A brief and incomplete Introduction to Social Choice Theory

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What is Social Choice Theory?

- Aim: study decision problems in which a *group* has to take a decision
- Abstract Theory
  - Nature of the decision
  - Size of the group
  - Nature of the group
- Many (deep) results
  - Economics, Political Science, Applied Mathematics, OR
  - Two Nobel Prizes (K. Arrow, A. Sen)

DA/AI and SCT?

- SCT is a general theory of aggregation
- Possible examples of application in DA/AI
  - Several agents with different priorities
  - Several decision rules indicating different actions
  - Several states of nature with different consequences
  - Several criteria

- DA/AI people may also be Citizens (Elections)

Outline

- Introduction
- Examples
  - What can go wrong?
- Some results
  - What can be expected?
- Extensions

Introduction: Vocabulary

- Group
  - *Society*
- Members of the Group
  - *Voters*
- Alternatives
  - *Candidates*
- Problem
  - Choice of *one* among several *Candidates*

Aside: Proportional representation

- We’ll study procedures selecting a *single* candidate
- Why not be interested in more refined procedures electing more than one candidate (Proportional Representation)?
  - PR does not solve the decision problem in the Parliament!
  - PR raises many difficult problems (What is a just PR? How to achieve it? PR and Power indices)

Introduction

- The choice of the candidate will affect all members of the society
- The choice of the candidate should take into account the *opinion* of the members of the society

Democracy $\Rightarrow$ Elections $\Rightarrow$ Majority

Elections

- "Philosophical problems"
  - General will and elections
  - Minorities vs. Majority
- "Political problems"
  - Direct vs. indirect democracy
  - Role of political parties
  - Who should vote? How often should we vote?
  - Who can be a candidate?
  - What mandate?

Technical problems

- Majority decisions
  - Candidate *a* should beat candidate *b* if more voters prefer *a* to *b*
- Two candidates ⇒ No problem: elect the candidate with more votes!
- How to extend the idea with more than 2 candidates?
  - Many ways to do so!

Types of Elections

- Type of ballot that the voters can cast
  - Indicate the name of a candidate
  - Rank order the set of candidates
  - Other (acceptable or unacceptable candidates, grades, veto, etc.)
- Aggregation method
  - Technique used to tabulate the ballots and to designate the winner

