathcal{A}} x_1$  and  $x_1 \succ_1^{\mathcal{F}} y_1$ . Hence, this ordered covering satisfies  $\Delta$ -linear<sub>1</sub> and  $C$ -linear<sub>1</sub> but violates  $\Delta C$ -linear<sub>1</sub>.  $\diamond$   
 $\square$

The following lemma shows that linear<sup>R</sup> is a necessary condition for model (M0) and connects the functions  $u_i$  in this model with the relations  $\succ_i^R$ .

**Lemma 64**

Suppose that an ordered covering  $\langle C^k \rangle_{k \in R}$  has a representation in model (M0). Then:

1. it is linear<sup>R</sup>,
2. for all  $i \in N$  and  $x_i, y_i \in X_i$ ,

$$x_i \succ_i^R y_i \Rightarrow u_i(x_i) > u_i(y_i). \quad (30)$$

PROOF

Part 1. Suppose that  $(x_i, a_{-i}) \in C^k$  and  $(y_i, b_{-i}) \in C^\ell$ , so that

$$\begin{aligned}\sigma^{k-1} &\leq F(u_1(a_1), \dots, u_{i-1}(a_{i-1}), u_i(x_i), u_{i+1}(a_{i+1}), \dots, u_n(a_n)) \leq \sigma^k \text{ and} \\ \sigma^{\ell-1} &\leq F(u_1(b_1), \dots, u_{i-1}(b_{i-1}), u_i(y_i), u_{i+1}(b_{i+1}), \dots, u_n(b_n)) \leq \sigma^\ell.\end{aligned}$$

We have either  $u_i(y_i) \geq u_i(x_i)$  or  $u_i(x_i) \geq u_i(y_i)$ . Using the nondecreasingness of  $F$ , this implies either

$$\begin{aligned}\sigma^{k-1} &\leq F(u_1(a_1), \dots, u_{i-1}(a_{i-1}), u_i(y_i), u_{i+1}(a_{i+1}), \dots, u_n(a_n)) \text{ or} \\ \sigma^{\ell-1} &\leq F(u_1(b_1), \dots, u_{i-1}(b_{i-1}), u_i(x_i), u_{i+1}(b_{i+1}), \dots, u_n(b_n)).\end{aligned}$$

Hence, model (M0) implies that we have either  $(y_i, a_{-i}) \in C^{\geq k}$  or  $(x_i, b_{-i}) \in C^{\geq \ell}$ , as required by  $C$ -linear <sub>$i$</sub> . The proof for  $\Delta$ -linear <sub>$i$</sub>  and  $\Delta C$ -linear <sub>$i$</sub>  is similar.

Part 2. Suppose that  $u_i(x_i) \leq u_i(y_i)$ . Using the nondecreasingness of  $F$ ,

$$\begin{aligned}\sigma^k &\leq F(u_1(a_1), \dots, u_{i-1}(a_{i-1}), u_i(x_i), u_{i+1}(a_{i+1}), \dots, u_n(a_n)) \leq \sigma^{k+1} \text{ implies} \\ \sigma^k &\leq F(u_1(a_1), \dots, u_{i-1}(a_{i-1}), u_i(y_i), u_{i+1}(a_{i+1}), \dots, u_n(a_n)).\end{aligned}$$

Similarly, we have

$$\begin{aligned} \sigma^k < F(u_1(a_1), \dots, u_{i-1}(a_{i-1}), u_i(x_i), u_{i+1}(a_{i+1}), \dots, u_n(a_n)) \leq \sigma^{k+1} \text{ implies} \\ \sigma^k < F(u_1(a_1), \dots, u_{i-1}(a_{i-1}), u_i(y_i), u_{i+1}(a_{i+1}), \dots, u_n(a_n)). \end{aligned}$$

This shows that, for all  $a_{-i} \in X_{-i}$ ,  $(x_i, a_{-i}) \in C^k$  implies  $(y_i, a_{-i}) \in C^{\geq k}$  and  $(x_i, a_{-i}) \in \Delta^k$  implies  $(y_i, a_{-i}) \in \Delta^{\geq k}$ , so that  $y_i \succsim_i^R x_i$ .  $\square$

