Additive conjoint measurement with ordered categories

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If you do not know Thierry. . .

Introduction

Context of today's talk

- preference modelling for MCDA

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Central model: additive value functions

$$x \succsim y \Leftrightarrow \sum_{i=1}^{n} v_i(x_i) \geq \sum_{i=1}^{n} v_i(y_i)$$

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Remarks

- firm theoretical background (Krantz et al., 1971)
- many assessment techniques (Keeney & Raiffa, 1976, von Winterfeld & Edwards, 1986)
- underlies many MCDM techniques

Need for extensions

Practical problems

- the output of the model is a preference relation, i.e., a relative evaluation model
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Theoretical problem

- can additive value functions be obtained on the basis of a different kind of information?

Outline

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Framework

Classical conjoint measurement setting

- $N = \{1, 2, \ldots, n\}$: set of attributes
- $X = \prod_{i=1}^{n} X_i$ with $n \geq 2$: set of alternatives
- notation: $(x_i, y_{-i}), (x_J, y_{-J}) \in X$

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Primitives: threefold ordered partitions $\langle \mathscr{A}, \mathscr{F}, \mathscr{U} \rangle$ of X

- $\mathscr{A}$: set of objects that are "good"
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Interpretation

- position of objects vis-à-vis a status quo
- objects in $\mathscr{A}$ ($\mathscr{U}$) are not equivalent

Model

Model: additive value functions with threshold

$$x \in \left\{ \begin{array}{c} \mathscr{A} \\ \mathscr{F} \\ \mathscr{U} \end{array} \right\} \Leftrightarrow \sum_{i=1}^{n} v_i(x_i) \left\{ \begin{array}{c} > \\ = \\ < \end{array} \right\} 0$$

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Non-degenerate

- $\langle \mathscr{A}, \mathscr{F}, \mathscr{U} \rangle$ is non-degenerate if $\mathscr{A} \neq \varnothing$ and $\mathscr{U} \neq \varnothing$

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Uniqueness and normalization

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Suppose that $n = 3$ so that $X = X_1 \times X_2 \times X_3$

- take any $(x_1^0, x_2^0, x_3^0) \in \mathscr{F}$
- normalize so that $v_1(x_1^0) = 0$, $v_2(x_2^0) = 0$ and $v_3(x_3^0) = 0$

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- take any $x_1^{-1} \in X_1$ such that $(x_1^{-1}, x_2^0, x_3^0) \in \mathscr{U}$
- normalize so that $v_1(x_1^{-1}) = -1$

Diagonal standard sequences

Assess equally-spaced points

$$(x_1^0, x_2^0, x_3^0) \in \mathscr{F} \qquad v_1(x_1^0) = v_2(x_2^0) = v_3(x_3^0) = 0$$
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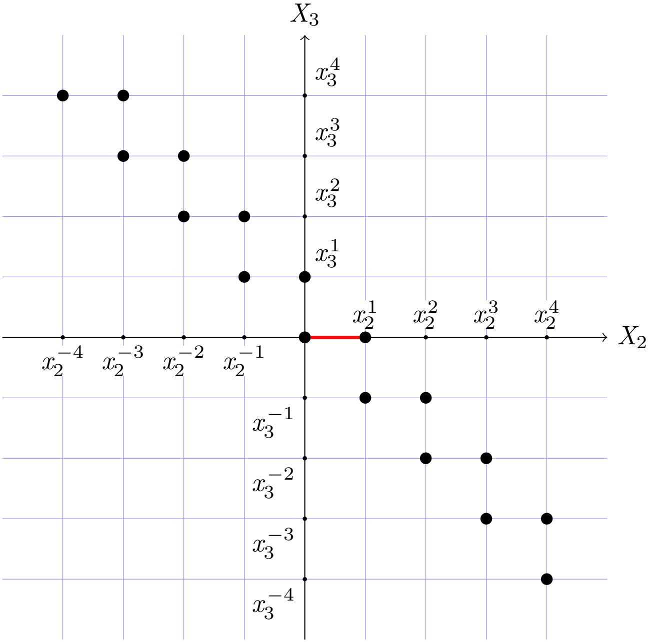