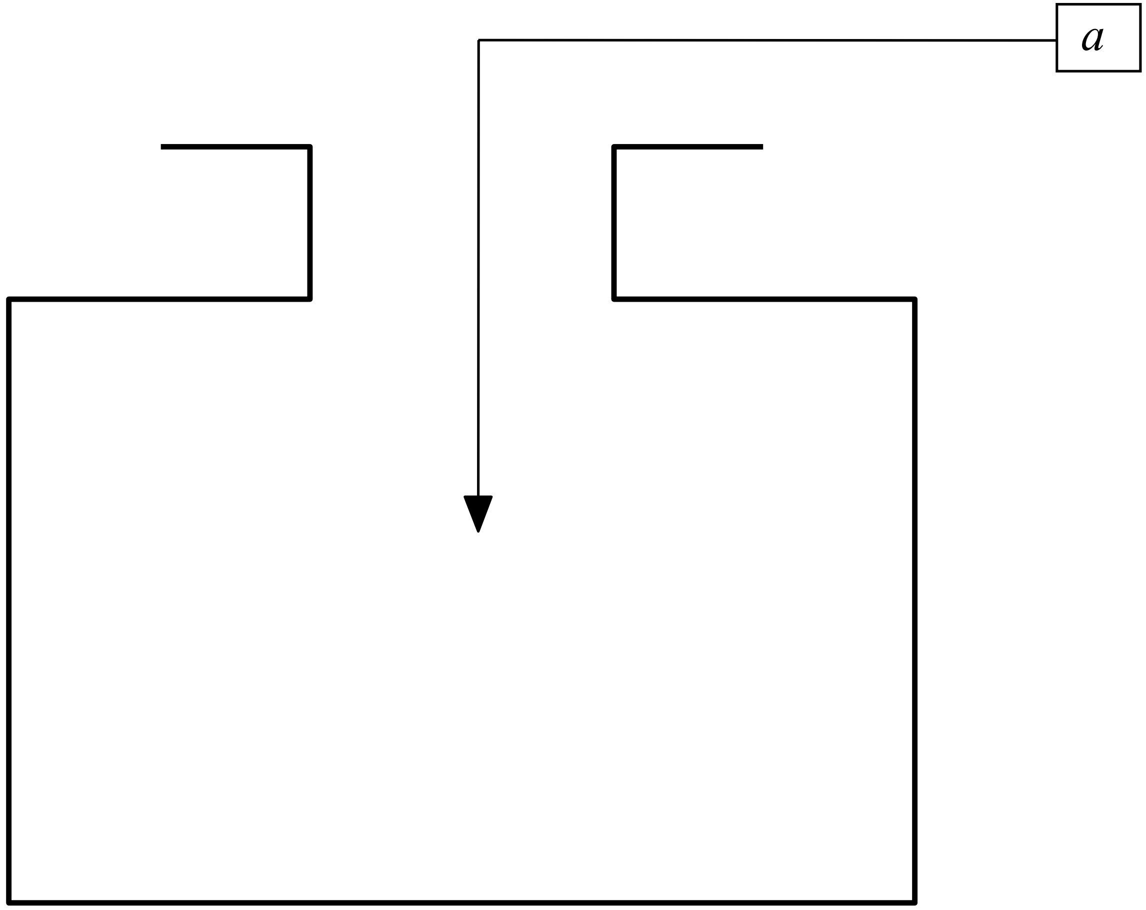
Hypothesis

- Each voter is able to rank order the set of candidates in terms of preference

  $a\ P\ b\ P\ [e\ I\ d]\ P\ c$

- Voters are *sincere*

Simple ballots



Plurality voting (UK)

- Ballots with a single name
- One round of voting
- The candidate with most votes is elected

ties (not likely) are neglected

Give some special tie-breaking power to one of the voter

Give some special special statute to one of the candidate

a : Tories

b : Labour

c : LibDem

3 candidates : $\{a, b, c\}$

21 voters (or 21 000 000 or 42 000 000)

Preferences of the voters

10 : $a\ P\ b\ P\ c$

6 : $b\ P\ c\ P\ a$

5 : $c\ P\ b\ P\ a$

Result

a : 10 b : 6 c : 5

a is elected

BUT…

An *absolute majority* of voters (11/21) prefer *all* other candidates to the candidate elected!

Is the UK system that democratic?

Can we expect the voters to be sincere?

Extra-democratic choice of only two candidates

Plurality voting with runoff (France – Presidential elections)

- Ballots with a single name
- 1st round of voting
  - The candidate with most votes is elected if he receives more than 50% of the votes
  - Otherwise go to a 2nd round of voting with the two candidates having received most votes in the first round
- 2nd round of voting
  - The candidate with most votes is elected

Preferences of the voters

10 : $a\ P\ b\ P\ \not{c}$

6 : $b\ P\ \not{c}\ P\ a$

5 : $\not{c}\ P\ b\ P\ a$

Apparently much better than the UK system

With little added complexity

1st round (absolute majority = 11)

a : 10 b : 6 c : 5

2nd round

a : 10 b : 11

b is elected (11/21)

AND

no candidate is preferred to b by a majority of voters

(a : 11/21, c : 16/21)

4 candidates: {*a*, *b*, *c*, *d*}

21 voters

10 : *b P* ~~*a*~~ *P c P* ~~*d*~~

6 : *c P* ~~*a*~~ *P* ~~*d*~~ *P b*

5 : ~~*a*~~ *P* ~~*d*~~ *P b P c*

1st Round (absolute majority = 11)

a : 5 *b* : 10 *c* : 6 *d* : 0

2nd Round

b : 15 *c* : 6

Result: *b* is (very well) elected (15/21)

BUT...

an absolute majority of voters (11/21) prefer candidates *a* and *d* to the candidate elected *b*!

The French system does only a little better than the UK system

Preferences used in the example are NOT bizarre

Sincerity?

Wasted votes

4 candidates : {*a*, *b*, *c*, *d*}

21 voters

10 : *b P a P c P d*

6 : *c P a P d P b*

5 : *a P d P b P c*

Result : *b* is elected

Non sincere voting

The 6 voters with *c P a P d P b*

decide to vote vote as if their preference was

a P c P d P b

(Do not waste your vote!)

Result : *a* is elected in the 1st round (11/21)

Voting non sincerely may be profitable

Method susceptible to **manipulation**

Manipulable methods ⇒ elections might not reveal the true opinion of the voters

Advantage to clever voters (knowing how to manipulate)

3 candidates: $\{a, b, c\}$

17 voters

Opinion poll

6 : $a\ P\ b\ P\ c$

5 : $c\ P\ a\ P\ b$

4 : $b\ P\ c\ P\ a$

2 : $b\ P\ a\ P\ c$

1st Round (absolute majority = 9)

a : 6 b : 6 c : 5

2nd Round

a : 11 b : 6

Nothing to worry about
up to now on this example

a starts a campaign against b

It works

2 voters: $b\ P\ a\ P\ c$
become
$a\ P\ b\ P\ c$

This change is favorable to a
which is the favorite

New preferences (after campaign)

6 : $a\ P\ b\ P\ c$

5 : $c\ P\ a\ P\ b$

4 : $b\ P\ c\ P\ a$

2 : $a\ P\ b\ P\ c$

Non monotonic method

Sincerity of voters?

1st Round (absolute majority = 9)

a : 8 b : 4 c : 5

2nd Round

a : 8 c : 9

c is elected!

The result of his succesful campaign is fatal to a

3 candidates: $\{a, b, c\}$

11 voters

4 : $a \; P \; b \; P \; c$

4 : $c \; P \; b \; P \; a$

3 : $b \; P \; c \; P \; a$

What if some voters abstain?

Abstention should NOT be profitable
(otherwise why vote?!)

