An axiomatic approach to concordance relations

Denis Bouyssou — CNRS
Marc Pirlot — FPMs

Brest – avril 2004

[Follow-up to the presentation by Marc in Durbuy]

Outline

I. Introduction and Motivation

II. Notation

III. Definitions and examples

IV. A general framework for conjoint measurement

V. Results

VI. Discussion

Introduction and motivation

Context: MCDM

2 traditions

- **decision theory**: conjoint measurement
- **pragmatic approach**: dominance and refinements
  - outranking methods (concordance-discordance)

Aims

- show that concordance relations can be fruitfully analyzed within a classical conjoint measurement framework
- characterization emphasizing the specific features of concordance relations

Notation

- $X = \prod_{i=1}^{n} X_i$ with $n \geq 2$: set of alternatives
- $N = \{1, 2, \ldots, n\}$: set of attributes
- abusing notation: (x_J, y_{-J}) and $(x_i, y_{-i}) \in X$

- $\succ$ asymmetric binary relation on X interpreted as "strict preference"
- for $J \subseteq N$, define:

$$x_J \succ_J y_J \text{ iff } (x_J, z_{-J}) \succ (y_J, z_{-J}) \text{ for all } z_{-J} \in X_{-J},$$

- $\succ$ is *independent* if

$$(x_J, z_{-J}) \succ (y_J, z_{-J}) \text{ for some } z_{-J} \in X_{-J} \Rightarrow x_J \succ_J y_J$$

Notation

- attribute $i \in N$ is *essential* if for some $x_i, y_i \in X_i$ and some $z_{-i} \in X_{-i}$

$$(x_i, z_{-i}) \succ (y_i, z_{-i})$$

- attribute $i \in N$ is *influent* if for some $x_i, y_i, z_i, w_i \in X_i$ and some $x_{-i}, y_{-i} \in X_{-i}$

$$\begin{cases} (x_i, x_{-i}) \succ (y_i, y_{-i}) \\ \text{and} \\ Not[(z_i, x_{-i}) \succ (w_i, y_{-i})] \end{cases}$$

- essential $\Rightarrow$ influent; influent $\not\Rightarrow$ essential
- all attributes will be supposed influent (wlog)

Strict Concordance Relations

$\succ$ is a strict concordance relation if:

- there is an asymmetric binary relation P_i on each attribute $i \in N$
- there is a binary relation $\rhd$ between disjoint subsets of N that is *monotonic* wrt inclusion, i.e. for all $A, B, C, D \subseteq N$ with $A \cap B = \varnothing$ and $C \cap D = \varnothing$,

$$\left.\begin{array}{c} A \rhd B \\ C \supseteq A \text{ and } B \supseteq D \end{array}\right\} \Rightarrow C \rhd D$$

such that, for all $x, y \in X$,

$$x \succ y \Leftrightarrow P(x, y) \rhd P(y, x)$$

where $P(x, y) = \{i \in N : x_i P_i y_i\}$

Starting with $\rhd$, we define $\trianglerighteq$ and $\triangleq$ as is usual

Examples

Simple majority

$$x \succ y \Leftrightarrow |\{i \in N : x_i P_i y_i\}| > |\{i \in N : y_i P_i x_i\}|$$

$$A \rhd B \Leftrightarrow |A| > |B|$$

Note that $P_i = \succ_i$ (all influent attributes are essential)

Examples

Weak majority

$$x \succ y \Leftrightarrow |\{i \in N : x_i P_i y_i\}| > \frac{|N|}{2}$$

$$A \rhd B \Leftrightarrow \Leftrightarrow |A| > \frac{|N|}{2}$$

Note that $P_i \neq \succ_i$ (influent are *not* essential)

Examples

TACTIC (Vansnick (1986))

$$x \succ y \Leftrightarrow \sum_{i \in P(x,y)} w_i > \rho \sum_{j \in P(y,x)} w_j \ + \ \varepsilon$$

$$A \rhd B \Leftrightarrow \sum_{i \in A} w_i > \rho \sum_{j \in B} w_j + \varepsilon$$

