An axiomatic approach to concordance relations

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[Follow-up to the presentation by Maïc in Durbuy]
Outline

I. Introduction and Motivation

II. Notation

III. Definitions and examples

IV. A general framework for conjoint measurement

V. Results

VI. Discussion
Introduction and motivation

Context: MCDM

2 traditions

- **decision theory**: conjoint measurement
- **pragmatic approach**: dominance and refinements
  - outranking methods (concordance-discordance)

Aims

- show that concordance relations can be fruitfully analyzed within a classical conjoint measurement framework
- characterization emphasizing the specific features of concordance relations
Notation

• $X = \prod_{i=1}^{n} X_i$ with $n \geq 2$: set of alternatives

• $N = \{1, 2, \ldots, n\}$: set of attributes

• abusing notation: $(x_J, y_{-J})$ and $(x_i, y_{-i}) \in X$

• $\succ$ asymmetric binary relation on $X$ interpreted as “strict preference”

• for $J \subseteq N$, define:

$$x_J \succ_J y_J \text{ iff } (x_J, z_{-J}) \succ (y_J, z_{-J}) \text{ for all } z_{-J} \in X_{-J},$$

• $\succ$ is independent if

$$(x_J, z_{-J}) \succ (y_J, z_{-J}) \text{ for some } z_{-J} \in X_{-J} \Rightarrow x_J \succ_J y_J$$
Notation

- attribute $i \in N$ is *essential* if for some $x_i, y_i \in X_i$ and some $z_{-i} \in X_{-i}$
  \[ (x_i, z_{-i}) \succ (y_i, z_{-i}) \]

- attribute $i \in N$ is *influent* if for some $x_i, y_i, z_i, w_i \in X_i$ and some $x_{-i}, y_{-i} \in X_{-i}$
  \[
  \left\{
  \begin{array}{l}
  (x_i, x_{-i}) \succ (y_i, y_{-i}) \\
  \text{and} \\
  \text{Not}[(z_i, x_{-i}) \succ (w_i, y_{-i})]
  \end{array}
  \right.
  \]

- essential $\Rightarrow$ influent; influent $\not\Rightarrow$ essential

- all attributes will be supposed influent (wlog)
Strict Concordance Relations

≻ is a strict concordance relation if:

- There is an asymmetric binary relation $P_i$ on each attribute $i \in N$
- There is a binary relation $\triangleright$ between disjoint subsets of $N$ that is monotonic with respect to inclusion, i.e. for all $A, B, C, D \subseteq N$ with $A \cap B = \emptyset$ and $C \cap D = \emptyset$,

$$
\begin{align*}
A \triangleright B \\
C \supseteq A \text{ and } B \supseteq D
\end{align*}
\Rightarrow C \triangleright D
$$

such that, for all $x, y \in X$,

$$
x \succ y \iff P(x, y) \triangleright P(y, x)
$$

where $P(x, y) = \{i \in N : x_i P_i y_i\}$

Starting with $\triangleright$, we define $\succeq$ and $\triangleq$ as is usual
Examples

Simple majority

\[ x \succ y \iff |\{i \in N : x_i P_i y_i \}| > |\{i \in N : y_i P_i x_i \}| \]

\[ A \rhd B \iff |A| > |B| \]

Note that \( P_i = \succ_i \) (all influent attributes are essential)
Examples

Weak majority

\[ x \succ y \iff |\{ i \in N : x_i P_i y_i \}| > \frac{|N|}{2} \]

\[ A \triangleright B \iff |A| > \frac{|N|}{2} \]

Note that \( P_i \neq \succ_i \) (influent are not essential)
Examples

TACTIC (Vansnick (1986))

\[ x \succ y \iff \sum_{i \in P(x,y)} w_i > \rho \sum_{j \in P(y,x)} w_j + \varepsilon \]

\[ A \triangleright B \iff \sum_{i \in A} w_i > \rho \sum_{j \in B} w_j + \varepsilon \]

Note that \( P_i \neq \succ_i \) (an influential attribute may not be essential)
Some elementary properties

If $\succ$ is a strict concordance relation with a representation $\langle P_i, \succ \rangle$, then:

1. For all $A, B \subseteq N$ such that $A \cap B = \emptyset$ exactly one of $A \succ B$, $B \succ A$ and $A \triangleq B$ holds and we have $\emptyset \triangleq \emptyset$

2. For all $A \subseteq N$, $A \succ \emptyset$ and $N \succ \emptyset$

3. $\succ$ is independent

4. For all $i \in N$, either $P_i = \succ_i$ or $\succ_i = \emptyset$

5. $\succ$ has a unique representation
Noncompensation à la Fishburn

\( \succ \) is noncompensatory à la Fishburn (1976) if:

\[
\begin{align*}
\succ (x, y) &= \succ (z, w) \\
\succ (y, x) &= \succ (w, z)
\end{align*}
\]

\( \Rightarrow \) \( [x \succ y \iff z \succ w] \)

where \( \succ (x, y) = \{i \in N : x_i \succ_i y_i\} \)

A strict concordance relation may not be noncompensatory

This happens as soon as \( P_i \not= \succ_i \), e.g. in the weak majority model \( o_i \) in TACTIC
Our approach

Use a general framework for conjoint measurement that would contain strict concordance relations as a particular case, but ... 

... a strict concordance relation:

- may not be transitive
- may have circuits

Traditional models of conjoint measurement are not suited for our purposes

Problem: find a conjoint measurement framework tolerating intransitivity
A general framework for conjoint measurement

Nontransitive Decomposable Measurement (Bouyssou & Pirlot (2002))

\[ x \succ y \iff F(p_1(x_1, y_1), p_2(x_2, y_2), \ldots, p_n(x_n, y_n)) > 0 \tag{M} \]

with

- \( p_i \) skew symmetric \( (p_i(x_i, y_i) = -p_i(y_i, x_i)) \)
- \( F \) is odd \( (F(x) = -F(-x)) \)
- \( F \) is nondecreasing in all its arguments

Many variants not introduced here
Axioms

$ARC_1$ if

\[
\begin{align*}
(x_i, a_{-i}) &\succ (y_i, b_{-i}) \\
\text{and} & \\
(z_i, c_{-i}) &\succ (w_i, d_{-i}) \\
\end{align*}
\Rightarrow
\begin{align*}
(x_i, c_{-i}) &\succ (y_i, d_{-i}) \\
\text{or} & \\
(z_i, a_{-i}) &\succ (w_i, b_{-i}) \\
\end{align*}
\]

$ARC_2$ if

\[
\begin{align*}
(x_i, a_{-i}) &\succ (y_i, b_{-i}) \\
\text{and} & \\
(y_i, c_{-i}) &\succ (x_i, d_{-i}) \\
\end{align*}
\Rightarrow
\begin{align*}
(z_i, a_{-i}) &\succ (w_i, b_{-i}) \\
\text{or} & \\
(w_i, c_{-i}) &\succ (z_i, d_{-i}) \\
\end{align*}
\]
Interpretation

\[(x_i, y_i) \succeq^*_i (z_i, w_i) \iff \]
\[\text{[for all } a_{-i}, b_{-i} \in X_{-i}, (z_i, a_{-i}) \succ (w_i, b_{-i}) \Rightarrow (x_i, a_{-i}) \succ (y_i, b_{-i})]\]

\[(x_i, y_i) \succeq^{**}_i (z_i, w_i) \iff [(x_i, y_i) \succeq^*_i (z_i, w_i) \text{ and } (w_i, z_i) \succeq^*_i (y_i, x_i)]\]

- \(\succeq^*_i\) and \(\succeq^{**}_i\) are always transitive (traces on preference differences)
- \(ARC1_i \iff \succeq^*_i\) is complete
- \(ARC1_i\) and \(ARC2_i \iff \succeq^{**}_i\) is complete
- \(ARC1\) and \(ARC2\) are independent conditions
- \(ARC2\) implies independence
Result

**Theorem.** If, for all \( i \in \mathbb{N} \), \( X_i^2 / \sim_i^{**} \) is finite or countably infinite (and, hence, if \( X \) is finite or countably infinite), \( \succ \) has a representation in model \( (M) \) iff