1st round (absolute majority = 6)

a : 4 b : 3 c : 4

2nd round

a : 4 c : 7

Result: c elected (7/11)

3 candidates: $\{a, b, c\}$

11 – 2 = 9 voters

2 : $a\ P\ b\ P\ c$

4 : $c\ P\ b\ P\ a$

3 : $b\ P\ c\ P\ a$

1st round (majority = 5)

a : 2 b : 3 c : 4

2nd round

b : 5 c : 4

Result: b elected (5/9)

2 voters among the 4 : $a\ P\ b\ P\ c$ abstain

Abstaing was VERY rational for our two voters (they prefer b to c)

Not participation incentive!

3 candidates: $\{a, b, c\}$

26 voters: 13 in district 1, 13 in district 2

District 1

13 voters

4 : $a\ P\ b\ P\ c$

3 : $b\ P\ a\ P\ c$

3 : $c\ P\ a\ P\ b$

3 : $c\ P\ b\ P\ a$

1st round (majority = 7)

a : 4 b : 3 c : 6

2nd round

a : 7 c : 6

Result: a elected (7/13) in district 1

District 2

13 voters

4 : $a\ P\ b\ P\ c$

3 : $c\ P\ a\ P\ b$

3 : $b\ P\ c\ P\ a$

3 : $b\ P\ a\ P\ c$

1st round (majority = 7)

a : 4 b : 6 c : 3

2nd round

a : 7 b : 6

Result: a elected (7/13) in district 2

a is elected in both district...

AND THUS should be elected

26 voters

4 : $a\ P\ b\ P\ c$

3 : $b\ P\ a\ P\ c$

3 : $c\ P\ a\ P\ b$

3 : $c\ P\ b\ P\ a$

4 : $a\ P\ b\ P\ c$

3 : $c\ P\ a\ P\ b$

3 : $b\ P\ c\ P\ a$

3 : $b\ P\ a\ P\ c$

Entire Society

a is elected in both districts

but looses when grouped

Non separable method

Decentralized decisions?

1st Round (majority = 14)

a : 8 b : 9 c : 9 ***a* looses in the first round!**

2nd Round

b : 17 c : 9

Result: *b* elected (17/26)

Summary

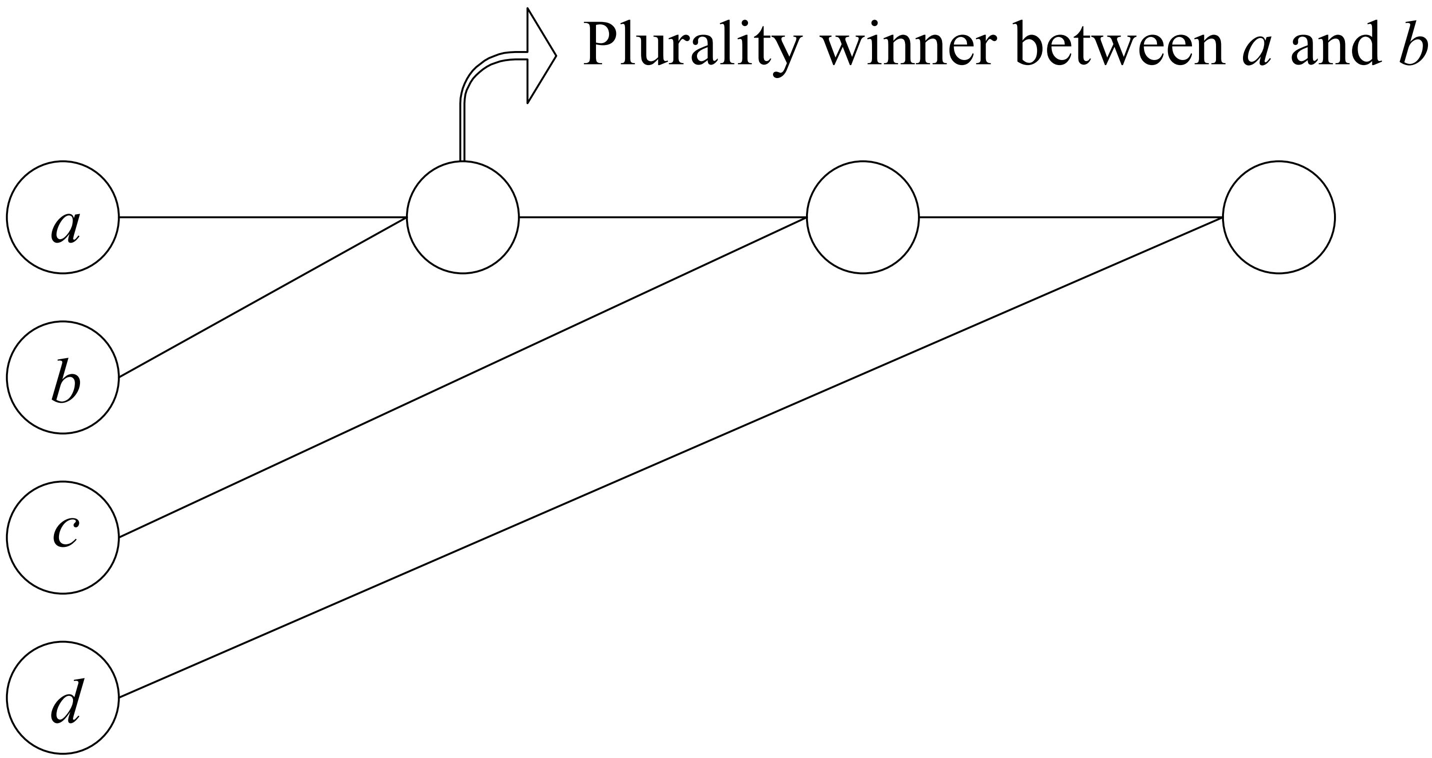
- The French system does only a little better better than the UK one on the "democratic side"
- It has many other problems
  - not monotonic
  - no incentive to participate
  - manipulable
  - non separable
- Other (better!) systems?