Note that $P_i \neq \succ_i$ (an influent attribute may not be essential)

Some elementary properties

If $\succ$ is a strict concordance relation with a representation $\langle P_i, \rhd \rangle$, then:

1. for all $A, B \subseteq N$ such that $A \cap B = \varnothing$ exactly one of $A \rhd B$, $B \rhd A$ and $A \triangleq B$ holds and we have $\varnothing \triangleq \varnothing$
2. for all $A \subseteq N$, $A \unrhd \varnothing$ and $N \rhd \varnothing$
3. $\succ$ is independent
4. for all $i \in N$, either $P_i = \succ_i$ or $\succ_i = \varnothing$
5. $\succ$ has a *unique* representation

Noncompensation *à la* Fishburn

$\succ$ is *noncompensatory à la* Fishburn (1976) if:

$$\left.\begin{array}{rcl} \succ(x,y) & = & \succ(z,w) \\ \succ(y,x) & = & \succ(w,z) \end{array}\right\} \Rightarrow [x \succ y \Leftrightarrow z \succ w]$$

where $\succ(x,y) = \{i \in N : x_i \succ_i y_i\}$

A strict concordance relation may not be noncompensatory

This happens as soon as $P_i \neq \succ_i$, e.g. in the weak majority model or in TACTIC

Our approach

Use a general framework for conjoint measurement that would contain strict concordance relations as a particular case, but ...

... a strict concordance relation:

- may not be transitive
- may have circuits

Traditional models of conjoint measurement are not suited for our purposes

Problem: find a conjoint measurement framework tolerating intransitivity

A general framework for conjoint measurement

Nontransitive Decomposable Measurement (Bouyssou & Pirlot (2002))

$$x \succ y \Leftrightarrow F(p_1(x_1, y_1), p_2(x_2, y_2), \ldots, p_n(x_n, y_n)) > 0 \qquad (M)$$

with

- p_i *skew symmetric* $(p_i(x_i, y_i) = -p_i(y_i, x_i))$
- F is *odd* $(F(\mathbf{x}) = -F(-\mathbf{x}))$
- F is *nondecreasing* in all its arguments

Many variants not introduced here

Axioms

$ARC1_i$ if

$$\left.\begin{array}{c}(x_i, a_{-i}) \succ (y_i, b_{-i}) \\ \text{and} \\ (z_i, c_{-i}) \succ (w_i, d_{-i})\end{array}\right\} \Rightarrow \left\{\begin{array}{c}(x_i, c_{-i}) \succ (y_i, d_{-i}) \\ \text{or} \\ (z_i, a_{-i}) \succ (w_i, b_{-i})\end{array}\right.$$

$ARC2_i$ if

$$\left.\begin{array}{c}(x_i, a_{-i}) \succ (y_i, b_{-i}) \\ \text{and} \\ (y_i, c_{-i}) \succ (x_i, d_{-i})\end{array}\right\} \Rightarrow \left\{\begin{array}{c}(z_i, a_{-i}) \succ (w_i, b_{-i}) \\ \text{or} \\ (w_i, c_{-i}) \succ (z_i, d_{-i})\end{array}\right.$$

Interpretation

$$(x_i, y_i) \succsim_i^* (z_i, w_i) \Leftrightarrow$$
$$[\text{for all } a_{-i}, b_{-i} \in X_{-i}, (z_i, a_{-i}) \succ (w_i, b_{-i}) \Rightarrow (x_i, a_{-i}) \succ (y_i, b_{-i})]$$

$$(x_i, y_i) \succsim_i^{**} (z_i, w_i) \Leftrightarrow [(x_i, y_i) \succsim_i^* (z_i, w_i) \text{ and } (w_i, z_i) \succsim_i^* (y_i, x_i)]$$

