Conjoint Measurement without additivity and transitivity

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I. Introduction and Motivation

The traditional way of modelling the preferences of a Decision-Maker consists in assuming the existence of a value function u such that an alternative a is at least as good as an alternative b ($a \succsim b$) if and only if $u(a) \geq u(b)$. This leads to a model of preference in which $\succsim$ is complete and transitive. Using such a preference model it is straightforward to establish a recommendation in a decision-aid study. The main task of the Analyst is therefore to assess u.

In a multicriteria/multiattribute (we will use these terms interchangeably here) context, the set of alternatives X is often modelled as some subset of a cartesian product $X_1 \text{x} X_2 \text{x} \ldots \text{x} X_n$; each alternative a is thus seen as a vector $(a_1, a_2, \ldots, a_n)$ of evaluations on n attributes. In such a context the assessment of a value function u is not an easy task. It usually requires the specification of a particular functional form for u. The additive decomposition of traditional conjoint measurement models in which:

$$a \succsim b \Leftrightarrow \sum_{i=1}^{n} u_i(a_i) \geq \sum_{i=1}^{n} u_i(b_i) \qquad (1)$$

(where u_i are real-valued functions on the sets X_i and it is understood that $a = (a_1, a_2, \ldots, a_n)$ and $b = (b_1, b_2, \ldots, b_n)$), is, by far, the most widely used. Modelling preferences using such a model amounts to assessing the "partial value functions" u_i. Many techniques have been proposed to do so (see *e.g.* Edwards and Newman (1982), Fishburn (1967), French (1993) or Keeney and Raiffa (1976)). This dominant model raises several difficulties however.

On a technical level, although many axiom systems have been proposed in order to obtain such a representation (see Krantz *et al.* (1971) or Wakker (1989)) their interpretation is not always easy. When X is finite, it is well-known that the system of axioms necessarily involves a denumerable number of “cancellation conditions” guaranteeing the existence of solutions to a system of (finitely many) linear inequalities through the use of various versions of the “theorem of the alternative”. When X is infinite the picture changes provided that conditions are imposed in order to guarantee that the structure of X is “close” to the structure of Re and that $\succsim$ behaves consistently in this continuum; this is traditionally done using either an “archimedean” axiom together with some solvability assumption or imposing some topological structure on X and a continuity requirement on $\succsim$. Under these conditions, it is well-known that model (1) obtains when a finite number of cancellation conditions are imposed. As opposed to the finite case, these structural assumptions allow to obtain nice uniqueness results for model (1): the functions u_i define interval scales with a common unit.

In the finite case the axiom system is hardly interpretable and testable. In the infinite case, it is not always easy to separate the respective roles of the (unnecessary) structural assumptions from the (necessary) cancellation conditions.

Besides these technical difficulties, traditional models of conjoint measurement might not always be appropriate for modelling preferences because:

- indifference (seen as the symmetric part of $\succsim$) may not be transitive;
- $\succsim$ may not be a complete relation, i.e. some alternatives may be incomparable;
- compensation effects between criteria may be more complex than with an additive model.

Let us finally mention that the framework of model (1) is too narrow to encompass a number of approaches developed since the early seventies : the so-called outranking methods (see Roy (1968), Roy and Bertier (1973) and for a recent presentation in English, Roy (1991, 1996), Vincke (1992)). In these approaches, the overall preference of a over b is usually determined by looking at the evaluation vectors $(a_1, a_2, \ldots, a_n)$ and $(b_1, b_2, \ldots, b_n)$ independently of the other alternatives and treating the "preference difference" between a_i and b_i in rather an ordinal way. They do not usually lead to a global preference relation that is complete or transitive (this being not unrelated to Arrow's theorem). Such methods therefore imply the application of specific "exploitation techniques" in order to derive a recommendation (choice of an alternative, ranking of all alternatives). Although these methods have been less widely used than the ones based on traditional conjoint measurement models, they are important and original tools for multiple criteria analysis. Our aim is to propose and to analyse a model that would be sufficiently flexible to encompass in the same framework the traditional models of conjoint measurement and the outranking methods while avoiding the technical difficulties encountered with conjoint measurement.