Amendment procedure

- The majority method works well with two candidates
- When there are more than two candidates, organize a series of confrontations between two candidates according to an *agenda*
- Method used in most parliaments
  - amendments to a bill
  - bill amended *vs.* status quo

4 candidates {*a*, *b*, *c*, *d*}

Agenda: *a, b, c, d*



Exemple: *c* is a bill, *a* and *b* are amendments, *d* is the status quo

3 candidates: {*a, b, c*}

3 voters

1 voter: *a P b P c*

1 voter: *b P c P a*

1 voter: *c P a P b*

Agenda: *a, b, c*	**Result**: *c*
Agenda: *b, c, a*	**Result**: *a*
Agenda: *c, a, b*	**Result**: *b*

Results depending on the arbitrary choice of an agenda
(power given to the agenda-setter)
Candidates are not treated equally (the later the better)

4 candidates: $\{a, b, c, d\}$

3 voters

1 voter: $b \; P \; a \; P \; d \; P \; c$

1 voter: $c \; P \; b \; P \; a \; P \; d$

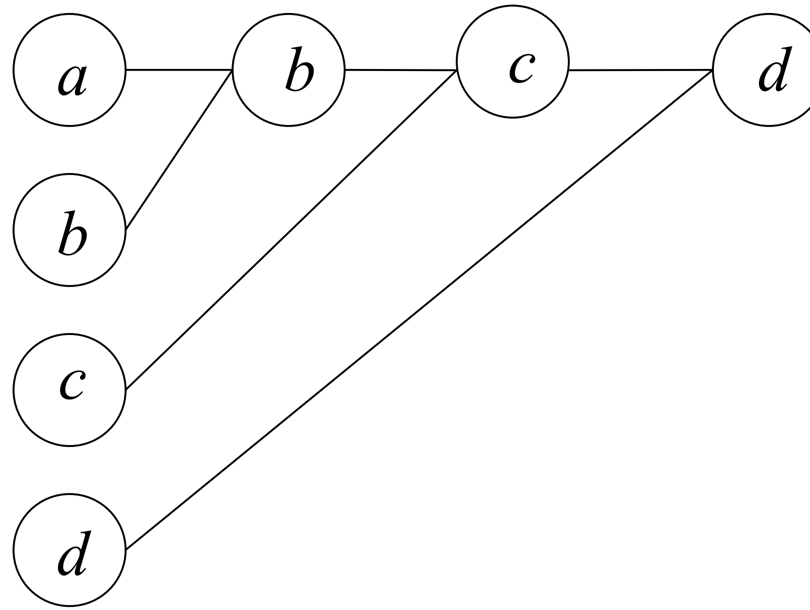
1 voter: $a \; P \; d \; P \; c \; P \; b$

Agenda: a, b, c, d

Result: d elected

BUT...

100% of voters prefer a to d !