- $\succsim_i^*$ and $\succsim_i^{**}$ are always transitive (traces on preference differences)
- $ARC1_i \Leftrightarrow \succsim_i^*$ is complete
- $ARC1_i$ and $ARC2_i \Leftrightarrow \succsim_i^{**}$ is complete
- $ARC1$ and $ARC2$ are independent conditions
- $ARC2$ implies independence

Result

Theorem. If, for all $i \in N$, $X_i^2/{\sim_i^{**}}$ is finite or countably infinite (and, hence, if X is finite or countably infinite), $\succ$ has a representation in model (M) *iff*

- $\succ$ is asymmetric
- $\succ$ satisfies $ARC1$ and $ARC2$

Can be generalized to sets of arbitrary cardinality

Strict concordance relations

Observations

- if $\succ$ is a strict concordance relation, it satisfies $ARC1$ and $ARC2$
- if $\succ$ has a representation in model (M) in which all functions p_i take at most 3 distinct values, it is a concordance relation

Consequences

- model (M) provide an adequate framework for characterizing strict concordance relations
  - all relations $\succsim_i^{**}$ have at most 3 equivalence classes
- model (M) is quite flexible (it also contains the additive utility model). Common grounds for quite different models

Axioms

$Maj1_i$ if

$$\left.\begin{array}{c}(x_i, a_{-i}) \succ (y_i, b_{-i}) \\ \text{and} \\ (z_i, a_{-i}) \succ (w_i, b_{-i}) \\ \text{and} \\ (z_i, c_{-i}) \succ (w_i, d_{-i})\end{array}\right\} \Rightarrow \left\{\begin{array}{c}(y_i, a_{-i}) \succ (x_i, b_{-i}) \\ \text{or} \\ (x_i, c_{-i}) \succ (y_i, d_{-i})\end{array}\right.$$

$Maj2_i$ if

$$\left.\begin{array}{c}(x_i, a_{-i}) \succ (y_i, b_{-i}) \\ \text{and} \\ (w_i, a_{-i}) \succ (z_i, b_{-i}) \\ \text{and} \\ (y_i, c_{-i}) \succ (x_i, d_{-i})\end{array}\right\} \Rightarrow \left\{\begin{array}{c}(y_i, a_{-i}) \succ (x_i, b_{-i}) \\ \text{or} \\ (z_i, c_{-i}) \succ (w_i, d_{-i})\end{array}\right.$$

Result

Theorem. $\succ$ on X is a strict concordance relation *iff*

- $\succ$ is asymmetric
- $\succ$ satisfies $ARC1$ and $ARC2$
- $\succ$ satisfies $Maj1$ and $Maj2$

In the class of asymmetric relations, conditions $ARC1$, $ARC2$, $Maj1$ and $Maj2$ are independent

In Durbuy, this was not yet available and $Maj1$ and $Maj2$ was bluntly replaced by saying that all relations $\succsim_i^{**}$ have at most 3 equivalence classes.

Discussion

What about strict concordance relation in which $\rhd$ is *strictly* monotonic? $(Not[B \rhd A]$ and $C \supsetneq A$ or $B \subsetneq D \Rightarrow C \rhd D)$

Answer. Replace $ARC1$ and $ARC2$ by TC

$$\left.\begin{array}{c}(x_i, a_{-i}) \succsim (y_i, b_{-i}) \\ \text{and} \\ (z_i, b_{-i}) \succsim (w_i, a_{-i}) \\ \text{and} \\ (w_i, c_{-i}) \succsim (z_i, d_{-i})\end{array}\right\} \Rightarrow (x_i, c_{-i}) \succsim (y_i, d_{-i})$$

Discussion

What about strict concordance relation in which P_i have nice transitivity properties?

