A NOTE ON THE 'MIN IN FAVOR' CHOICE PROCEDURE FOR FUZZY PREFERENCE RELATIONS

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Abstract

This note deals with the problem of choosing on the basis of a fuzzy preference relation. In the area of MCDM, such a problem occurs with outranking methods using fuzzy relations such as ELECTRE III. In this note, we study a choice procedure based on the 'min in favor'. We show that it is the only one to satisfy a system of three independent axioms.

Keywords: Decision Making, Fuzzy Preferences, Choice, Outranking Methods, MCDM.

I - Introduction

Let A be a finite set of objects with at least two elements. We interpret the elements of A as alternatives among which a choice is to be made taking into account different points of view, *e.g.* several criteria or the opinion of several voters. A common practice in such a situation is to associate with each ordered pair (a, b) of alternatives a number indicating the strength or the credibility of the proposition "a is at least as good as b", *e.g.* the sum of the weights of the criteria favoring a or the percentage of voters declaring that a is preferred or indifferent to b. This leads to a fuzzy (large) preference relation on A. In the area of MCDM, ELECTRE III (Roy (1978)) is a typical illustration of such a process. It is well-known that, when the different points of view taken into account are conflictual, such a fuzzy preference relation does not, in general, possess "nice properties". In fact, Bouyssou (1994) has shown that any reflexive fuzzy binary relation can be obtained as the result of ELECTRE III. Choosing alternatives on the basis of such information is therefore far from being an easy task. This calls for the study of choice procedures for fuzzy preference relations.

A fuzzy (binary) relation on a set A is a function R associating with each ordered pair of alternatives $(a, b) \in A^2$ an element of [0, 1]. A fuzzy binary relation R on A is said to be reflexive if R(a, a) = 1 for all $a \in A$. The set of all fuzzy reflexive relations on A will be denoted by F(A). With ELECTRE III in mind, it will be supposed here any reflexive fuzzy binary relation on A may be the basis for choice. Therefore, we define a choice procedure for fuzzy preference relations (on a set A) as function associating a nonempty subset of A, the "choice set", with each fuzzy reflexive binary relation on A.

In this note, we study "choice procedures" instead of the more classical notion of "choice functions", *i.e.* functions associating a choice set with any subset of A. This is due to the fact that, following Sen (1993), we shall not make use of any "consistency property" of choice when the set of alternatives shrinks or expands (on such properties see, *e.g.*, Aizerman (1985) or Moulin (1985)).

If a fuzzy relation $R \in F(A)$ is such that $R(a, b) \in \{0, 1\}$, for all $a, b \in A$, we say that R is crisp. In this case, we write a R b instead of R(a, b) = 1. We denote by U(A) the set of all crisp

(Unfuzzy) relations in F(A), *i.e.* the set of all crisp reflexive binary relations on A. Throughout this note we adhere to the terminology of Roubens and Vincke (1985) concerning crisp relations.

The classical problem of defining "reasonable" choice procedures for crisp relations is not an easy one. It has generated numerous studies, in particular in the case of tournaments (*i.e.*, crisp, asymmetric and complete binary relations, see Moulin (1986)). This difficulty is largely due to the fact that when a crisp preference relation is not complete and/or has cycles in its asymmetric part, the very notion of a "good" alternative is not easy to define. Such relations are commonly found in Social Choice Theory (see McGarvey (1952) or Deb (1976)).

Turning now to fuzzy preference relations, the situation appears even more complex. This increased complexity stems in particular from:

-the possibility to generalize to the fuzzy case the classical properties of crisp binary relations (completeness, transitivity, absence of circuits, etc.) in many different ways, (see, *e.g.*, Barrett and Pattanaik (1989), Billot (1991), Dutta *et al.* (1986), Jain (1990), Montero and Tejada (1987), Ovchinnikov (1990) or Perny (1992));

-the difficulty to define in a consistent way the symmetric and asymmetric parts of a fuzzy relation and, hence, to separate strict preference from indifference (see, *e.g.*, Fodor (1991), Ovchinnikov and Roubens (1991, 1992) or Perny and Roy (1992));

-the difficulty to interpret the meaning of the numbers R(a, b): are they a cardinal measure of credibility or do they only represent an ordinal information on credibility ? (see, *e.g.*, Barrett *et al.* (1992), Basu *et al.* (1992) or Perny (1992)).