Non unanimous method

26 candidates: $\{a, b, c, ..., z\}$

100 voters

51 voters: $a \ P \ b \ P \ c \ P \ ... \ P \ y \ P \ z$

49 voters: $z \ P \ b \ P \ c \ P \ ... \ P \ y \ P \ a$

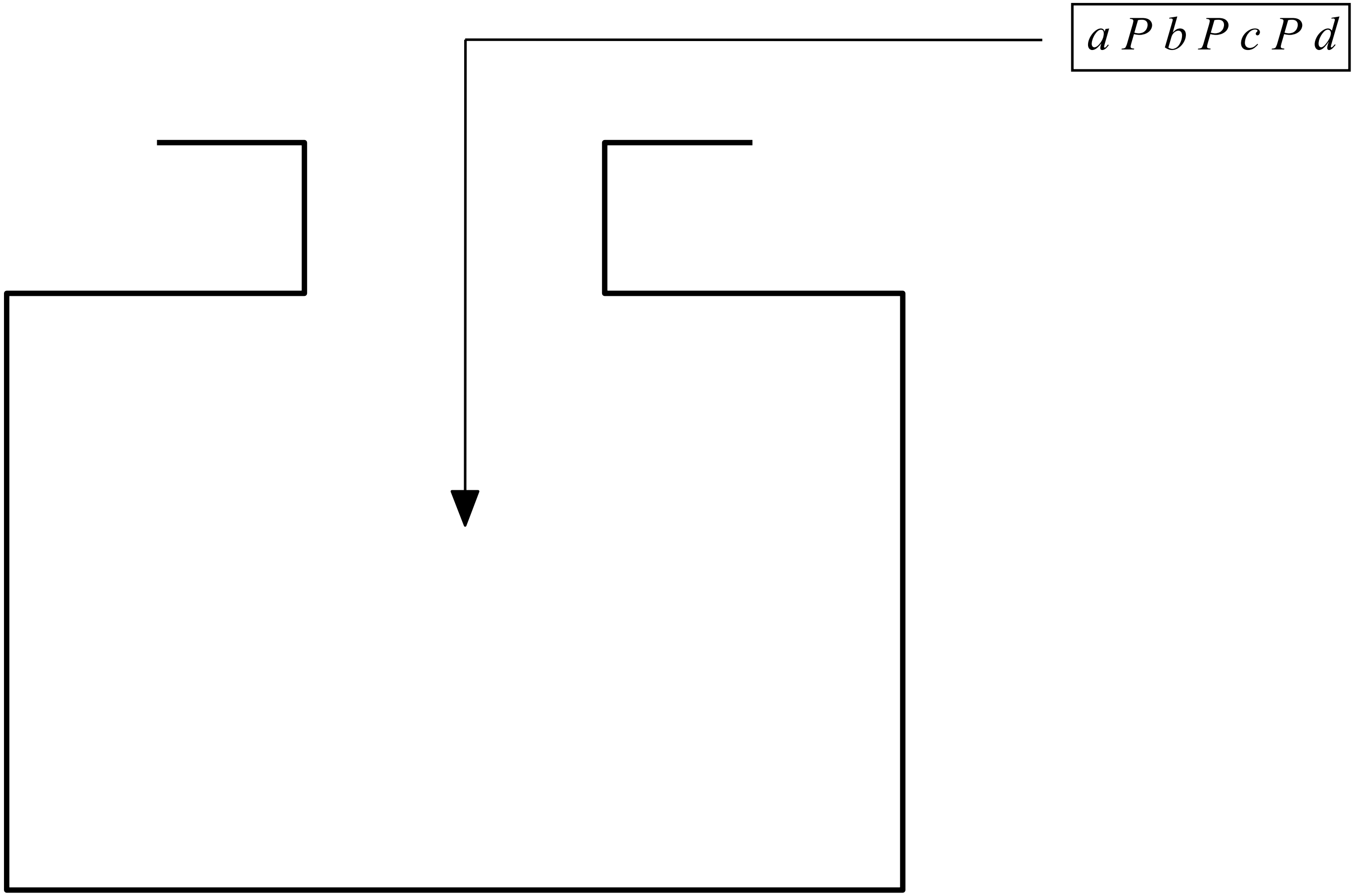
With sincere voters and with all majority-based systems with only one name per ballot, *a* is elected and the "compromise" candidate *b* is rejected

Dictature of the majority

(recent European history?)

$\Rightarrow$ look for more refined ballots

Ballots: Ordered lists



Remarks

- Much richer information
  - practice?
- Ballots with one name are a particular case

Condorcet

- Compare all candidates by pair
- Declare that a is "socially preferred" to b if (strictly) more voters prefer a to b

  (social indifference in case of a tie)
- **Condorcet's principle**: if one candidate is preferred to *all other* candidates, it should be elected.
- **Condorcet Winner** (must be unique)

Remarks

- UK and French systems violate Condorcet’s principle
- The UK system may elect a Condorcet looser
- Condorcet’s principle does not solve the “dictature of the majority” difficulty
- A Condorcet winner is not necessarily “ranked high” by voters
- An attractive concept however... BUT

3 candidates: {*a*, *b*, *c*}

21 voters

Preferences of the voters

10 : *a P b P c*

6 : *b P c P a*

5 : *c P b P a*

a is the plurality winner

b is the Condorcet Winner (11/21 over *a*, 16/21 over *c*)

a is the Condorcet Looser (10/21 over *b*, 10/21 over *c*)

4 candidates: $\{a, b, c, d\}$

21 voters

10 : $b\ P\ a\ P\ c\ P\ d$

6 : $c\ P\ a\ P\ d\ P\ b$

5 : $a\ P\ d\ P\ b\ P\ c$

b is the plurality with runoff winner

a is the Condorcet Winner

(11/21 over *b*, 15/21 over *c*, 21/21 over *d*)

5 candidates: $\{a, b, c, d, e\}$

5 voters

1 voter: $a\ P\ b\ P\ c\ P\ d\ P\ e$

1 voter: $b\ P\ c\ P\ e\ P\ d\ P\ a$

1 voter: $e\ P\ a\ P\ b\ P\ c\ P\ d$

1 voter: $a\ P\ b\ P\ d\ P\ e\ P\ c$

1 voter: $b\ P\ d\ P\ c\ P\ a\ P\ e$

Ranks	1	2	3	4	5
a	2	1	0	1	1
b	2	2	1	0	0

a is the Condorcet winner

(3:2 win on all other candidates)