Several extensions of model (1) have been proposed in the literature. The first extension consists in replacing the additivity requirement by a mere decomposability requirement. Krantz *et al.* (1971, chap. 7) introduced the following decomposable model:

$$a \succsim b \Leftrightarrow F(u_1(\mathrm{a}_1), u_2(a_2), \ldots, u_\mathrm{n}(a_\mathrm{n})) \geq F(u_1(b_1), u_2(b_2), \ldots, u_\mathrm{n}(b_\mathrm{n})) \qquad (2)$$

where F is increasing in all its arguments. Such a model clearly allows interactions between attributes that are more complex than with an additive model. In the denumerable (*i.e.* finite or countably infinite) case, necessary and sufficient conditions for (2) consist in a transitivity and completeness requirement together with a single cancellation condition requiring that the preference between objects differing on a single attribute is independent from their common level on the remaining n–1 attributes. In the non denumerable case these conditions turn out to have identical implication when supplemented with the obviously necessary requirement that a numerical representation exists for $\succsim$. Though (2) may appear as exceedingly general when compared to (1), it allows to deal with the finite and the infinite case in a unified way using a simple axiom system while imposing nontrivial restrictions on $\succsim$.

Both (1) and (2) imply that $\succsim$ is complete and transitive. Among many others May (1954) and Tversky (1969) have argued that the transitivity hypothesis is most unlikely to hold when subjects are asked to compare objects evaluated on several attributes. Hence the need for measurement models accommodating intransitivities. Tversky (1969) was one of the first to propose such a model generalising (1), known as the “additive difference model” in which:

$$a \succeq b \Leftrightarrow \sum_{i=1}^{n} \Phi_i(u_i(a_i) - u_i(b_i)) \geq 0 \qquad (3)$$

where Φ_i are increasing and odd functions. It is clear that (3) allows for intransitive $\succeq$ but implies its completeness. When attention is restricted to the comparison of objects that only differ on one attribute, (3) as well as (2) and (1) imply that the preference relation between these objects is independent from their common level on the remaining n–1 attributes. As noted by Bouyssou (1986), an unpleasant feature of (3), for a model aiming at accommodating intransitivities, is that it implies that these “partial preference relations” are complete and transitive, therefore excluding the possibility of any “perception threshold” on each attribute.

As suggested by Bouyssou (1986), Fishburn (1990, 1991a) and Vind (1991), the subtractivity requirement in (3) can be relaxed. This leads to non transitive additive conjoint measurement models in which:

$$a \succeq b \Leftrightarrow \sum_{i=1}^{n} p_i(a_i, b_i) \geq 0 \qquad (4)$$

where the p_i are real-valued functions on X_i^2 and may have several additional properties (*e.g.* skew-symmetry).

This model is an obvious generalisation of the additive difference model. It allows for intransitive and incomplete preference relations $\succeq$ as well as for intransitive and incomplete partial preferences. Fishburn (1991b) gives an excellent overview of these non transitive models and recalls the several axiom systems that have been proposed to characterise them.

It should be noticed that even the “weakest” model presented so far, *i.e.* (4), involves an addition operation. Therefore it is unsurprising that the difficulties that we mentioned concerning the axiomatic analysis of traditional models are still present here. Except in the special case in which n = 2, this case relating more to ordinal than to conjoint measurement, the various axiom systems that have been proposed involve:

- a denumerable number of cancellation conditions in the finite case or
- a finite number of cancellation conditions together with unnecessary structural assumptions in the infinite case (these structural assumptions generally allow to obtain nice uniqueness results for (4): the functions p_i are unique up to a positive linear transformation).

The models that we study in this paper may be seen both as a generalisation of (2) dropping transitivity and completeness and as a generalisation of (4) dropping additivity. In their most general form they are of the type (see also Goldstein (1991)):

$$a \succsim b \Leftrightarrow F(p_1(a_1, b_1), p_2(a_2, b_2), \ldots, p_n(a_n, b_n)) \geq 0 \quad (5)$$

where F is non decreasing in all its arguments. This type of non transitive decomposable conjoint models may seem exceedingly general. However we shall see that this model and its specialisations:

- imply substantive requirements on $\succsim$,
- may be axiomatised in a simple way avoiding the use of a denumerable number of conditions in the finite case and of unnecessary structural assumptions in the infinite case,
- allow to study the "pure consequences" of cancellation conditions in the absence of transitivity, completeness and structural requirements on X,
- are sufficiently general to include as particular cases many aggregation rules that have been proposed in the literature.