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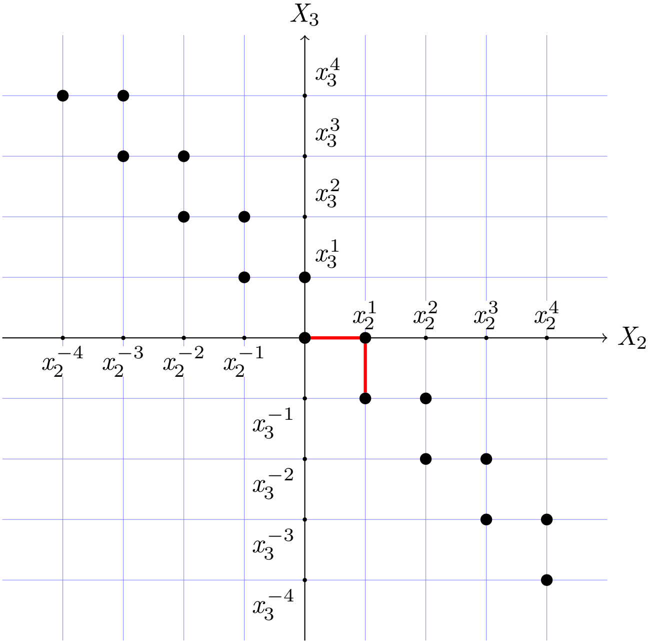

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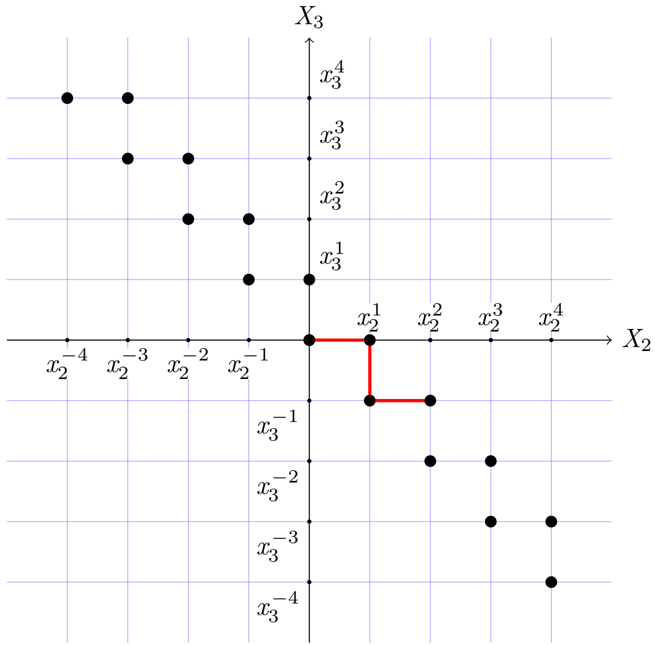

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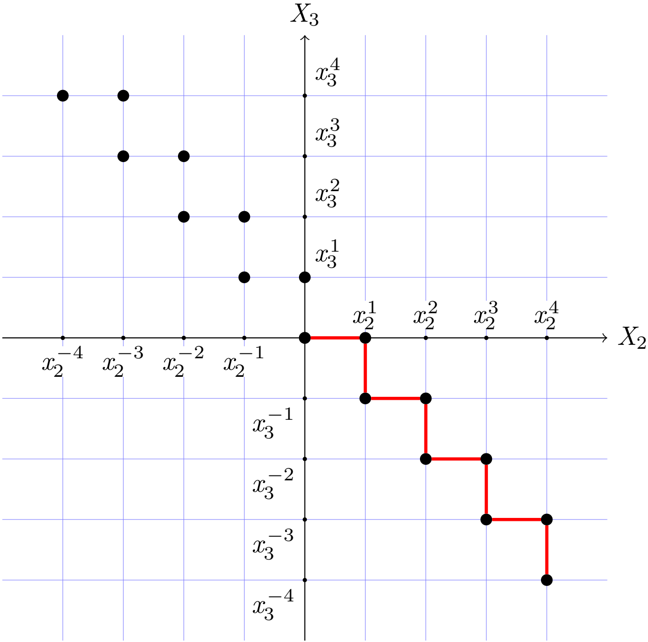