3 candidates: $\{a, b, c\}$

3 voters

1 : $a \; P \; b \; P \; c$

1 : $b \; P \; c \; P \; a$

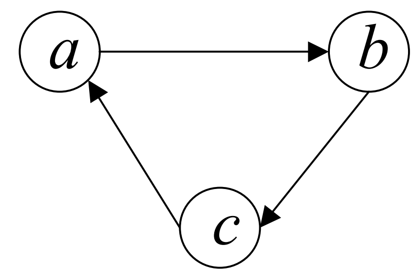
1 : $c \; P \; a \; P \; b$

Condorcet's Paradox

a is socially preferred to b

b is socially preferred to c

c is socially preferred to a



As the social preference relation may have cycles, a Condorcet winner does not always exist (probability 40% with 7 candidates and a large number of voters)

McGarvey's Theorem

Condorcet

- Weaken the principle so as to elect candidates that are not strictly beaten

  (Weak CW)
  - they may not exist
  - there may be more than one
- Find what to do when there is no (weak) Condorcet winner

Schwartz

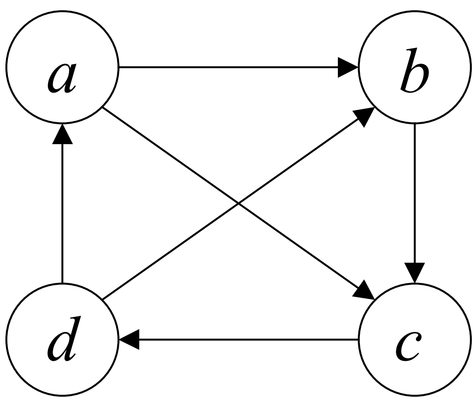
- The strict social preference may not be transitive
  – Take its transitive closure
  – Take the maximal elements of the resulting weak order

4 candidates: {*a*, *b*, *c*, *d*}, 3 voters

1 : *a P b P c P d*

1 : *d P a P b P c*

1 : *c P d P a P b*



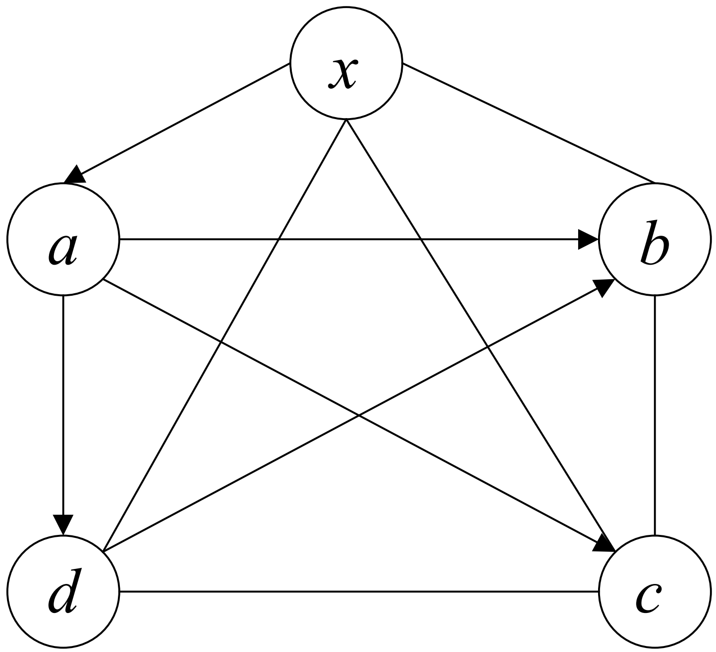
Taking the transitive closure, all alternatives are indiffrent

BUT....

100% of the voters prefer *a* to *b*

Copeland

- Count the number of candidates that are beaten by one candidate minus the number of candidates that beat him (Copeland score)
- Elect the candidate with the highest score
- Sports league
  - +2 for a victory, +1 for a tie
  - equivalent to Copeland's rule (round robin tournaments)



x	**1**
a	**2**
b	**-2**
c	**-1**
d	**0**

x is the only unbeaten candidate but is not elected

Borda

- Each ballot is an ordered list of candidates (exclude ties for simplicity)
- On each ballot compute the rank of the candidates in the list
- Rank order the candidates according to the decreasing sum of their ranks

4 candidates: {a, b, c, d}

3 voters

2 : $b\ P\ a\ P\ c\ P\ d$

1 : $a\ P\ c\ P\ d\ P\ b$

	1st	2nd	3rd	4th
a	1	2	0	0
b	2	0	0	1
c	0	1	2	0
d	0	0	1	2

Borda Scores

a : 2×2 + 1×1 = 5 b : 6 c : 8 d : 11

Result: a elected

Remark: b is the (obvious) Condorcet winner

Borda

- Simple
- Efficient: always lead to a result
- Separable, monotonic, participation incentive

BUT...