II. Outline of Results

In this section we give, without proof, a number of sample results concerning model (5) and show how they can be used. Let $\succsim$ be a binary relation on a set $X = \prod_{i=1}^{n} X_i$. This relation is said to satisfy:

$\mathrm{RC_i}$ if

$$\left.\begin{array}{r}(x_i, a_{-i}) \succsim (y_i, b_{-i}) \\ \text{and} \\ (z_i, c_{-i}) \succsim (w_i, d_{-i})\end{array}\right\} \Rightarrow \left\{\begin{array}{l}(x_i, c_{-i}) \succsim (y_i, d_{-i}) \\ \text{or} \\ (z_i, a_{-i}) \succsim (w_i, b_{-i}),\end{array}\right.$$

for all $x_i, y_i, z_i, w_i \in X_i$ and all $a_{-i}, b_{-i}, c_{-i}, d_{-i} \in X_{-i}$, with $X_{-i} = \prod_{j \neq i} X_j$. We say that $\succsim$ satisfies RC if it satisfies $\mathrm{RC_i}$ for $i = 1, 2, \ldots, n$.

Condition $\mathrm{RC_i}$ (inteR-attribute Cancellation) suggests that $\succsim$ induces on X_i a relation that compares "preference differences" in a well-behaved way: if (x_i, y_i) is a "larger preference difference" than (z_i, w_i) then if $(z_i, c_{-i}) \succsim (w_i, d_{-i})$ we should also have $(x_i, c_{-i}) \succsim (y_i, d_{-i})$ and *vice versa*. The idea that comparison of "preference differences" is central to the analysis of conjoint measurement models was already powerfully stressed by Wakker (1989). The obvious extension of condition $\mathrm{RC_i}$ to subsets of attributes is central to the analysis of a special case of (4) by Vind (1991) where this condition is called "independence". It is called "weak cancellation" in Bouyssou (1986). Technically $\mathrm{RC_i}$ amounts to defining a biorder in the sense of Doignon *et al.* (1984) between the sets X_i and X_{-i}.

It is not difficult to see that model (5) implies RC. The converse is also true when X is finite or countably infinite and we have:

Proposition 1. Let $\succsim$ be a binary relation on a finite or countably infinite set $X = \prod_{i=1}^{n} X_i$. Then $\succsim$ satisfies model (5) iff RC holds.

We show in Bouyssou and Pirlot (1998) how this result can be generalised to the non denumerable case and extended in various directions. In particular, it is possible to characterise by axioms similar to $\mathrm{RC_i}$ the situation where the preference difference (x_i, y_i) is "the opposite" of the preference difference (y_i, x_i); in such a case F will be an odd function and the p_i's will be skew-symmetric, *i. e.*

$$p_i(x_i, y_i) = -p_i(y_i, x_i)$$

and

$$F(-p_1, -p_2, \ldots, -p_n) = -F(p_1, p_2, \ldots, p_n)$$

Further detail about this type of models may be found in Bouyssou and Pirlot (1998).

Among the other possible extensions, an interesting one consists in specifying a functional form for the functions $p_i(x_i,y_i)$. We envisage here the simplest one in which all functions $p_i(x_i, y_i)$ in (5) are such that:

$$p_i(x_i, y_i) = \varphi_i(u_i(x_i), u_i(y_i)), \quad (6)$$

where u_i is a real-valued function on X_i and φ_i is a real-valued function on $u_i(X_i)^2$ being non decreasing in its first argument and non increasing in its second argument. The intuition behind model (5) and (6) is that the "weight" of the "difference" between elements of X_i may be understood via a linear arrangement on these elements. We say that $\succsim$ satisfies:

AC1$_i$ if

$$\left.\begin{array}{r}(x_i, a_{-i}) \succsim (y_i, b_{-i})\\ \text{and}\\ (z_i, c_{-i}) \succsim (w_i, d_{-i})\end{array}\right\} \Rightarrow \left\{\begin{array}{l}(z_i, a_{-i}) \succsim (w_i, b_{-i})\\ \text{or}\\ (x_i, c_{-i}) \succsim (w_i, d_{-i}),\end{array}\right.$$

AC2$_i$ if

$$\left.\begin{array}{r}(x_i, a_{-i}) \succsim (y_i, b_{-i})\\ \text{and}\\ (z_i, c_{-i}) \succsim (w_i, d_{-i})\end{array}\right\} \Rightarrow \left\{\begin{array}{l}(x_i, a_{-i}) \succsim (w_i, b_{-i})\\ \text{or}\\ (z_i, c_{-i}) \succsim (y_i, d_{-i}),\end{array}\right.$$