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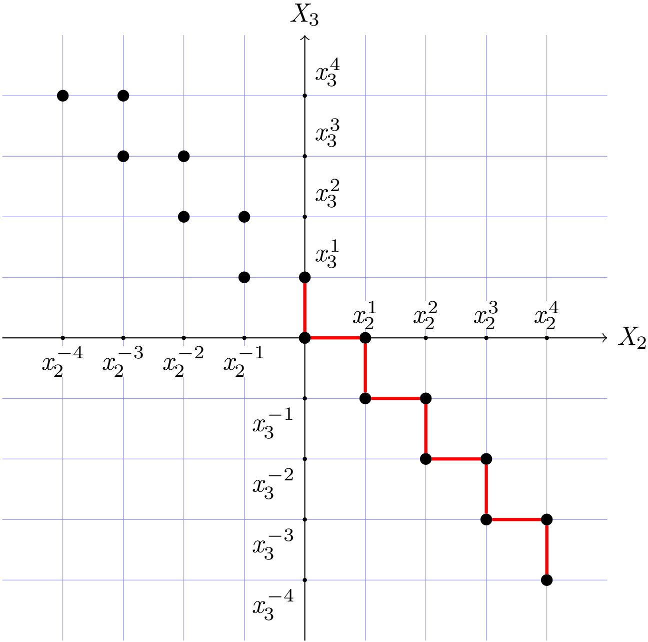
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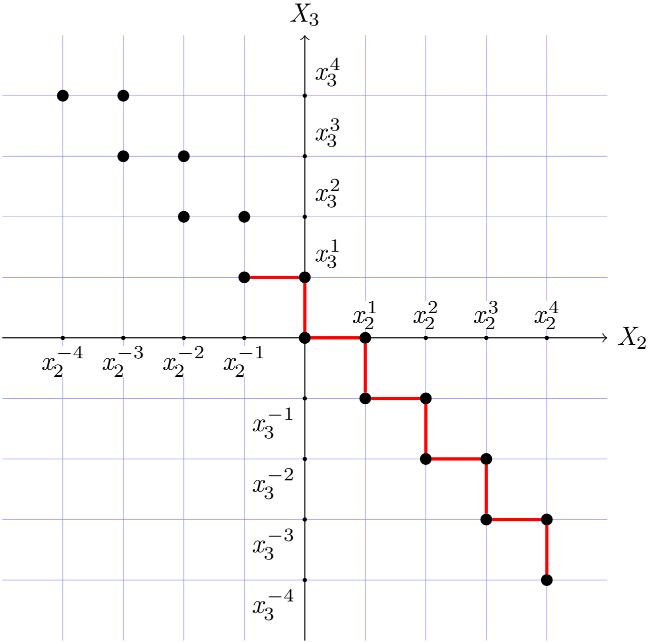
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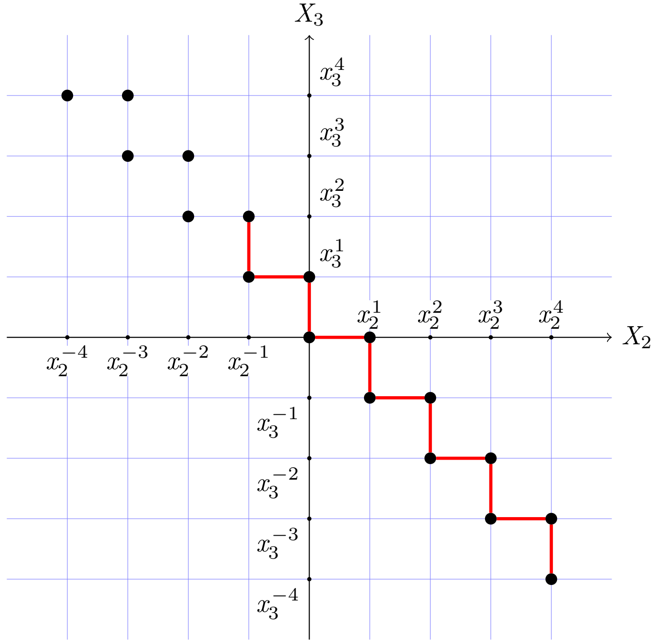