- Violates Condorcet's Principle
- Has other problems
  - consistency of choice in case of withdrawals

4 candidates: $\{a, b, c, d\}$

3 voters

2 : $b\ P\ a\ P\ c\ P\ d$

1 : $a\ P\ c\ P\ d\ P\ b$

Borda Scores

$a : 2\times2 + 1\times1 = 5 \qquad b : 6 \qquad c : 8 \qquad d : 11$

Result: *a* elected

Suppose that *c* and *d* withdraw from the competition

Borda Scores

$a : 2\times2 + 1\times1 = 5 \qquad b : 4$

Result: *b* elected

Is the choice of a method important?

4 candidates: $\{a, b, c, d\}$, 27 voters

5 : $a \, P \, b \, P \, c \, P \, d$

4 : $a \, P \, c \, P \, b \, P \, d$

2 : $d \, P \, b \, P \, a \, P \, c$

6 : $d \, P \, b \, P \, c \, P \, a$

8 : $c \, P \, b \, P \, a \, P \, d$

2 : $d \, P \, c \, P \, b \, P \, a$

d is the plurality winner

a is the plurality with runoff winner

b is the Borda winner

c is the Condorcet winner

What are we looking for?

- "Democratic method"
  - always giving a result like Borda
  - always electing the Condorcet winner
  - consistent wrt withdrawals
  - monotonic, separable, incentive to participate, not manipulable, etc.

Arrow

- n ≥ 3 candidates (otherwise use plurality)
- m voters (m ≥ 2 and finite)
- ballots = ordered list of candidates

- Problem: find all “methods” respecting a small number of “desirable” principles

- **Universality**: the method should be able to deal with any configuration of ordered lists
- **Transitivity**: the result of the method should be an ordered list of candidates
- **Unanimity**: the method should respect a unanimous preference of the voters
- **Absence of dictator**: the method should not allow for dictators
- **Independence**: the comparison of two candidates should be based only on their respective standings in the ordered lists of the voters

Arrow's Theorem (1951)

- Theorem: There is no method respecting the five principles
  - Borda is
    - universal, transitive, unanimous with no dictator

  $\Rightarrow$ it cannot be independent
  - Condorcet is
    - universal, unanimous, independent with no dictator

  $\Rightarrow$ it cannot be transitive

Sketch of the proof

- $V \subseteq N$ is *decisive* for (a,b) if whenever $a\ P_i\ b$ for all $i \in V$ then $a\ P\ b$
- $V \subseteq N$ is *almost decisive* for (a,b) if whenever

  $a\ P_i\ b$ for all $i \in V$ and

  $b\ P_j\ a$ for all $j \notin V$

  then $a\ P\ b$

Lemma 1

- If V is almost decisive over some ordered pair (a,b), it is decisive over all ordered pairs.

$\{a, b, x, y\}$ and use *universality* to obtain:

$V : x\ P\ a\ P\ b\ P\ y$

$N\backslash V : x\ P\ a,\ b\ P\ y,\ b\ P\ a$ (position of x and y unspecified)

Unanimity $\Rightarrow x\ P\ a$ and $b\ P\ y$

V is almost decisive for $(a,b) \Rightarrow a\ P\ b$

$\Rightarrow x\ P\ y$ (*transitivity*)

Independence $\Rightarrow$ the ordering of a and b is irrelevant

Lemma 2

- If V is decisive and card(V) > 1, then some proper subset of V is decisive

$\{x, y, z\}$ use *universality* to obtain:

$V1 : x \; P \; y \; P \; z$

$V2 : y \; P \; z \; P \; x$

$N \backslash V : z \; P \; x \; P \; y$

V decisive $\Rightarrow y \; P \; z$

If $x \; P \; z$ then $V1$ is almost decisive for (x, z) and thus decisive (lemma 1)

If $z \; R \; x$ then $y \; P \; x$ (*transitivity*) and $V2$ is almost decisive for (y, x) and thus decisive (lemma 1)

Proof

- *Unanimity* $\Rightarrow$ *N* is decisive
- Since *N* is finite the iterated use of lemma 2 leads to the existence of a dictator

Principles

- *Unanimity*: no apparent problem
- *Absence of dictator*: minimal requirement of democracy!
- *Universality*: a group adopting functioning rules that would not function in "difficult situations" could be in big trouble!

Independence

- no *intensity of preference* considerations
  - I "intensely" or "barely" prefer *a* to *b*
    - practice, manipulation, interpersonal comparisons?
- no consideration of a third alternative to rank order *a* and *b*

Borda and Independence

4 candidates: $\{a, b, c, d\}$, 3 voters

2 voters: $c\ P\ a\ P\ b\ P\ d$

1 voter: $a\ P\ b\ P\ c\ P\ d$

Borda: $a\ P\ c\ P\ b\ P\ d$ (scores : 5, 6, 7 and 11)