AC3$_i$ if

$$\left.\begin{array}{r}(x_i, a_{-i}) \succsim (y_i, b_{-i})\\ \text{and}\\ (z_i, c_{-i}) \succsim (x_i, d_{-i})\end{array}\right\} \Rightarrow \left\{\begin{array}{l}(w_i, a_{-i}) \succsim (y_i, b_{-i})\\ \text{or}\\ (z_i, c_{-i}) \succsim (w_i, d_{-i})\end{array}\right.$$

for all $x_i, y_i, z_i, w_i \in X_i$ and all $a_{-i}, b_{-i}, c_{-i}, d_{-i} \in X_{-i}$.

Condition $AC1_i$ suggests that the elements of X_i can be linearly ordered considering "upward dominance": if x_i "upward dominates" z_i then $(z_i, c_{-i}) \succsim w$ entails $(x_i, c_{-i}) \succsim w$ and *vice versa*. Condition $AC2_i$ has a similar interpretation considering now "downward dominance". Condition $AC3_i$ ensures that the linear arrangements of the elements of X_i obtained considering upward and downward dominance are not incompatible. It is not difficult to see that AC1 (*i.e.* $AC1_i$ holding for $i = 1, 2, \ldots, n$) AC2 and AC3 are necessary conditions for model (5)-(6). When coupled with RC they turn out to be also sufficient when X is finite or countably infinite and we have:

Proposition 2. Let $\succsim$ be a binary relation on a finite or countably infinite set $X = \prod_{i=1}^{n} X_i$. Then $\succsim$ satisfies model (5) and (6) iff RC, AC1, AC2 and AC3 hold.

Several extensions of this result, including the case of a non denumerable set X, are discussed in Bouyssou and Pirlot (1998).

III. Examples and discussion

In this section we give some examples of aggregation rules used in well-known methods fitting in our framework and discuss the interest of our results.

a) Examples

Example 1. Model (1) is the simplest case. The preference difference (a_i, b_i) is represented through an algebraic difference $p_i(a_i, b_i) = \varphi_i(u_i(a_i), u_i(b_i)) = \Phi_i\,(u_i(a_i) - u_i(b_i)) = u_i(a_i) - u_i(b_i)$ and F is a sum. (See also on Figure 1).

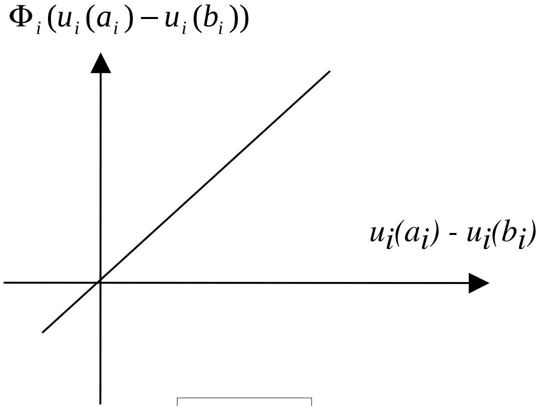

Fig. 1

Example 2 Another family of models could be termed *Difference Threshold models*. The range of $u_i(a_i) - u_i(b_i)$ is partitioned by means of a number of points q_{ij}, $j=1,\ldots N_i$ of the real line and Φ_i is constant on each interval $[q_{ij}, q_{i(j+1)}[$; more precisely (see Figure 2):

$$\varphi_i(u_i(a_i), u_i(b_i)) = \Phi_i (u_i(a_i) - u_i(b_i)) = \Phi_{ij} \quad \text{if} \quad q_{ij} \leq u_i(a_i) - u_i(b_i) \leq q_{i(j+1)},$$

with $\Phi_{ij} < \Phi_{i(j+1)}$ and $q_{ij} < q_{i(j+1)}$, for all $j=1,\ldots N_i$; F is any function of n variables, non decreasing in each argument. It is not difficult to see that, on a finite set, model (1) can always be represented as a "difference threshold model".

