A brief and incomplete Introduction to Social Choice Theory

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What is Social Choice Theory?

- Aim: study decision problems in which a *group* has to take a decision
- Abstract Theory
  - Nature of the decision
  - Size of the group
  - Nature of the group
- Many (deep) results
  - Economics, Political Science, Applied Mathematics, OR
  - Two Nobel Prizes (K. Arrow, A. Sen)
DA/AI and SCT?

- SCT is a general theory of aggregation
- Possible examples of application in DA/AI
  - Several agents with different priorities
  - Several decision rules indicating different actions
  - Several states of nature with different consequences
  - Several criteria
- DA/AI people may also be Citizens (Elections)
Outline

• Introduction
• Examples
  – What can go wrong?
• Some results
  – What can be expected?
• Extensions
Introduction: Vocabulary

• Group
  – Society

• Members of the Group
  – Voters

• Alternatives
  – Candidates

• Problem
  – Choice of one among several Candidates
Aside: Proportional representation

• We’ll study procedures selecting a single candidate
• Why not be interested in more refined procedures electing more than one candidate (Proportional Representation)?
  – PR does not solve the decision problem in the Parliament!
  – PR raises many difficult problems (What is a just PR? How to achieve it? PR and Power indices)
Introduction

• The choice of the candidate will affect all members of the society

• The choice of the candidate should take into account the opinion of the members of the society

Democracy ⇒ Elections ⇒ Majority
Elections

• “Philosophical problems”
  – General will and elections
  – Minorities vs. Majority

• “Political problems”
  – Direct vs. indirect democracy
  – Role of political parties
  – Who should vote? How often should we vote?
  – Who can be a candidate?
  – What mandate?
Technical problems

• Majority decisions
  – Candidate $a$ should beat candidate $b$ if more voters prefer $a$ to $b$

• Two candidates $\Rightarrow$ No problem: elect the candidate with more votes!

• How to extend the idea with more than 2 candidates?
  – Many ways to do so!
Types of Elections

• Type of ballot that the voters can cast
  – Indicate the name of a candidate
  – Rank order the set of candidates
  – Other (acceptable or unacceptable candidates, grades, veto, etc.)

• Aggregation method
  – Technique used to tabulate the ballots and to designate the winner
Hypothesis

- Each voter is able to rank order the set of candidates in terms of preference
  \[ a P b P [e I d] P c \]
- Voters are sincere
Simple ballots
Plurality voting (UK)

- Ballots with a single name
- One round of voting
- The candidate with most votes is elected

Ties (not likely) are neglected

Give some special tie-breaking power to one of the voter

Give some special special statute to one of the candidate
3 candidates: \{a, b, c\}
21 voters (or 21 000 000 or 42 000 000)

Preferences of the voters
10 : a P b P c
6 : b P c P a
5 : c P b P a

Result
a : 10  b : 6  c : 5
a is elected

BUT…

An *absolute majority* of voters (11/21) prefer *all* other candidates to the candidate elected!
Plurality voting with runoff
(France – Presidential elections)

- Ballots with a single name
- 1st round of voting
  - The candidate with most votes is elected if he receives more than 50% of the votes
  - Otherwise go to a 2nd round of voting with the two candidates having received most votes in the first round
- 2nd round of voting
  - The candidate with most votes is elected
Preferences of the voters

10 : $a P b P \not\in$
6 : $b P \not\in P a$
5 : $\not\in P b P a$

1st round (absolute majority = 11)
$a : 10 \quad b : 6 \quad c : 5$

2nd round
$a : 10 \quad b : 11$

$b$ is elected (11/21)

AND

no candidate is preferred to $b$ by a majority of voters
$(a : 11/21, c : 16/21)$

Apparently much better than the UK system

With little added complexity
4 candidates: \{a, b, c, d\}

21 voters

preferences used in the example are NOT bizarre

1st Round (absolute majority = 11)

a: 5 \quad b: 10 \quad c: 6 \quad d: 0

2nd Round

b: 15 \quad c: 6

Result: b is (very well) elected (15/21)

BUT...

an absolute majority of voters (11/21) prefer candidates a and d to the candidate elected b!
Manipulable methods ⇒ elections might not reveal the true opinion of the voters

Advantage to clever voters (knowing how to manipulate)

4 candidates: \{a, b, c, d\}

21 voters

10: b P a P c P d

6: c P a P d P b

5: a P d P b P c

Result: b is elected

Non sincere voting

The 6 voters with c P a P d P b decide to vote as if their preference was a P c P d P b

(Do not waste your vote!)