Omitting the cumbersome formulation of the order-denseness condition in terms of the partition  $\langle C^k \rangle_{k \in R}$ , this leads to:

**Proposition 65**

*An ordered covering  $\langle C^k \rangle_{k \in R}$  has a representation in model (M0) iff it is linear<sup>R</sup> and, for all  $i \in N$ , there is a finite or countably infinite set  $X'_i \subseteq X_i$  that is dense in  $X_i$  for  $\succsim_i^R$ .*

PROOF

The necessity of linearity follows from Part 1 of Lemma 64. Using Part 2 of Lemma 64, we know that the the weak order induced on  $X_i$  by  $u_i$  always refines  $\succsim_i^R$ . Hence, there is a finite or countably infinite set  $X'_i \subseteq X_i$  that is dense in  $X_i$  for  $\succsim_i^R$ .

Sufficiency. Using Part 1 of Lemma 60, we know that  $\succsim_i^R$  is a weak order. Since there is a finite or countably infinite set  $X'_i \subseteq X_i$  that is dense in  $X_i$  for  $\succsim_i^R$ , there is a real-valued function  $u_i$  on  $X_i$  such that, for all  $x_i, y_i \in X_i$ ,

$$x_i \succsim_i^R y_i \Leftrightarrow u_i(x_i) \geq u_i(y_i). \tag{31}$$

Consider, on each  $i \in N$  any function  $u_i$  satisfying (31) and take any  $\sigma^0, \sigma^1, \dots, \sigma^r \in \mathbb{R}$  such that  $\sigma^0 < \sigma^1 < \dots < \sigma^r$ . For all  $k \in \{1, 2, \dots, r\}$ , consider any increasing function  $\phi_k$  mapping  $\mathbb{R}$  into  $(\sigma^{k-1}, \sigma^k)$ . Define  $F$  on  $\prod_{i=1}^n u_i(X_i)$  letting, for all  $x \in X$  and all  $k \in R$ ,

$$F(u_1(x_1), u_2(x_2), \dots, u_n(x_n)) = \begin{cases} \phi_k(\sum_{i=1}^n u_i(x_i)) & \text{if } x \in \Delta^k, \\ \sigma^k & \text{if } x \in \Upsilon^k. \end{cases} \tag{32}$$

The well-definedness of  $F$  follows from Part 3 of Lemma 57 together with (31). Its nondecreasingness is easily shown using the definition of  $u_i$  and Parts 1 and 2 of Lemma 57.  $\square$

The above proposition generalizes Part 3 of Theorem 1 in Goldstein (1991, p. 70) to the case of an arbitrary number of category.

**Remark 66**

The uniqueness of the representation in model (M0) is easily analyzed. Observe first that the thresholds  $\sigma^2, \sigma^3, \dots, \sigma^{r-1}$  maybe be chosen arbitrarily provided that they satisfy:

$$\sigma^2 < \sigma^3 < \dots < \sigma^{r-1}. \quad (33)$$

Part 2 of Lemma 64 has shown that in all representations of  $\langle C^k \rangle_{k \in R}$  in model (M0),  $u_i$  must satisfy (30). Consider, independently on each attribute any function  $u_i$  satisfying (30). For  $k = 2, 3, \dots, r - 1$ , consider any function  $f_k$  from  $\mathbb{R}^n$  into  $(\sigma^{k-1}, \sigma^k)$  being nondecreasing in each of its arguments. Take any function  $f_1$  (resp.  $f_r$ ) from  $\mathbb{R}^n$  into  $(-\infty, \sigma^1)$  (resp.  $(\sigma^{r-1}, +\infty)$ ) that is nondecreasing in each of its arguments. Define the function the real-valued  $F$  on  $\prod_{i=1}^n u_i(X_i)$  letting, for all  $x \in X$ ,

$$F(u_1(x_1), \dots, u_n(x_n)) = \begin{cases} f_k(u_1(x_1), \dots, u_n(x_n)) & \text{if } x \in \Delta^k, \\ \sigma^k & \text{if } x \in \Upsilon^k. \end{cases} \quad (34)$$