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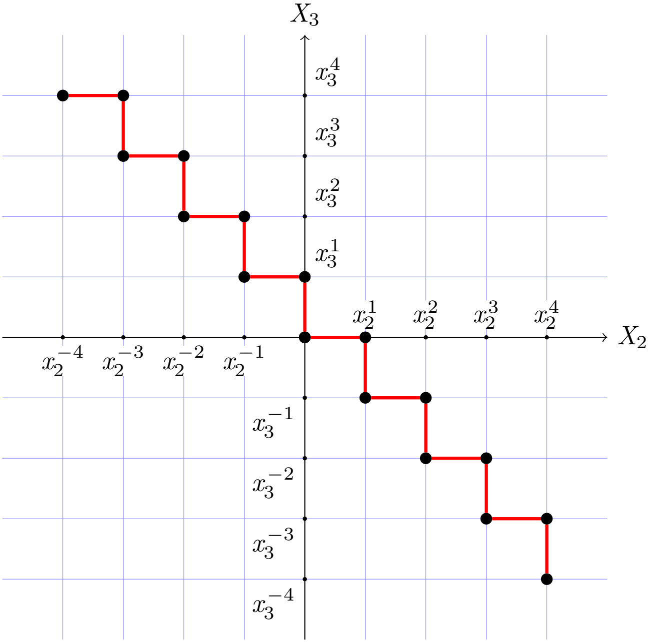
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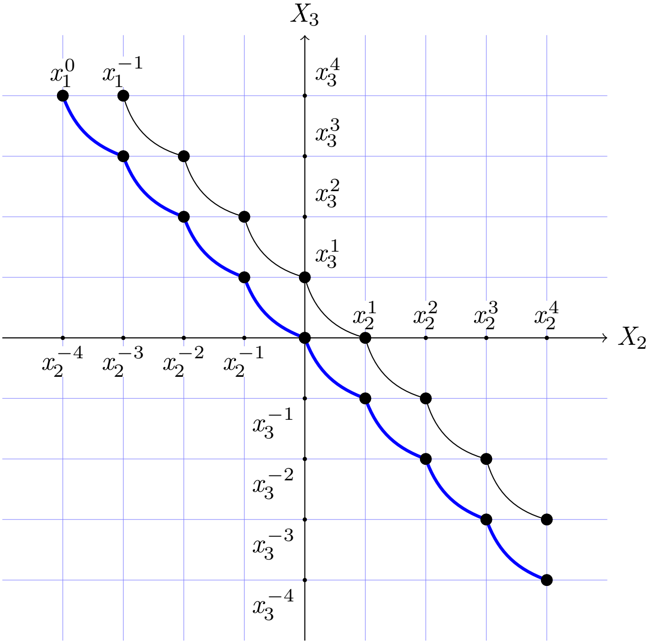
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Conditions

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- the function v_i orders the element of X_i
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Archimedean condition

- a diagonal standard sequence is able to “reach” all diagonal points

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Structural conditions

- solvability: we can find $(x_1^0, x_2^0, x_3^0) \in \mathscr{F}$
- influence of attribute 1: we can find x_1^{-1} below x_1^0

Thomsen condition

Problems

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- is the construction sound?

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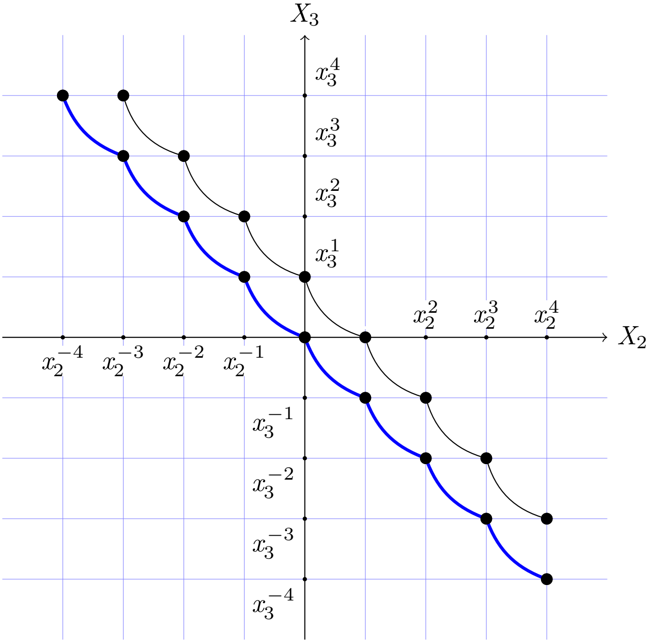
Interpretation