Result: a is elected in the 1st round (11/21)

Voting non sincerely may be profitable

Method susceptible to manipulation
3 candidates: \( \{a, b, c\} \)
17 voters

**Opinion poll**

6 : \( a P b P c \)
5 : \( c P a P b \)
4 : \( b P c P a \)
2 : \( b P a P c \)

1st Round (absolute majority = 9)

\( a : 6 \quad b : 6 \quad c : 5 \)

2nd Round

\( a : 11 \quad b : 6 \)

Nothing to worry about up to now on this example

\( a \) starts a campaign against \( b \)

It works

2 voters: \( b P a P c \) become \( a P b P c \)

This change is favorable to \( a \) which is the favorite
New preferences (after campaign)

6 : \(a P b P c\)
5 : \(c P a P b\)
4 : \(b P c P a\)
2 : \(a P b P c\)

1st Round (absolute majority = 9)
\[a : 8 \quad b : 4 \quad c : 5\]

2nd Round
\[a : 8 \quad c : 9\]

\(c\) is elected!

The result of his successful campaign is fatal to \(a\)

Non monotonic method
Sincerity of voters?
3 candidates: \{a, b, c\}

11 voters

4 : a P b P c
4 : c P b P a
3 : b P c P a

1st round (absolute majority = 6)

\begin{align*}
a & : 4 \\
b & : 3 \\
c & : 4
\end{align*}

2nd round

\begin{align*}
a & : 4 \\
c & : 7
\end{align*}

Result: \textit{c} elected (7/11)

Abstention should NOT be profitable (otherwise why vote?!)

What if some voters abstain?
3 candidates: \{a, b, c\}
11 − 2 = 9 voters
2 : a P b P c
4 : c P b P a
3 : b P c P a

1st round (majority = 5)
a : 2  b : 3  c : 4

2nd round
b : 5  c : 4

Result: b elected (5/9)

2 voters among the 4 : a P b P c  abstain
Abstaining was VERY rational for our two voters (they prefer b to c)
Not participation incentive!
3 candidates: \( \{a, b, c\} \)

26 voters: 13 in district 1, 13 in district 2

**District 1**

13 voters

\[
\begin{align*}
4 & : a P b P c \\
3 & : b P a P c \\
3 & : c P a P b \\
3 & : c P b P a \\
\end{align*}
\]

**1st round** (majority = 7)

\[
\begin{align*}
a & : 4 \\
b & : 3 \\
c & : 6 \\
\end{align*}
\]

**Result:** \(a\) elected (7/13) in district 1

\[
\begin{align*}
a & : 7 \\
c & : 6 \\
\end{align*}
\]

**2nd round**
District 2
13 voters
4 : a P b P c
3 : c P a P b
3 : b P c P a
3 : b P a P c

1st round (majority = 7)
a : 4   b : 6   c : 3

2nd round
a : 7   b : 6

Result: a elected (7/13) in district 2

a is elected in both district...

AND THUS should be elected
26 voters
4 : a P b P c
3 : b P a P c
3 : c P a P b
3 : c P b P a
4 : a P b P c
3 : c P a P b
3 : b P c P a
3 : b P a P c

1st Round (majority = 14)
a : 8  b : 9  c : 9  a looses in the first round!

2nd Round
b : 17  c : 9

Result: b elected (17/26)

Entire Society

a is elected in both districts but looses when grouped

Non separable method
Decentralized decisions?
Summary

• The French system does only a little better than the UK one on the “democratic side”

• It has many other problems
  – not monotonic
  – no incentive to participate
  – manipulable
  – non separable

• Other (better!) systems?
Amendment procedure

- The majority method works well with two candidates.
- When there are more than two candidates, organize a series of confrontations between two candidates according to an agenda.
- Method used in most parliaments:
  - amendments to a bill
  - bill amended vs. status quo
4 candidates \{a, b, c, d\}

Agenda: a, b, c, d

Exemple: c is a bill, a and b are amendments, d is the status quo
3 candidates: \{a, b, c\}

3 voters
1 voter: \(a \ Preference \ b \ Preference \ c\)
1 voter: \(b \ Preference \ c \ Preference \ a\)
1 voter: \(c \ Preference \ a \ Preference \ b\)

Agenda: \(a, b, c\) \hspace{1cm} \textbf{Result}: \(c\)
Agenda: \(b, c, a\) \hspace{1cm} \textbf{Result}: \(a\)
Agenda: \(c, a, b\) \hspace{1cm} \textbf{Result}: \(b\)

Results depending on the arbitrary choice of an agenda (power given to the agenda-setter)
Candidates are not treated equally (the later the better)
4 candidates: \( \{a, b, c, d\} \)

3 voters

1 voter: \( b \ P a \ P d \ P c \)
1 voter: \( c \ P b \ P a \ P d \)
1 voter: \( a \ P d \ P c \ P b \)

Agenda: \( a, b, c, d \)

Result: \( d \) elected

BUT...