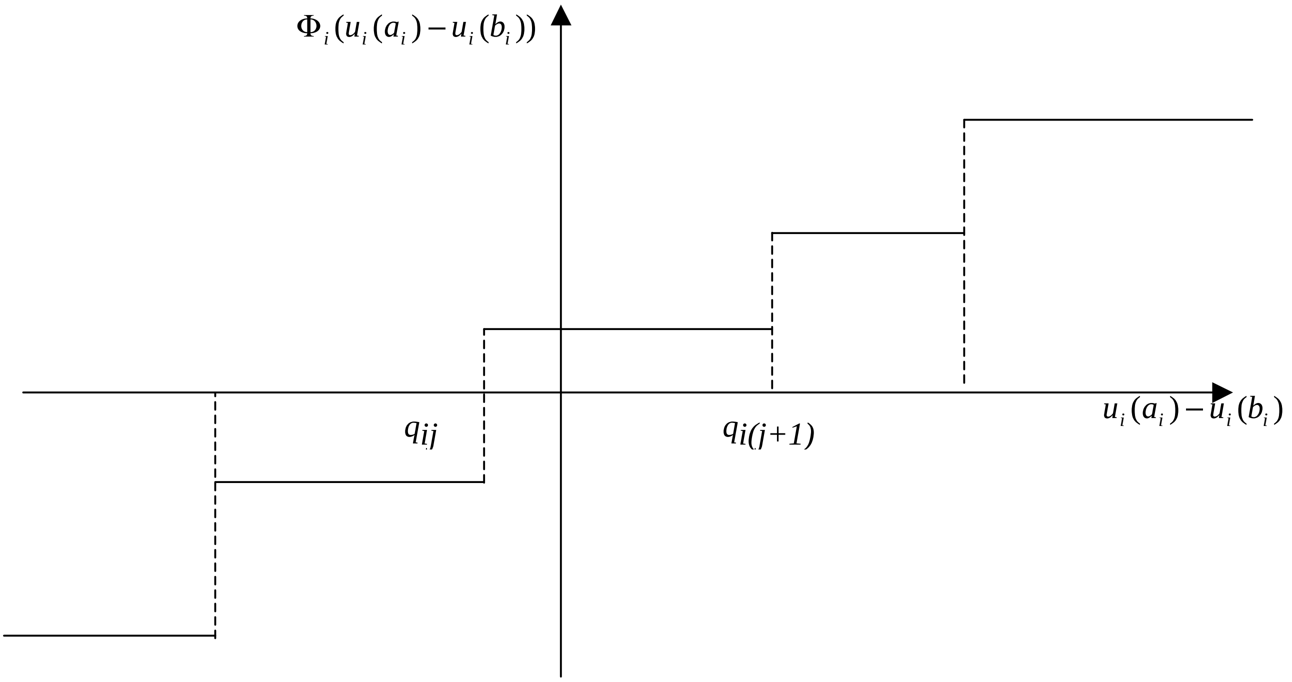

Fig. 2

The model just described encompasses aggregation rules used in outranking methods of the ELECTRE type (see Roy (1991) or Vincke (1992)). In this approach it is assumed that differences of preference are imprecisely known; in the most elementary methods, ELECTRE I and II, the Decision- Maker is supposed to be able to categorize them in a small number of classes; those classes are "significant" at the level of the overall preference since going from a class to another on an attribute may have an incidence on the overall preference. The aggregation function is then essentially a rule (or a set of rules) telling when a profile of differences of preferences between two alternatives *a* and *b* is in favour of *a*.

The aggregation procedure of ELECTRE I fits into this framework. The idea behind it can be summarised as follows: if there is no criterion on which the difference of preference is so much in favour of *b* that it would prevent declaring *a* better than *b* ("veto of *b* against *a*"), then the overal preference of *a* over *b* is declared when the coalition of criteria on which a_i is better than b_i is strong enough (i.e. the sum of their weights passes a threshold *s*). Formally, we can take for instance :

$$p_i(x_i, y_i) = \Phi_i(u_i(a_i) - u_i(b_i)) = \begin{cases} 1 \text{ if } i \in I_1 \text{ i.e. } u_i(a_i) - u_i(b_i) < q_{i1} \\ 2 \text{ if } i \in I_2 \text{ i.e. } q_{i1} \leq u_i(a_i) - u_i(b_i) < q_{i2} \\ 3 \text{ if } i \in I_3 \text{ i.e. } q_{i2} \leq u_i(a_i) - u_i(b_i) \end{cases}$$

and,

$$f(a,b) = F(\Phi_i(u_i(a_i) - u_i(b_i)), i = 1,...n) = \sum_{i \in I_3} w_i - M|I_1| - s \qquad (7)$$