It is clear that this defines a representation of  $\langle C^k \rangle_{k \in R}$  in model (M0). It is not difficult to see that only such representations may be used. Therefore, the combination of (30), (33), and (34) describe the set of all representations of  $\langle C^k \rangle_{k \in R}$  in model (M0). •

### F.3 Model (M1)

Since model (M0) only requires the function  $F$  to be nondecreasing, it should be apparent that this model does not deal with alternatives at the frontier between two categories in a special way. Things change with model (M1) that requires  $F$  to be increasing in all its arguments. This motivates the introduction of the following condition.

**Definition 67**

We say that the covering  $\langle C^k \rangle_{k \in R}$  satisfies  $\text{thin}_i^R$  if,

$$\left. \begin{array}{l} (x_i, a_{-i}) \in \Upsilon^k \\ \text{and} \\ (y_i, a_{-i}) \in \Upsilon^k \end{array} \right\} \Rightarrow [(x_i, b_{-i}) \in C^\ell \Leftrightarrow (y_i, b_{-i}) \in C^\ell]$$

for all  $x_i, y_i \in X_i$ ,  $k, \ell \in R$  and  $a_{-i}, b_{-i} \in X_{-i}$ . We say that  $\langle C^k \rangle_{k \in R}$  is  $\text{thin}^R$  if satisfies  $\text{thin}_i^R$  on all attributes  $i \in N$ .

This condition is identical to the condition called in ‘‘Generalized 1-thinness’’ in Appendix E. Intuitively, a covering satisfies  $\text{thin}_i^R$  if, as soon as two distinct levels on  $X_i$  are caught in a frontier when they are associated with the same evaluations

on the attributes other than  $i$ , it will be impossible for these two levels to be distinguished, i.e., they will be linked by  $\sim_i^R$ . It is easy to see that  $\text{thin}_i^R$  is a necessary condition for model (M1). Indeed, the premise of  $\text{thin}_i^R$  implies that:

$$\begin{aligned} F(u_1(a_1), \dots, u_{i-1}(a_{i-1}), u_i(x_i), u_{i+1}(a_{i+1}), \dots, u_n(a_n)) = \\ F(u_1(a_1), \dots, u_{i-1}(a_{i-1}), u_i(y_i), u_{i+1}(a_{i+1}), \dots, u_n(a_n)). \end{aligned}$$

Because  $F$  is increasing, this can only happen if  $u_i(x_i) = u_i(y_i)$ , which leads to the desired conclusion.

The following lemma takes note of some important consequences of thinness when combined with linearity.

**Lemma 68**

1. Suppose that  $\langle C^k \rangle_{k \in R}$  satisfies  $\text{linear}_i^R$  and  $\text{thin}_i^R$  on attribute  $i \in N$ . Then

$$\begin{aligned} [(x_i, a_{-i}) \in \Upsilon^k \text{ and } y_i \succ_i^R x_i] &\Rightarrow (x_i, a_{-i}) \in \Delta^{\geq k}, \\ [(x_i, a_{-i}) \in \Upsilon^k \text{ and } x_i \succ_i^R z_i] &\Rightarrow (z_i, a_{-i}) \notin \Delta^{\geq k}. \end{aligned}$$

for all  $x_i, y_i, z_i \in X_i$ ,  $k \in R^*$  and  $a_{-i} \in X_{-i}$ .

2. Conditions  $\Delta$ - $\text{linear}_i$ ,  $C$ - $\text{linear}_i$ ,  $\Delta C$ - $\text{linear}_i$  and  $\text{thin}_i^R$  are independent.