100% of voters prefer \( a \) to \( d \)!

Non unanimous method
26 candidates: \{a, b, c, ..., z\}

100 voters

51 voters: \( a \ P b \ P c \ P \ ... \ P y \ P z \)

49 voters: \( z \ P b \ P c \ P \ ... \ P y \ P a \)

With sincere voters and with all majority-based systems with only one name per ballot, \( a \) is elected and the “compromise” candidate \( b \) is rejected

Dictature of the majority

(recent European history?)

\( \Rightarrow \) look for more refined ballots
Ballots: Ordered lists
Remarks

• Much richer information
  – practice?
• Ballots with one name are a particular case
Condorcet

- Compare all candidates by pair
- Declare that \( a \) is “socially preferred” to \( b \) if (strictly) more voters prefer \( a \) to \( b \) (social indifference in case of a tie)
- **Condorcet’s principle**: if one candidate is preferred to *all other* candidates, it should be elected.
- **Condorcet Winner** (must be unique)
Remarks

- UK and French systems violate Condorcet’s principle
- The UK system may elect a Condorcet looser
- Condorcet’s principle does not solve the “dictature of the majority” difficulty
- A Condorcet winner is not necessarily “ranked high” by voters
- An attractive concept however... BUT
3 candidates: \{a, b, c\}
21 voters

Preferences of the voters
10 : a \ P \ b \ P \ c
6 : b \ P \ c \ P \ a
5 : c \ P \ b \ P \ a

a is the plurality winner
b is the Condorcet Winner (11/21 over a, 16/21 over c)
a is the Condorcet Looser (10/21 over b, 10/21 over c)
4 candidates: \( \{a, b, c, d\} \)
21 voters
10 : \( b \ P \ a \ P \ c \ P \ d \)
6 : \( c \ P \ a \ P \ d \ P \ b \)
5 : \( a \ P \ d \ P \ b \ P \ c \)

\( b \) is the plurality with runoff winner
\( a \) is the Condorcet Winner
(11/21 over \( b \), 15/21 over \( c \), 21/21 over \( d \))
5 candidates: \{a, b, c, d, e\}
5 voters

1 voter: \(a P b P c P d P e\)
1 voter: \(b P c P e P d P a\)
1 voter: \(e P a P b P c P d\)
1 voter: \(a P b P d P e P c\)
1 voter: \(b P d P c P a P e\)

<table>
<thead>
<tr>
<th>Ranks</th>
<th>1</th>
<th>2</th>
<th>3</th>
<th>4</th>
<th>5</th>
</tr>
</thead>
<tbody>
<tr>
<td>a</td>
<td>2</td>
<td>1</td>
<td>0</td>
<td>1</td>
<td>1</td>
</tr>
<tr>
<td>b</td>
<td>2</td>
<td>2</td>
<td>1</td>
<td>0</td>
<td>0</td>
</tr>
</tbody>
</table>

\(a\) is the Condorcet winner
(3:2 win on all other candidates)
3 candidates: \{a, b, c\}

3 voters

1 : a \succ b \succ c

1 : b \succ c \succ a

1 : c \succ a \succ b

\[ a \text{ is socially preferred to } b \]

\[ b \text{ is socially preferred to } c \]

\[ c \text{ is socially preferred to } a \]

As the social preference relation may have cycles, a Condorcet winner does not always exist (probability 40% with 7 candidates and a large number of voters)

McGarvey’s Theorem
Condorcet

• Weaken the principle so as to elect candidates that are not strictly beaten (Weak CW)
  – they may not exist
  – there may be more than one

• Find what to do when there is no (weak) Condorcet winner
Schwartz

• The strict social preference may not be transitive
  – Take its transitive closure
  – Take the maximal elements of the resulting weak order
4 candidates: \{a, b, c, d\}, 3 voters
1 : a P b P c P d
1 : d P a P b P c
1 : c P d P a P b