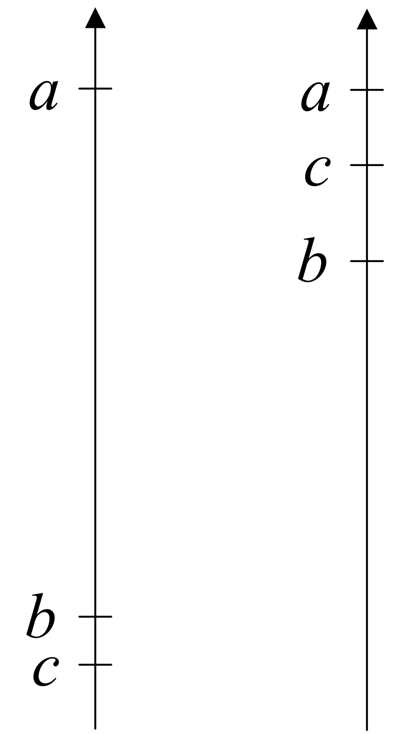
2 voters: $c\ P\ a\ P\ b\ P\ d$

1 voter: $a\ P\ c\ P\ b\ P\ d$

Borda: $c\ P\ a\ P\ b\ P\ d$ (scores : 4, 5, 9 and 12)

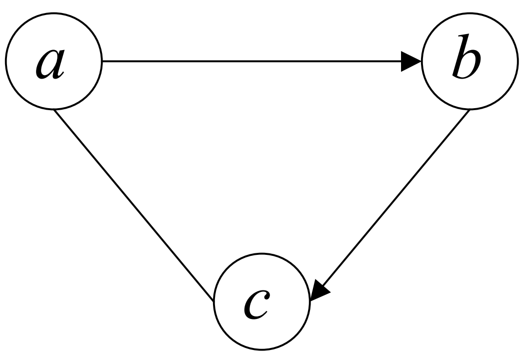
The ranking of *a* and *c* is reversed

BUT... the respective positions of *a* and *c* is unchanged in the individual lists



Transitivity

- maybe too demanding if the only problem is to elect a candidate
- BUT... guarantees consistency



In $\{a, b, c\}$, a is elected

In $\{a, c\}$, both a and c are elected

Relaxing transitivity

- Semi-orders and interval order
  - no change (if more than 4 candidates)
- Transitivity of strict preference
  - oligarchy: group O of voters st

    $a \, P_i \, b \; \forall \, i \in O \Rightarrow a \, P \, b$

    $i \in O$ and $a \, P_i \, b \Rightarrow \text{Not}[b \, P \, a]$
- Absence of cycles
  - some voter has a veto power

    $a \, P_i \, b \Rightarrow \text{Not}[b \, P \, a]$

Message?

- Despair
  - no “ideal” method (this would be dull!)

BUT...

- A group is more complex than an individual
- Analyze the pros and cons of each method
- Beware of “method-sellers”

Extensions

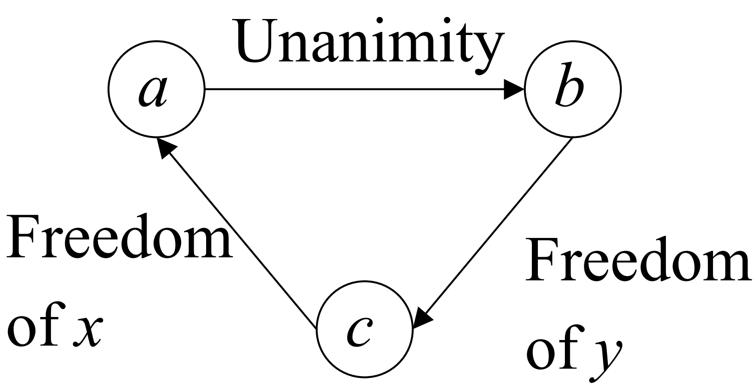
- Impossibility results
  - Arrow
  - Gibbard-Sattherthwaite
    - All “reasonable methods” may be manipulated (more or less easily or frequently)
  - Moulin
    - No separable method can be Condorcet
    - No Condorcet method can give an incentive to participate
  - Sen
    - tensions between unanimity and individual freedom

Paretian Liberal Paradox

- There are obvious tensions between the majority principle and the respect of individual rights
- Paradox: there are tensions between the respect of individual rights and the unanimity principle
- Theorem: Unanimity+universality+respect of individual rights $\Rightarrow$ Problems

Example

- 2 individuals (males) on a desert island
  - Mr. x the Puritan and Mr. y the Liberal
- A pornographic brochure
  - 3 social states
    - a : x reads
    - b : y reads
    - c : nobody reads
  - Preferences
    - x : $c \; P \; a \; P \; b$
    - y : $a \; P \; b \; P \; c$



Extensions

- Characterization results
  - find a list of properties that a method is the only one to satisfy simultaneously
    - Borda
    - Copeland
    - Plurality
  - Neutral, anonymous and separable method are of Borda-type (Young 1975)
- Analysis results
  - find a list of desirable properties
  - fill up the methods×properties table

Conclusion

- Little hope to find THE method
- Immense literature: DO NOT re-invent the wheel
  - these problems and results generalize easily to other settings
    - fuzzy preference
    - states of nature
    - etc.

Other aspects

- Institutional setting
- Welfare judgments
- Direct vs. indirect democracy
  - Ostrogorski paradox
  - Referendum paradox
- Electoral platforms
- Paradox of voting (why vote?)