$$(x_2, x_3) \sim_{23} (y_2, y_3) \Leftrightarrow \left\{\begin{array}{l}(a_1, x_2, x_3) \in \mathscr{A} \Leftrightarrow (a_1, y_2, y_3) \in \mathscr{A} \\ (a_1, x_2, x_3) \in \mathscr{F} \Leftrightarrow (a_1, y_2, y_3) \in \mathscr{F} \\ (a_1, x_2, x_3) \in \mathscr{U} \Leftrightarrow (a_1, y_2, y_3) \in \mathscr{U}\end{array}\right.$$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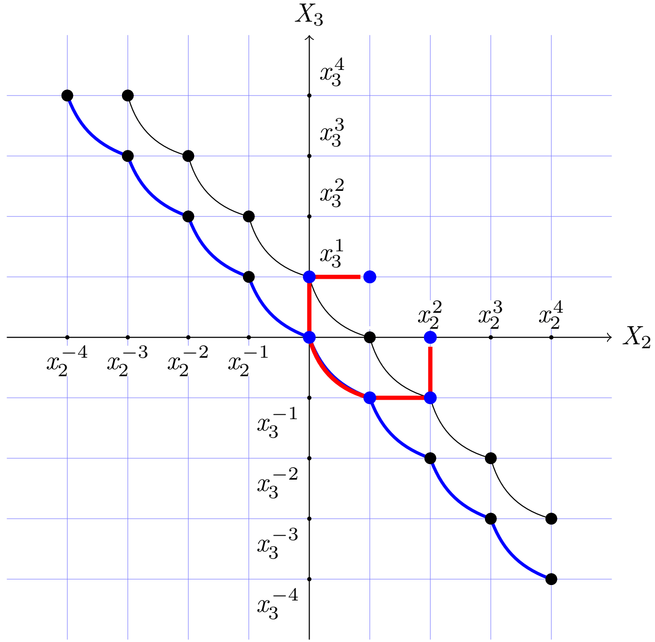

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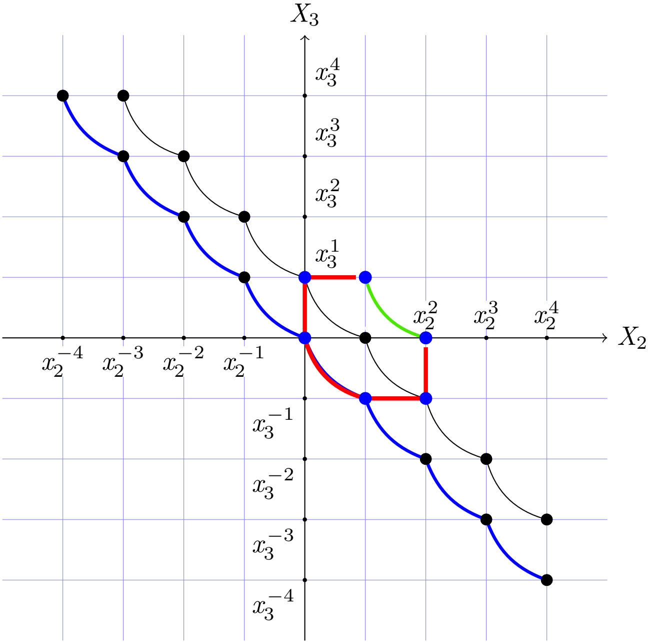
x_2^2
x_2^3
x_2^4
X_2
x_2^{-4}
x_2^{-3}
x_2^{-2}
x_2^{-1}
x_3^{-1}
x_3^{-2}
x_3^{-3}
x_3^{-4}


X_3
x_3^4
x_3^3
x_3^2
x_3^1
x_2^2
x_2^3
x_2^4
X_2
x_2^{-4}
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x_3^{-1}
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Outline

$\mathscr{A}$-linear

$\langle \mathscr{A}, \mathscr{F}, \mathscr{U} \rangle$ is $\mathscr{A}$-linear on $I \subset N$ if

$$\left.\begin{array}{c} (x_I, a_{-I}) \in \mathscr{A} \\ \text{and} \\ (y_I, b_{-I}) \in \mathscr{A} \end{array}\right\} \Rightarrow \left\{\begin{array}{c} (y_I, a_{-I}) \in \mathscr{A} \\ \text{or} \\ (x_I, b_{-I}) \in \mathscr{A} \end{array}\right.$$

$\mathscr{F}$-linear

$\langle \mathscr{A}, \mathscr{F}, \mathscr{U} \rangle$ is $\mathscr{F}$-linear on $I \subset N$ if

$$\left.\begin{array}{c} (x_I, a_{-I}) \in \mathscr{F} \\ \text{and} \\ (y_I, b_{-I}) \in \mathscr{F} \end{array}\right\} \Rightarrow \left\{\begin{array}{c} (y_I, a_{-I}) \in \mathscr{A}\mathscr{F} \\ \text{or} \\ (x_I, b_{-I}) \in \mathscr{A}\mathscr{F} \end{array}\right.$$

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$\langle \mathscr{A}, \mathscr{F}, \mathscr{U} \rangle$ is $\mathscr{AF}$-linear on $I \subset N$ if