with $q_{i1} \leq q_{i2} \leq 0$ (q_{i1} is a veto threshold; q_{i2} is an indifference threshold); *s* is the so-called concordance threshold, M is a "large" positive number (large enough for preventing *F* from being positive as soon as I_i is not empty) and w_i , the (normalised) weight of criterion *i* . Using formula (5) and (6), we get the definition of the overal preference in ELECTRE I :

$$a \succeq b \text{ iff } |I_1| = 0 \text{ and } \sum_{i \in I_3} w_i \geq s$$

TACTIC (Vansnick 1986) is another approach which can be described by model (5)-(6) (see Bouyssou *et al* (1997) for more detail). In ELECTRE II, there is a further distinction between two levels of overall preference (weak and strong outranking); those levels are characterised by specific values of the thresholds q_{i1} and *s* but both the weak and the strong outranking relations fit into the model. An interesting feature here is that *F* is no longer a sum but the minimum of two functions of the differences of preferences.

The strong outranking relation of ELECTRE II distinguishes eventually four classes of differences of preference:

$$\Phi_i(u_i(a_i)-u_i(b_i)) = \begin{cases} 1 & \text{if} \quad i\in I_1 \quad \text{i.e.} \quad u_i(a_i)-u_i(b_i) < q_{i1} \\ 2 & \text{if} \quad i\in I_2 \quad \text{i.e.} \quad q_{i1} \le u_i(a_i)-u_i(b_i) < q_{i2} \\ 3 & \text{if} \quad i\in I_3 \quad \text{i.e.} \quad q_{i2} \le u_i(a_i)-u_i(b_i) < -q_{i2} = q_{i3} \\ 4 & \text{if} \quad i\in I_4 \quad \text{i.e.} \quad -q_{i2} = q_{i3} \le u_i(a_i)-u_i(b_i) \end{cases}$$

with $q_{i1} \le q_{i2} \le 0 \le q_{i3} = -q_{i2}$. The first two classes are the same as in ELECTRE I, while the third class of ELECTRE I is further partitioned in two subclasses. We have

$a \succeq b$ iff $f(a, b) \geq 0$

where $f(a, b) = \min\{ f_1(a, b), f_2(a, b)\}$; $f_1(a, b)$ is identical to $f(a, b)$ in (7)

$$f_1(a,b) = \sum_{i\in I_3\cup I_4} w_i - M|I_1| - s$$

and

$$f_2(a,b) = \sum_{i\in I_4} w_i - \sum_{i\in I_1\cup I_2} w_i \ .$$

The above description is also valid for the weak outranking relation in ELECTRE II; weak and strong outranking relations only differ in the values given to the thresholds q_{i1}, q_{i2} and *s*.

b) Discussion

Compensation versus non-compensation. ELECTRE I is characterised by a very rough distinction between differences of preference since no more than three classes may be distinguished (only two in case no veto threshold is specified). The “richness” (number of classes) of the relation comparing differences of preference on each attribute is intimately related to the more or less compensatory character of the aggregation procedure. Intuitively, in a compensatory method, any difference of preference in favour of *b* against *a* on a criterion can be compensated by a sufficiently large difference of preference in favour of *a* on another criterion. The model in example 1 is fully compensatory. Defining non-compensation is more delicate; according to Fishburn (1976), in a non-compensatory method, the overal preference of *a* over *b* is determined by the list of criteria on which *a* is better than *b* and no consideration at all is paid to the “amplitude” of the difference between evaluations of the alternatives on each criterion. The ELECTRE I aggregation rule without veto threshold is a non-compensatory method in Fishburn’s sense but obviously, strictly non-compensatory methods have very poor practical value since they enable to distinguish only between preference or no-preference on each criterion. The framework has been extended in order to encompass methods with vetoes (three classes of differences of preference) by Bouyssou and Vansnick (1986).

There is a need for a gradual definition of non-compensation that would allow to characterise the methods as more or less compensatory. In our model, if the overal preference satisfies RC, the comparison of difference of preferences is transitive and complete; this implies in particular that differences of preference can be grouped in classes that are totally ordered; differences of preference that belong to the same class are equivalent from a global preference point of view. As suggested in Bouyssou *et al* (1997), this allows to situate the methods in a sort of continuum which ranges from purely compensatory to purely non-compensatory: purely non-compensatory methods distinguish only two classes on each criterion; the more numerous the classes, the more compensatory the method.