This note will not attempt to answer these difficult questions which are still widely open. Its purpose is to characterize what appears to be a "reasonable" choice procedure in the particular case in which the meaning of fuzziness is strictly "ordinal". This somewhat radical interpretation of fuzziness allows to considerably simplify the problem of defining "reasonable" choice procedures. This is because, within this interpretation, the answers to the first two questions are not so crucial.

II - Some properties of choice procedures

Consider a fuzzy relation $R \in F(A)$, *i.e.* a function associating with each ordered pair of alternatives $(a, b) \in A^2$ an element of [0, 1]. Suppose that R(a, b) = 0.2 and R(c, d) = 0.8. Should we conclude that the proposition "c is at least as good as d" is four times more credible than the proposition "a is at least as good as b" ? In some situations, *e.g.* when the numbers R(a, b) represent a proportion of voters or of criteria favoring the proposition "a is at least as good as b", this may seem reasonable and a choice procedure should take into account such considerations (for an example of such a procedure see Bouyssou (1992a)). In other situations, *e.g.* when the fuzzy relation has been obtained on a purely introspective basis or when the weights of the criteria only reflect an ordinal information about their respective importance, this may well lead to a somewhat unrealistic preference model. Many attempts have been made to propose an "ordinal" theory of fuzziness (see, *e.g.*, Goguen (1967) or, more recently, Basu *et al.* (1992)). In this note we shall remain in the ordinary framework of the theory of fuzzy sets but impose that a choice procedure should only make use of the underlying ordinal information on credibility conveyed by a fuzzy preference relation.

We say that a choice procedure C is **ordinal** if, for all $R \in F(A)$ and all strictly increasing and one-to-one transformation ϕ from [0, 1] to [0, 1],

$C(R) = C(\phi[R])$,

where $\phi[R]$ is the element of F(A) such that $\phi[R](c, d) = \phi(R(c, d))$ for all $c, d \in A$.

It is clear that an ordinal choice procedure does not make use of the cardinal properties of the numbers R(a, b). Many ordinal choice procedures can be envisaged. Let us mention one of them that has often been discussed in the literature and may be seen as a direct extension to the fuzzy case of the classical notion of the "greatest elements" of a crisp preference relation (see Switalski (1988)). Let $R \in F(A)$ and, for all $a \in A$, define, using the same notation as in Barrett *et al.* (1990), the 'min in Favor' score of alternative a letting:

$$mF(a, R) = \underset{c \in A \setminus \{a\}}{\mathrm{Min}} R(a, c)$$

A clearly ordinal choice procedure is defined by:

$C_{mF}(R) = \{a \in A: mF(a, R) \geq mF(b, R) \text{ for all } b \in A\}$.

Many other ordinal choice procedures can be envisaged (see Barrett *et al.* (1990)).

A choice procedure for fuzzy relations should generate "reasonable choices" when applied to crisp relations. Given a crisp relation R on A define $G(R) = \{a \in A: a\,R\,b \text{ for all } b \in A\}$ as the subset of the R-greatest elements in A. Unless the crisp relation R has "nice properties", the set G(R) will often be empty. When it is not however, there is little interest in choosing alternatives outside G(R), since these alternatives are "at least as good" as every other alternatives in A. This is the *raison d'être* of the following axiom.

We say that a choice procedure C is **faithful** if, for all $R \in F(A)$,

$[R \in U(A) \text{ and } G(R) \neq \emptyset] \Rightarrow C(R) \subseteq G(R)$.

It is not difficult to see that C_{mF} is indeed faithful. A more restrictive version of faithfulness has been introduced by Barrett *et al.* (1990).

Faithfulness imposes a constraint on the result of a choice procedure when applied to (some) crisp relations. Ordinality imposes that the result of choice procedure is identical when applied to two "ordinally-equivalent" relations, *i.e.* to two relations $R, S \in F(A)$ such that, for all a, b $\in A$, $R(a, b) = \phi(S(a, b))$ for some strictly increasing and one-to-one transformation ϕ on [0, 1]. It should be noticed that no relation in F(A)\U(A) can be "ordinally-equivalent" to a relation in U(A) since only one-to-one transformations are invoked by ordinality. Thus, these two axioms impose very few constraints on the desirable behavior of a choice procedure when applied to fuzzy relations outside U(A). In particular, they leave room for "discontinuities", which seem rather paradoxical. Let us illustrate the possibility of discontinuities on a simple example involving a crisp relation and an "almost crisp" one. Consider the relations R and R′ on A = {a, b, c} defined by the following tables (to be read from row to column):

R	a	b	c
a	1	1	1
b	0	1	0
c	0	0	1

R′	a	b	c
a	1	1	λ
b	0	1	0
c	0	0	1

where $0 < \lambda < 1$.