PROOF

Part 1. Suppose that  $(x_i, a_{-i}) \in \Upsilon^k$  and  $y_i \succ_i^R x_i$ . Because  $y_i \succ_i^R x_i$ , we know that  $(y_i, a_{-i}) \in C^{\geq k}$ . Suppose that  $(y_i, a_{-i}) \in \Upsilon^k$ . Using  $\text{thin}_i^R$ ,  $(x_i, a_{-i}) \in \Upsilon^k$  and  $(y_i, a_{-i}) \in \Upsilon^k$  imply  $x_i \sim_i^R y_i$ , a contradiction. The proof of the other implication is similar.

Part 2. It is easy to check that in Examples 61, 62 and 63 the covering considered satisfy  $\text{thin}^R$ . We give below the remaining example, using notation introduced before Example 61.

**Example 69**

Consider the ordered covering such that:

$$\begin{aligned} \mathcal{A} &= \{(x_1, x_2, x_3), (x_1, y_2, x_3), (y_1, x_2, x_3)\}, \\ \mathcal{F} &= \{(x_1, x_2, y_3), (y_1, x_2, y_3)\}, \\ \mathcal{U} &= X \setminus [\mathcal{A} \cup \mathcal{F}]. \end{aligned}$$

Conditions  $\text{thin}_2^R$  and  $\text{thin}_3^R$  are trivially satisfied. Condition  $\text{thin}_1^R$  is violated since  $(x_1, x_2, y_3) \in \mathcal{F}$  and  $(y_1, x_2, y_3) \in \mathcal{F}$ ,  $(x_1, y_2, x_3) \in \mathcal{A}$  and  $(y_1, y_2, x_3) \in \mathcal{U}$ . It is simple to check that we have  $x_i \succ_i^{\mathcal{A}} y_i$  and  $x_i \succ_i^{\mathcal{F}} y_i$ , for all  $i \in N$ , which show that the covering satisfies  $\text{linear}^R$ . ◇

□

The following proposition generalizes Part 4 of Theorem 1 in Goldstein (1991, p. 70) to the case of an arbitrary number of category.

**Proposition 70**

A covering  $\langle C^k \rangle_{k \in R}$  has a representation in model (M1) iff it satisfies linear<sup>R</sup>, thin<sup>R</sup> and, for all  $i \in N$ , there is a finite or countably infinite set  $X'_i \subseteq X_i$  that is dense in  $X_i$  for  $\succsim_i^R$ .

PROOF

Necessity. Since model (M1) implies model (M0), the necessity of linearity and the order-denseness conditions follows from Proposition 65. The necessity of thinness was shown above.

Sufficiency. Define  $u_i$  and  $F$  as in the proof of Proposition 65. The well-definedness of  $F$  follows from Part 3 of Lemma 57 together with (31). Suppose that  $u_i(x_i) > u_i(y_i)$ , so that  $x_i \succ_i^R y_i$ . If  $(y_i, a_{-i}) \in \Delta^k$ , for some  $k \in R$ , Part 3 of Lemma 57 implies that  $(y_i, a_{-i}) \in \Delta^{\geq k}$ . The conclusion therefore follows from the definition of  $F$ . If  $(y_i, a_{-i}) \in \Upsilon^k$ , for some  $k \in R$ , Part 1 of Lemma 68 implies that  $(x_i, a_{-i}) \in \Delta^{\geq k}$ . The conclusion therefore follows from the definition of  $F$ .  $\square$

**Remark 71**

The uniqueness of the representation in model (M1) is only slightly stronger than what was the case with model (M0). It is first clear that the thresholds can be chosen arbitrarily provided that they satisfy (33). Similarly, the functions  $u_i$  must satisfy (30). If, for some  $x_i, y_i \in X_i$  and some  $a_{-i} \in X_{-i}$ , we have  $(x_i, a_{-i}) \in \Upsilon^k$  and  $(y_i, a_{-i}) \in \Upsilon^k$ , for some  $k \in R$ , model (M1) implies that we must have  $u_i(x_i) = u_i(y_i)$ . It is not difficult to see that these are the only constraints on  $u_i$ . Therefore, we may, independently on each attribute  $i \in N$ , choose any function  $u_i$  such that, for all  $x_i, y_i \in X_i$ ,