Valued preference relations. In a number of methods, the aggregation phase ends up with a valued relation on X, i.e. a function $X^2 \rightarrow E$; in most cases E is a subset of Re , *e.g.* $E = [0,1]$. This is the situation with valued versions of ELECTRE (ELECTRE III or IV) as well as with PROMETHEE (see Brans *et al* (1984), Brans and Vincke (1985) or Vincke (1992)). A striking fact is that our model (5)-(6) is essentially a valued preference model since $f(a, b) = F(\varphi_i(u_i(a_i), u_i(b_i)), i=1, \ldots , n)$ is a valued relation. Up to this point, we have considered that the whole overal preference information is contained in a crisp binary relation which can be obtained, in our model, by encoding whether a function F passes 0 or not. We could of course obtain much richer information, namely a chain of relations $\{(\succeq_\alpha)\}$, by cutting f (or F) at all possible levels $\alpha \in$ Re, i.e.

$$a \succeq_\alpha b \quad \text{iff} \quad f(a, b) \geq \alpha$$

So, as a natural extension of the framework studied in this paper, overal preference structures consisting of chains of relations can be considered; imposing conditions RC, AC1, AC2 and AC3 to each relation of the chain yields preference relations $\succeq_\alpha$ that satisfy model (5)-(6) for all α; of course the functional representation will in general depend on the index α. A case of particular interest (that will be studied in a subsequent paper) is when all relations $\succeq_\alpha$ can be obtained, as follows, on the basis of a common representation of preference differences:

$$a \succeq_\alpha b \quad \text{iff} \quad F_\alpha(\varphi_i(u_i(a_i), u_i(b_i)), i=1, \ldots n) = F(\varphi_i(u_i(a_i), u_i(b_i)), i=1, \ldots n) - \alpha \geq 0$$

i.e.

$$a \succeq_\alpha b \quad \text{iff} \quad f(a,b) - \alpha \geq 0 \,;$$

in other words, each relation $\succeq_\alpha$ results from cutting a function *F* of the differences of preference at some level α. Note that any increasing function of *f* would determine the same chain of relations (if the value of the index associated to each cut is not considered as part of the definition of the chain); more explicitly, if σ: Re$\rightarrow$Re is increasing, we have

$$a \succeq_\alpha b \quad \text{iff} \quad f(a, b) \geq \alpha \quad \text{iff} \quad \sigma(f(a, b)) \geq \sigma(\alpha)$$

Let us call *ordinal* valued preference relation a valued relation *f* determined up to a (strictly) increasing transformation or equivalently the chain of relations ($\succeq_\alpha$) obtained by cutting *f* at all possible levels. A stronger structure of preference is called *cardinally valued* if the numerical value of *f(a,b)* does matter; in such a case, the structure is not completely known when the chain of cuts is given, the value of the index associated with each cut is relevant too; arithmetic operations on the values *f(a,b)* can then be meaningfully performed.

In view of this, the valued outranking relation of PROMETHEE is a function *f(a, b)* as in model (5)-(6) since

$$f(a,b) = \sum_{i=1}^{n} w_i \Phi_i(u_i(a_i) - u_i(b_i))$$

where w_i is a normalised weight associated with criterion *i* and Φ_i recodes the differences of preference on criterion *i* according to one of six pre-defined coding functions (see e.g. Vincke (1992), p. 74). All pre-defined Φ_i functions take 0 value when $u_i(a_i)-u_i(b_i) \leq 0$ and are otherwise non-negative and non-decreasing.

In the subsequent exploitation phase, PROMETHEE fully makes usage of the cardinal properties of *f(a,b)*; for instance, sums and differences of such numbers are considered (for computing the so-called *leaving* and *entering flows* or the *net flow*). Hence, in this model, additional assumptions are made, explicitly or implicitly, guaranteeing that *f* is determined up to a positive affine transformation, which of course is more demanding than a *f* being determined up to a strictly increasing transformation; in other words, not only the chain of relations ($\succeq_\alpha$) is important but the labelling α also matters.

In contrast, the ELECTRE methods (especially ELECTRE I and II) appear as mostly ordinal, in the sense that the valued preference (outranking) relations can be considered as determined up to an increasing transformation and the results of the subsequent exploitation phase are invariant when such a transformation is applied to the valued relation; in other words, those results only depend on the chain of relations $\{\succeq_\alpha\}$, not on their labelling.