It is easy to see that R is crisp and that G(R) = {a}. Let C be a faithful choice procedure. We have C(R) = {a}. Even if C is ordinal, it may happen that $a \notin C(R')$ whatever the value of λ. As a result $C(R) \cap C(R')$ will be empty even when R′ is arbitrarily "close" to R. Our final axiom is designed to prevent such situations.

Consider a sequence of valued relations ($R_i \in F(A)$, i = 1, 2, ...). We say that this sequence converges to converges to $R \in F(A)$ if, for all $\varepsilon \in \mathbb{R}$ with $\varepsilon > 0$, there is an integer k such that, for all $j \geq k$ and all $a, b \in A$, $|R_j(a, b) - R(a, b)| < \varepsilon$.

A choice procedure C is said to be **continuous** if, for all $R \in F(A)$ and all sequences ($R_i \in F(A)$, i = 1, 2, ...) converging to R,

[$a \in C(R_i)$ for all R_i in the sequence] $\Rightarrow$ [$a \in C(R)$].

Our definition of continuity implies that an alternative that is always chosen with fuzzy relations arbitrarily close to a given relation should remain chosen with this relation. It is not difficult to see that C_{mF} is continuous.

The result presented in the next section combines ordinality and continuity. This may appear awkward since ordinality implies that the cardinal properties of the numbers R(a, b) should not be used whereas continuity involves a measure of distance between fuzzy relations using these properties. In presence of ordinality, it would be more satisfactory to formulate a continuity requirement in purely ordinal terms, *e.g.* using a notion of distance between fuzzy relations based on the crisp relations in terms of credibility that they induce on A$\times$A. Though it is fairly easy to reformulate our continuity axiom in this way, such a reformulation would involve a significant loss of intuitive appeal and would lead us outside the traditional framework of the theory of fuzzy sets. In this note, we do not pursue any further in this direction.

Ordinality and continuity impose severe restrictions on a choice procedure. These restrictions are easily understood using the notion of λ-cut. Let $R \in F(A)$ and $\lambda \supseteq\in [0, 1]$. The λ-cut of R is the crisp relation $R_\lambda \in U(A)$ such that, for all $a, b \in A$, $a R_\lambda b$ if and only if $R(a, b) \geq \lambda$. It is not difficult to see that, for any $\lambda \in (0, 1]$, there is a sequence of one-to-one and strictly increasing transformations (ϕ_i, i = 1, 2, ...) on [0, 1] such that $\phi_i[R]$ converges to R_λ. Thus, ordinality and continuity implies that alternatives that are chosen with a fuzzy relation should also be chosen with all of its (strictly positive) λ-cuts. The result in the following section is based on this simple observation coupled with the fact that, since λ-cuts are crisp relations, the result of a choice procedure with such relations may be constrained by faithfulness.

III - Results and Discussion

The main purpose of this note is to prove the following:

Proposition

C_{mF} is the only choice procedure that is ordinal, continuous and faithful.

Proof

We already observed that the C_{mF} is ordinal, continuous and faithful.

Let us now show that if a choice procedure C is ordinal, continuous and faithful then, for all $R \in F(A)$ and all $a, b \in A$:

$mF(a, R) > mF(b, R) \Rightarrow b \notin C(R)$ and(α)

$mF(a, R) = mF(b, R)$ and $b \in C(R) \Rightarrow a \in C(R)$.($\beta$)

which will complete the proof.