Taking the transitive closure, all alternatives are indifferent

BUT....
100% of the voters prefer a to b
Copeland

- Count the number of candidates that are beaten by one candidate minus the number of candidates that beat him (Copeland score)
- Elect the candidate with the highest score
- Sports league
  - +2 for a victory, +1 for a tie
  - equivalent to Copeland’s rule (round robin tournaments)
$x$ is the only unbeaten candidate but is not elected
Borda

- Each ballot is an ordered list of candidates (exclude ties for simplicity)
- On each ballot compute the rank of the candidates in the list
- Rank order the candidates according to the decreasing sum of their ranks
4 candidates: \{a, b, c, d\}

3 voters

2 : b \ P \ a \ P \ c \ P \ d
1 : a \ P \ c \ P \ d \ P \ b

Borda Scores

\[
\begin{align*}
a : 2 \times 2 + 1 \times 1 &= 5 \\
b : 6 \\
c : 8 \\
d : 11
\end{align*}
\]

Result: \( a \) elected

Remark: \( b \) is the (obvious) Condorcet winner
Borda

- Simple
- Efficient: always lead to a result
- Separable, monotonic, participation incentive

BUT...
- Violates Condorcet’s Principle
- Has other problems
  - consistency of choice in case of withdrawals
4 candidates: \{a, b, c, d\}
3 voters
2 : b P a P c P d
1 : a P c P d P b

Borda Scores
\[
\begin{align*}
a : 2 \times 2 + 1 \times 1 &= 5 \\
b : 6 \\
c : 8 \\
d : 11
\end{align*}
\]
Result: \(a\) elected

Suppose that \(c\) and \(d\) withdraw from the competition

Borda Scores
\[
\begin{align*}
a : 2 \times 2 + 1 \times 1 &= 5 \\
b : 4
\end{align*}
\]
Result: \(b\) elected
Is the choice of a method important?

4 candidates: \{a, b, c, d\}, 27 voters

5 : a P b P c P d
4 : a P c P b P d
2 : d P b P a P c
5 : d P b P c P a
8 : c P b P a P d
2 : d P c P b P a

\(d\) is the plurality winner

\(a\) is the plurality with runoff winner

\(b\) is the Borda winner

\(c\) is the Condorcet winner
What are we looking for?

- “Democratic method”
  - always giving a result like Borda
  - always electing the Condorcet winner
  - consistent wrt withdrawals
  - monotonic, separable, incentive to participate, not manipulable, etc.
Arrow

• $n \geq 3$ candidates (otherwise use plurality)
• $m$ voters ($m \geq 2$ and finite)
• ballots = ordered list of candidates

• Problem: find all “methods” respecting a small number of “desirable” principles
• **Universality**: the method should be able to deal with any configuration of ordered lists

• **Transitivity**: the result of the method should be an ordered list of candidates

• **Unanimity**: the method should respect a unanimous preference of the voters

• **Absence of dictator**: the method should not allow for dictators

• **Independence**: the comparison of two candidates should be based only on their respective standings in the ordered lists of the voters
Arrow’s Theorem (1951)

• Theorem: There is no method respecting the five principles
  – Borda is
    • universal, transitive, unanimous with no dictator
    ⇒ it cannot be independent
  – Condorcet is
    • universal, unanimous, independent with no dictator
    ⇒ it cannot be transitive
Sketch of the proof

- $V \subseteq N$ is **decisive** for $(a,b)$ if whenever $a P_i b$ for all $i \in V$ then $a P b$
- $V \subseteq N$ is **almost decisive** for $(a,b)$ if whenever $a P_i b$ for all $i \in V$ and $b P_j a$ for all $j \notin V$ then $a P b$
Lemma 1

• If $V$ is almost decisive over some ordered pair $(a,b)$, it is decisive over all ordered pairs.