IV. Conclusion

Models (5) and (5)-(6) are of interest in several respects:

- they offer a general framework of conjoint measurement that does not require the preference relation $\succsim$ to be complete or transitive,
- they can be simply analysed on a axiomatic level,
- they are sufficiently general to contain as particular cases many different aggregation rules (as shown in Bouyssou *et al* (1997) and Bouyssou and Pirlot (1998)) while capturing what seems to us at the heart of any multiattribute aggregation: a modelling of "preference differences" on each attribute with reference to an underlying linear arrangement of the various levels of the attribute.

In view of this, our two sets of conditions (RC and AC1-2-3) could thus be considered as providing a common and simple foundation of most multiattribute aggregation procedures.

References

Bouyssou D. (1986), Some remarks on the notion of compensation in MCDM, *European Journal of Operational Research* **26**, 150-160.

Bouyssou D. and J.-Cl. Vansnick (1986), Noncompensatory and generalized noncompensatory preference structures, *Theory and Decision* **21**, 251-266.

Bouyssou D. and M. Pirlot (1998), Non Transitive Decomposable Conjoint Measurement: Non Additive Representations of Non transitive Preferences on Product Sets. Working Paper, ESSEC 1998.

Bouyssou D., M. Pirlot and Ph. Vincke (1997a), A general model of preference aggregation, in J. Spronk (Ed.), *Essays in Decision-Making,* Springer, Berlin.

Brans J.-P., B. Mareschal and Ph. Vincke (1984), PROMETHEE: a niew family of outranking methods in multicriteria analysis, in J.-P. Brans (Ed.), *OR' 84*, 408-421, North Holland, Amsterdam.

Brans J.-P. and Ph. Vincke (1985), A preference ranking organization method, *Management Science*, **31**, 647-656.

Doignon J.P, A. Ducamp and J.-C. Falmagne (1984), On realizable biorders and the biorder dimension of a relation, *Journal of Mathematical Psychology* **28**, 73-109.

Edwards W. and J.R. Newman (1982), *Multiattribute evaluation*, Sage, Beverly Hills, CA.

Fishburn P.C. (1967), Methods for estimating additive utilities, *Management Science* **13**, 435-453.

Fishburn P.C. (1976), Noncompensatory preferences, *Synthese* **33**, 393-403.

Fishburn P.C. (1990), Continuous nontransitive additive conjoint measurement, *Mathematical Social Sciences* **20**, 165-193.

Fishburn P.C. (1991a), Nontransitive additive conjoint measurement, *Journal of Mathematical Psychology* **35**, 1-40

Fishburn P.C. (1991b), Nontransitive Preferences in Decision Theory, *Journal of Risk and Uncertainty*, **4**, 113-134.

French S. (1993), *Decision Theory – An Introduction to the Mathematics of Rationality*, Ellis Horwood.

Goldstein W.M. (1991), Decomposable Threshold Models, *Journal of Mathematical Psychology* **35**, 64-79.

Keeney R.L. and H. Raiffa (1976), *Decisions with Multiple Objectives – Preferences and value tradeoffs*, Wiley, New York.

Krantz D..H., R.D. Luce , P. Suppes and A. Tversky (1971): *Foundations of measurement*, Volume 1: Additive and polynomial representations, Academic Press Inc., New-York.

May K.O. (1954), Intransitivity, utility and the aggregation of preference patterns, *Econometrica* **22**, 1-13.

Roy B. (1968), Classement et choix en présence de points de vue multiples (la méthode ELECTRE), *Revue d'Informatique et de Recherche Opérationnelle (RIRO)* **8**, 57-75.

Roy B. (1991), The outranking approach and the foundations of ELECTRE methods, *Theory and Decision* **31**, 49-73.

Roy B. (1996), *Multicriteria Methodology for Decision Aiding*, Kluwer, Dordrecht.

Roy B. and P. Bertier (1973), La méthode ELECTRE II – Une application au media-planning, in M. Ross (Ed.), *OR' 72*, 291-302, North Holland, Amsterdam.

Tversky A. (1969), Intransitivity of preferences, *Psychological Review* **76**, 31-48.

Vansnick J.-Cl. (1986), On the problem of weights in multiple criteria decision making, *European Journal of Operational Research* **24**, 288-294.

Vincke Ph. (1992), *Multicriteria Decision-Aid*, Wiley, Chichester.

Vind K. (1991), Independent preferences, *Journal of Mathematical Economics* **20**, 119-135.

Wakker P. (1989), *Additive Representations of Preferences – A New Foundation of Decision Analysis*, Theory and Decision Library, Series C, Kluwer, Dordrecht.