In contradiction with (α), suppose that $mF(a, R) > mF(b, R)$ and $b \in C(R)$ for some ordinal, continuous and faithful choice procedure C, some $R \in F(A)$ and some $a, b \in A$. Let $\lambda \in (mF(b, R), mF(a, R))$ and consider any sequence of strictly increasing and one-to-one transformations (ϕ_i, i = 1, 2, ...) on [0, 1] such that:

$$\lim_{i\to\infty} \phi_i(x) = \begin{cases} 1 \text{ if } x \geq \lambda \\ 0 \text{ otherwise.} \end{cases}$$

(A simple example of such a sequence of functions is:

$$\phi_i(x) = \begin{cases} \dfrac{\lambda^{1/i} + (x-\lambda)^{1/i}}{\lambda^{1/i} + (1-\lambda)^{1/i}} \text{ if } x \geq \lambda \\[2ex] \dfrac{\lambda^{1/i} + (\lambda - x)^{1/i}}{\lambda^{1/i} + (1-\lambda)^{1/i}} \text{ otherwise).} \end{cases}$$

By construction, the sequence ($\phi_i[R]$, i = 1, 2, ...) converges to the λ-cut R_λ of R. It is clear that $a \in G(R_\lambda)$ and $b \notin G(R_\lambda)$. By ordinality and continuity we know that $b \in C(R_\lambda)$ which contradicts faithfulness and proves (α).

In order to prove (β), suppose that $mF(a, R) = mF(b, R)$ and $b \in C(R)$, for some ordinal, continuous and faithful choice procedure C, some $R \in F(A)$ and some $a, b \in A$.

Since $b \in C(R)$, we know, using part (α), that $mF(b, R) \geq mF(c, R)$ for all $c \in A$.

Consider the sequence ($R_i \in F(A)$, i = 1, 2 ...) converging to R, where R_i is identical to R except that $R_i(b, c) = Max(0 ; R(b, c) - 1/i)$ for all $c \in A\backslash\{b\}$ and $R_i(a, d) = Min(1 ; R(a, d) + 1/i)$ for all $d \in A\backslash\{a\}$.

By construction, we have $mF(a, R_i) > mF(c, R_i)$ for all $c \in A\backslash\{a\}$. Since $C(R_i)$ is nonempty, (α) implies that $C(R_i) = \{a\}$ for all R_i in the sequence. Thus continuity implies $a \in C(R)$. This proves (β) and completes the proof. ◊

We conclude this note with some remarks.

a) The three axioms that we used to characterize C_{mF} are independent as shown by the following examples (we use the notations of Barrett *et al.* (1990).

i- Let C_{SF} be the ("Sum in Favor") choice procedure defined as:

$C_{SF}(R) = \{a \in A: SF(a, R) \geq SF(b, R) \text{ for all } b \in A\}$ where

$$SF(a,R) = \sum_{c \in A \setminus \{a\}} R(a,c)$$

This choice procedure is continuous and faithful but not ordinal.

ii- Let C_{MA} be the ("Max Against") choice procedure defined as:

$C_{MA}(R) = \{a \in A: MA(a, R) \leq MA(b, R) \text{ for all } b \in A\}$ where

$$MA(a,R) = \operatorname*{Max}_{c \in A \setminus \{a\}} R(c,a)$$

This choice procedure is ordinal and continuous but not faithful.

iii- Let C_L be the choice procedure defined by:

$C_L(R) = \{a \in C_{mF}(R): MF(a, R) \geq MF(b, R) \text{ for all } b \in C_{mF}(R)\}$, where

$$MF(a,R) = \operatorname*{Max}_{c \in A \setminus \{a\}} R(a,c)$$

This choice procedure is ordinal and faithful. It is not difficult to see that it is not continuous.

b) The mF score can be used not only to define a choice procedure – selecting the alternatives with the highest score – but also to rank order alternatives according to their scores. Independent characterizations of such a ranking procedure have been obtained by Pirlot (1992, 1994) and Bouyssou (1992b). Though the ranking procedure for fuzzy preference relations based on the mF score has many interesting properties, applying this ranking procedure to a complete and transitive crisp relation leads a complete and transitive crisp relation that may be different from the first one, thus violating a possible interpretation of "faithfulness" for ranking procedures (it should be noted that a "faithful" ranking procedure based on the mF score can be obtained by an iterative use of C_{mF}; such a "faithful" ranking procedure would not behave very well in terms of monotonicity however – see Perny (1992)). Thus the mF score seems to be more adapted to choice problems than to ranking problems. Indeed, we have shown that if the reasonableness of ordinality, faithfulness and continuity is admitted then C_{mF} is the only "reasonable" choice procedure.

c) Let us finally mention that a truly satisfactory fuzzy model of preferences probably lies in between strict ordinality – as was supposed here – and full cardinality allowing all possible operations on the numbers R(a, b). A precise formalization of such intermediate situations certainly deserves further investigations.

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