$\{a, b, x, y\}$ and use universality to obtain:

$V : x P a \land b P y$

$N\backslash V : x P a, b P y, b P a$ (position of $x$ and $y$ unspecified)

Unanimity $\Rightarrow x P a$ and $b P y$

$V$ is almost decisive for $(a,b) \Rightarrow a P b$

$\Rightarrow x P y$ (transitivity)

Independence $\Rightarrow$ the ordering of $a$ and $b$ is irrelevant
Lemma 2

- If $V$ is decisive and $\text{card}(V) > 1$, then some proper subset of $V$ is decisive

$\{x, y, z\}$ use *universality* to obtain:

$V1 : x P y P z$

$V2 : y P z P x$

$N\backslash V : z P x P y$

$V$ decisive $\Rightarrow y P z$

If $x P z$ then $V1$ is almost decisive for $(x, z)$ and thus decisive (lemma 1)

If $z R x$ then $y P x$ (*transitivity*) and $V2$ is almost decisive for $(y, x)$ and thus decisive (lemma 1)
Proof

- *Unanimity* $\Rightarrow$ $N$ is decisive
- Since $N$ is finite the iterated use of lemma 2 leads to the existence of a dictator
Principles

- *Unanimity*: no apparent problem
- *Absence of dictator*: minimal requirement of democracy!
- *Universality*: a group adopting functioning rules that would not function in “difficult situations” could be in big trouble!
Independence

- no intensity of preference considerations
  - I “intensely” or “barely” prefer $a$ to $b$
    - practice, manipulation, interpersonal comparisons?

- no consideration of a third alternative to rank order $a$ and $b$
Borda and Independence

4 candidates: \{a, b, c, d\}, 3 voters

2 voters: \(c \ P \ a \ P \ b \ P \ d\)
1 voter: \(a \ P \ b \ P \ c \ P \ d\)
Borda: \(a \ P \ c \ P \ b \ P \ d\) (scores: 5, 6, 7 and 11)

2 voters: \(c \ P \ a \ P \ b \ P \ d\)
1 voter: \(a \ P \ c \ P \ b \ P \ d\)
Borda: \(c \ P \ a \ P \ b \ P \ d\) (scores: 4, 5, 9 and 12)

The ranking of \(a\) and \(c\) is reversed
BUT... the respective positions of \(a\) and \(c\) is unchanged in the individual lists
Transitivity

- maybe too demanding if the only problem is to elect a candidate
- BUT... guarantees consistency

In \{a, b, c\}, a is elected

In \{a, c\}, both a and c are elected
Relaxing transitivity

- Semi-orders and interval order
  - no change (if more than 4 candidates)

- Transitivity of strict preference
  - oligarchy: group $O$ of voters st
    \[ a \, P_i \, b \quad \forall \, i \in O \implies a \, P \, b \]
    \[ i \in O \text{ and } a \, P_i \, b \implies \neg[b \, P \, a] \]

- Absence of cycles
  - some voter has a veto power
    \[ a \, P_i \, b \implies \neg[b \, P \, a] \]
Message?

• Despair
  – no “ideal” method (this would be dull!)

BUT...

• A group is more complex than an individual
• Analyze the pros and cons of each method
• Beware of “method-sellers”
Extensions

• **Impossibility results**
  
  – *Arrow*
  
  – *Gibbard-Satterthwaite*
    
    • All “reasonable methods” may be manipulated
      (more or less easily or frequently)

  – *Moulin*
    
    • No separable method can be Condorcet
    
    • No Condorcet method can give an incentive to participate

  – *Sen*
    
    • tensions between unanimity and individual freedom
Paretian Liberal Paradox

- There are obvious tensions between the majority principle and the respect of individual rights
- Paradox: there are tensions between the respect of individual rights and the unanimity principle
- Theorem: Unanimity + universality + respect of individual rights $\implies$ Problems
Example

- 2 individuals (males) on a desert island
  - Mr. \(x\) the Puritan and Mr. \(y\) the Liberal
- A pornographic brochure
  - 3 social states
    - \(a\) : \(x\) reads
    - \(b\) : \(y\) reads
    - \(c\) : nobody reads
  - Preferences
    - \(x \in c \overset{P}{\rightarrow} a \overset{P}{\rightarrow} b\)
    - \(y \in a \overset{P}{\rightarrow} b \overset{P}{\rightarrow} c\)
Extensions

• Characterization results
  – find a list of properties that a method is the only one to satisfy simultaneously
    • Borda
    • Copeland
    • Plurality
  – Neutral, anonymous and separable method are of Borda-type (Young 1975)

• Analysis results
  – find a list of desirable properties
  – fill up the methods×properties table
Conclusion

• Little hope to find THE method
• Immense literature: DO NOT re-invent the wheel
  – these problems and results generalize easily to other settings
    • fuzzy preference
    • states of nature
    • etc.
Other aspects

- Institutional setting
- Welfare judgments
- Direct vs. indirect democracy
  - Ostrogorski paradox
  - Referendum paradox
- Electoral platforms
- Paradox of voting (why vote?)