Answer. Add appropriate axioms. These new axioms are independent from the previous ones

Underlying model

$$x \succ y \Leftrightarrow F(\varphi_1(u_1(x_1), u_1(y_1)), \ldots, \varphi_n(u_n(x_n), u_n(y_n))) > 0 \tag{M'}$$

with φ_i nondecreasing (increasing) in its first argument and nonincreasing in its second argument

Additional axioms

$AAC1_i$ if

$$\left.\begin{array}{c} x \succ y \\ \text{and} \\ z \succ w \end{array}\right\} \Rightarrow \left\{\begin{array}{c} (z_i, x_{-i}) \succ y \\ \text{or} \\ (x_i, z_{-i}) \succ w \end{array}\right.$$

$AAC3_i$ if

$$\left.\begin{array}{c} z \succ (x_i, a_{-i}) \\ \text{and} \\ (x_i, b_{-i}) \succ y \end{array}\right\} \Rightarrow \left\{\begin{array}{c} z \succ (w_i, a_{-i}) \\ \text{or} \\ (w_i, b_{-i}) \succ y \end{array}\right.$$

Result

Theorem. $\succ$ on X is a strict concordance relation having a representation in which all P_i are strict semiorders *iff*

- $\succ$ is asymmetric
- $\succ$ satisfies $ARC1$ and $ARC2$
- $\succ$ satisfies $AAC1$ and $AAC3$
- $\succ$ satisfies $Maj1$ and $Maj2$

In the class of asymmetric relations, conditions $ARC1$, $ARC2$, $AAC1$, $AAC3$, $Maj1$ and $Maj2$ are independent

Discussion

What about strict concordance relation in which $\triangleright$ has nice properties?

Answer. Complex ... but it exists

Discussion

It is easy to generalize Arrow-like theorem to the case of MCDM using noncompensation ...

... is it so with strict concordance relation (which may not be noncompensatory?)

Answer:

YES because in a strict concordance relation it is always true that

$$\left.\begin{array}{rcl} P(x,y) & \subseteq & P(z,w) \\ P(y,x) & \supseteq & P(w,z) \end{array}\right\} \Rightarrow [x \succ y \Rightarrow z \succ w]$$

Discussion

Does the analysis generalize to reflexive concordance relations?

Answer:

YES with an alternative general model:

$$x \succsim y \Leftrightarrow F(p_1(x_1, y_1), p_2(x_2, y_2), \ldots, p_n(x_n, y_n)) \geq 0$$

- p_i skew symmetric
- $F(\mathbf{0}) \geq 0$
- F nondecreasing in all its arguments

Related Literature (1/2)

- Bouyssou and Vansnick (1986) use Fishburn's definition of noncompensatory preferences to characterize TACTIC, Bouyssou (1986) generalizes the analysis to reflexive relations, Bouyssou (1992) shows that Arrow-like theorems can easily be transferred to noncompensatory preferences
- Perny and Fargier (2000), Dubois et al. (2002, 2003) generalize the analysis of Bouyssou and Vansnick (1986) to reflexive relations

All these works are based on Fishburn's (1976) analysis of noncompensatory preferences

- only a subset of (strict) concordance relations is studied
- conditions are quite specific to concordance relations

Related Literature (2/2)

Greco et al. (2001) use a related approach in order to characterize a subset of concordance relations that are of ELECTRE I type ($\succsim_i^*$ has only two distinct equivalence classes, $\succ$ is independent, (x_i, x_i) belong to the last equivalence class of $\succsim_i^*$)

- Advantage: discordance is easily captured using a very clever condition
- Drawbacks:
  - characterizing conditions are strong
  - they do not allow to recast concordance relations into a broader framework

Conclusion

Open problems

- discordance?
- simpler conditions for obtaining properties on $\rhd$?

Purpose and usefulness of axiomatic analysis?

- not to characterize models
- show structures

Aggregation procedure *à la* ELECTRE can be analyzed using standard conjoint measurement techniques, including numerical representations

References

- Bouyssou & Pirlot (2002) (volume in honour of B. Roy)
- Bouyssou & Pirlot (2004) (EJOR forthcoming)
- Bouyssou & Pirlot (2004) Working Paper

  `http://www.lamsade.dauphine.fr/~bouyssou/`