$$\begin{aligned} x_i \succ_i^R y_i &\Rightarrow u_i(x_i) > u_i(y_i), \\ (x_i, a_{-i}) \in \Upsilon^k \text{ and } (y_i, a_{-i}) \in \Upsilon^k, \text{ for some } k \in R, &\Rightarrow u_i(x_i) = u_i(y_i). \end{aligned} \tag{35}$$

For  $k = 2, 3, \dots, r - 1$ , consider any function  $f_k$  from  $\mathbb{R}^n$  into  $(\sigma^{k-1}, \sigma^k)$  being increasing in each of its arguments. Take any function  $f_1$  (resp.  $f_r$ ) from  $\mathbb{R}^n$  into  $(-\infty, \sigma^1)$  (resp.  $(\sigma^{r-1}, +\infty)$ ) that is increasing in each of its arguments. Define the function using (34). It is easy to see that this leads to representation of  $\langle C^k \rangle_{k \in R}$  in model (M1) and that only such representations may be used.  $\bullet$

Consider an ordered covering  $\langle C^k \rangle_{k \in R}$  such that, for all  $k \in R$ ,  $\Upsilon^k = \emptyset$ , i.e., an ordered partition. This implies that, for all  $k \in R$ ,  $C^k = \Delta^k$  and  $C^{\geq k} = \Delta^{\geq k}$ . It is then easy to see that, for all  $i \in N$ , conditions  $\Delta$ -linear<sub>i</sub>,  $C$ -linear<sub>i</sub> and  $\Delta C$ -linear<sub>i</sub> become equivalent. Furthermore, condition thin<sub>i</sub><sup>R</sup> is always trivially satisfied. This allows to state the following corollary of Propositions 65 and 70.

**Proposition 72**

Let  $\langle C^k \rangle_{k \in R}$  be an ordered partition of  $X$ . There are real numbers  $\sigma^0 < \sigma^1 < \dots < \sigma^r$  and real-valued functions  $u_i$  on  $X_i$  and  $F$  on  $\prod_{i=1}^n u_i(X_i)$  with  $F$  being increasing in all its arguments such that, for all  $x \in X$  and all  $k \in R$ ,

$$x \in C^k \Leftrightarrow \sigma^{k-1} < F(u_1(x_1), u_2(x_2), \dots, u_n(x_n)) < \sigma^k, \quad (36)$$

iff  $\langle C^k \rangle_{k \in R}$  is  $\Delta$ -linear and, for all  $i \in N$  there is a finite or countably infinite set  $X'_i \subseteq X_i$  that is dense in  $X_i$  for  $\succsim_i^R$ .

This proposition generalizes Theorem 2 in Goldstein (1991, p. 72) to the case of an arbitrary number of category. Greco et al. (2001) and Słowiński et al. (2002) state a version of this result when  $X$  is finite or countably infinite. Bouyssou and Marchant (2007b) have given a direct proof of Proposition 72. It is easy to see that for an ordered partition, there is a representation (36) with  $F$  increasing in all its arguments iff there is a representation (36) with  $F$  nondecreasing in all its arguments. Again, the distinction between increasingness and nondecreasingness only matters if there are alternatives at the frontier between categories.

## F.4 Extensions

In the  $r = 2$  case, Goldstein (1991) has suggested two weaker forms of model (M0) and has given a complete characterization of these two variants. It is not difficult to extend this analysis to the general case.