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Linearity

$\langle \mathscr{A}, \mathscr{F}, \mathscr{U} \rangle$ is $\mathscr{A}\mathscr{F}$-linear on $I \subset N$ if

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Strong linearity

- $\mathscr{A}$-linear, $\mathscr{F}$-linear and $\mathscr{A}\mathscr{F}$-linear, for all $I \subset N$

Linearity

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$$x_I \succsim_I y_I \Leftrightarrow \text{for all } a_{-I} \in X_{-I}, \begin{cases}(y_I, a_{-I}) \in \mathscr{A} \Rightarrow (x_I, a_{-I}) \in \mathscr{A} \\ (y_I, a_{-I}) \in \mathscr{F} \Rightarrow (x_I, a_{-I}) \in \mathscr{AF}\end{cases}$$

$\succsim_I$ is complete $\Leftrightarrow$ $\mathscr{A}$-linear, $\mathscr{F}$-linear, $\mathscr{AF}$-linear on $I \subset N$

Thinness

$\langle \mathscr{A}, \mathscr{F}, \mathscr{U} \rangle$ is thin$_I$ if

$$\left.\begin{array}{c}(x_I, a_{-I}) \in \mathscr{F} \\ \text{and} \\ (y_I, a_{-I}) \in \mathscr{F}\end{array}\right\} \Rightarrow \begin{cases}(x_I, b_{-I}) \in \mathscr{A} \Leftrightarrow (y_I, b_{-I}) \in \mathscr{A} \\ (x_I, b_{-I}) \in \mathscr{U} \Leftrightarrow (y_I, b_{-I}) \in \mathscr{U}\end{cases}$$

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Strong thinness

- thin$_I$, for all $I \subset N$

Thomsen

$\langle \mathscr{A}, \mathscr{F}, \mathscr{U} \rangle$ on X satisfies the Thomsen condition if

$$\left.\begin{array}{c} (x_i, x_j, a_{-ij}) \in \mathscr{F} \ \& \ (x_i, x_j) \sim_{ij} (y_i, y_j) \\ (y_i, z_j, b_{-ij}) \in \mathscr{F} \ \& \ (y_i, z_j) \sim_{ij} (z_i, x_j) \end{array}\right\} \Rightarrow (x_i, z_j) \sim_{ij} (z_i, y_j)$$

Archimedean and Unrestricted solvability

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$\langle \mathscr{A}, \mathscr{F}, \mathscr{U} \rangle$ satisfies the Archimedean condition if a diagonal standard sequence that is strictly bounded must be finite

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$\langle \mathscr{A}, \mathscr{F}, \mathscr{U} \rangle$ satisfies unrestricted solvability if, for all $i \in N$ and all $x_{-i} \in X_{-i}$, $(x_i, x_{-i}) \in \mathscr{F}$, for some $x_i \in X_i$

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- unrestricted solvability is a strong condition
- can be weakened to restricted solvability with some connectedness spice

Outline

Main result

Theorem, B & Marchant (2009)

$\langle \mathscr{A}, \mathscr{F}, \mathscr{U} \rangle$ is an ordered partition on $X = X_1 \times X_2 \times \cdots \times X_n$, $n \geq 3$.
Suppose that $\langle \mathscr{A}, \mathscr{F}, \mathscr{U} \rangle$ is

- non-degenerate
- satisfies unrestricted solvability
- satisfies strong linearity
- satisfies strong thinness
- satisfies Archimedean condition
- (if $n = 3$) satisfies Thomsen

Then there is an additive representation $\langle v_i \rangle_{i \in N}$ of $\langle \mathscr{A}, \mathscr{F}, \mathscr{U} \rangle$

The functions $\langle v_i \rangle_{i \in N}$ are unique up to the choice of origins and unit

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- do you have something to say for the case of two attributes?

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- do you have something to say when there are more than two categories with a frontier?
  - yes: our main results generalize without major problem

Outline

Summary

Additive conjoint measurement

- additive value functions with tight uniqueness properties an be obtained on the basis of rather poor information: $\langle \mathscr{A}, \mathscr{F}, \mathscr{U} \rangle$
- reasonably simple conditions that can be tested in experiments

Usefulness to MCDM

All this is theory. . . but

- axioms lead to an assessment technique
- axiomatic analysis as a tool to compare models
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Growing literature on the foundations of sorting methods

- decomposable models and decision rules, GMS (2002)
- ELECTRE TRI
  - surprising relation to a Sugeno integral, B & Marchant (2007a, 2007b)

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C'EST TOUT POUR AUJOURD'HUI.