The weakest form, called model (M), is obtained from model (M0) by removing the requirement that  $F$  is nondecreasing in each of its arguments. It is easy to characterize this model. Remember that the relation  $\sim_i^R$  is an equivalence. Suppose that model (M) holds and that  $\text{Not}[x_i \sim_i^R y_i]$ . This clearly implies that we must have  $u_i(x_i) \neq u_i(y_i)$ . Since  $u_i$  is real-valued, this implies that there must exist a one-to-one correspondence between  $X_i / \sim_i^R$  and some subset of  $\mathbb{R}$ . Conversely, suppose that this condition is satisfied, for all  $i \in N$ . This implies that, for all  $i \in N$ , there is a real-valued function  $u_i$  such that

$$x_i \sim_i^R y_i \Leftrightarrow u_i(x_i) = u_i(y_i). \quad (37)$$

Take on each attribute  $i \in N$  any function  $u_i$  satisfying (37) and define  $F$  as in the proof of Proposition 65. The well-definedness of  $F$  follows from Part 3 of Lemma 57 together with (37). This shows that as soon as the cardinality of  $X_i / \sim_i^R$  is not too large, all ordered coverings have a representation in model (M).

Another variant of model (M0), called model (M') is obtained replacing the fact that  $F$  is nondecreasing in each of its arguments by the fact that  $F$  is one-to-one

in each of its arguments, i.e., is such that for all  $i \in N$  and all  $\alpha_i, \beta_i, \gamma_i \in u_i(X_i)$ ,

$$\left. \begin{aligned} F(\alpha_1, \dots, \alpha_{i-1}, \beta_i, \alpha_{i+1}, \dots, \alpha_n) &= \\ F(\alpha_1, \dots, \alpha_{i-1}, \gamma_i, \alpha_{i+1}, \dots, \alpha_n) & \end{aligned} \right\} \Rightarrow \beta_i = \gamma_i.$$

Because model (M') implies model (M), we know that, for all  $i \in N$ , there must exist a one-to-one correspondence between  $X_i/\sim_i^R$  and some subset of  $\mathbb{R}$ . More importantly, this model implies condition  $\text{thin}_i^R$ , for all  $i \in N$ . Indeed suppose that  $(x_i, a_{-i}) \in \Upsilon^k$  and  $(y_i, a_{-i}) \in \Upsilon^k$ . This implies, abusing notation in an obvious way,  $F(u_i(x_i), \mathbf{K}) = F(u_i(y_i), \mathbf{K}) = \sigma^k$ . Using one-to-one decomposability implies  $u_i(x_i) = u_i(y_i)$ , so that, for all  $\ell \in R$  and  $b_{-i} \in X_{-i}$ ,  $(x_i, b_{-i}) \in C^\ell \Leftrightarrow (y_i, b_{-i}) \in C^\ell$ .

Conversely, let us show that as soon as an ordered covering is thin and satisfies the order-denseness condition, it has a representation in model (M'). Using the order-denseness condition, choose on each  $X_i$  a real-valued function satisfying (37) and define  $F$  as in the proof of Proposition 65. It remains to show that such a function  $F$  is one-to-one in each variable. Suppose that

$$\begin{aligned} F(u_1(a_1), \dots, u_{i-1}(a_{i-1}), u_i(x_i), u_{i+1}(a_{i+1}), \dots, u_n(a_n)) &= \\ F(u_1(a_1), \dots, u_{i-1}(a_{i-1}), u_i(y_i), u_{i+1}(a_{i+1}), \dots, u_n(a_n)) &= \lambda. \end{aligned}$$

If  $\lambda \notin \{\sigma^0, \sigma^1, \dots, \sigma^r\}$ , the definition of  $F$  implies that  $u_i(x_i) = u_i(y_i)$ . If  $\lambda \in \{\sigma^0, \sigma^1, \dots, \sigma^r\}$ , we have  $(x_i, a_{-i}) \in \Upsilon^k$  and  $(y_i, a_{-i}) \in \Upsilon^k$ . Therefore, condition  $\text{thin}_i^R$  implies  $x_i \sim_i^R y_i$ , so that  $u_i(x_i) = u_i(y_i)$ .

The uniqueness of the representations in models (M) and (M') is obviously extremely weak. It can be easily analyzed along the lines sketched in Bouyssou and Marchant (2007a).

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