- \( \succ \) is asymmetric
- \( \succ \) satisfies \( ARC1 \) and \( ARC2 \)

Can be generalized to sets of arbitrary cardinality.
Strict concordance relations

Observations

• if ⊳ is a strict concordance relation, it satisfies $ARC1$ and $ARC2$

• if ⊳ has a representation in model $(M)$ in which all functions $p_i$ take at most 3 distinct values, it is a concordance relation

Consequences

• model $(M)$ provide an adequate framework for characterizing strict concordance relations
  – all relations $\succsim_i^{**}$ have at most 3 equivalence classes

• model $(M)$ is quite flexible (it also contains the additive utility model).

Common grounds for quite different models
Axioms

**Maj1** if

\[
(x_i, a_{-i}) \succ (y_i, b_{-i}) \\
\quad \text{and} \\
(z_i, a_{-i}) \succ (w_i, b_{-i}) \quad \Rightarrow \quad \begin{cases} 
(y_i, a_{-i}) \succ (x_i, b_{-i}) \\
\text{or} \\
(x_i, c_{-i}) \succ (y_i, d_{-i}) \end{cases}
\]

\[
(z_i, c_{-i}) \succ (w_i, d_{-i})
\]

**Maj2** if

\[
(x_i, a_{-i}) \succ (y_i, b_{-i}) \\
\quad \text{and} \\
(w_i, a_{-i}) \succ (z_i, b_{-i}) \quad \Rightarrow \quad \begin{cases} 
(y_i, a_{-i}) \succ (x_i, b_{-i}) \\
\text{or} \\
(z_i, c_{-i}) \succ (w_i, d_{-i}) \end{cases}
\]

\[
(y_i, c_{-i}) \succ (x_i, d_{-i})
\]
Result

**Theorem.** $\succ$ on $X$ is a strict concordance relation iff

- $\succ$ is asymmetric
- $\succ$ satisfies $ARC1$ and $ARC2$
- $\succ$ satisfies $Maj1$ and $Maj2$

In the class of asymmetric relations, conditions $ARC1$, $ARC2$, $Maj1$ and $Maj2$ are independent.

In Durbuy, this was not yet available and $Maj1$ and $Maj2$ was bluntly replaced by saying that all relations $\succsim^*_i$ have at most 3 equivalence classes.
Discussion

What about strict concordance relation in which $\triangleright$ is strictly monotonic? ($\neg[B \triangleright A]$ and $C \supseteq A$ or $B \subsetneq D \Rightarrow C \triangleright D$)

Answer. Replace $ARC1$ and $ARC2$ by $TC$

$$(x_i, a_{-i}) \succ (y_i, b_{-i})$$
and
$$(z_i, b_{-i}) \succ (w_i, a_{-i})$$
and
$$(w_i, c_{-i}) \succ (z_i, d_{-i})$$

$\Rightarrow (x_i, c_{-i}) \succ (y_i, d_{-i})$$

$$\begin{align*}
(x_i, a_{-i}) &\succ (y_i, b_{-i}) \\
\text{and} \\
(z_i, b_{-i}) &\succ (w_i, a_{-i}) \\
\text{and} \\
(w_i, c_{-i}) &\succ (z_i, d_{-i})
\end{align*}$$
Discussion

What about strict concordance relation in which $P_i$ have nice transitivity properties?

Answer. Add appropriate axioms. These new axioms are independent from the previous ones.

Underlying model

$$x > y \Leftrightarrow F(\varphi_1(u_1(x_1), u_1(y_1)), \ldots, \varphi_n(u_n(x_n), u_n(y_n))) > 0 \quad (M')$$

with $\varphi_i$ nondecreasing (increasing) in its first argument and nonincreasing in its second argument.
Additional axioms

**AAC1\(_i\)** if

\[
\begin{aligned}
  x \succ y \\
  \text{and} \\
  z \succ w
\end{aligned} \implies \begin{aligned}
  (z_i, x_{-i}) \succ y \\
  \text{or} \\
  (x_i, z_{-i}) \succ w
\end{aligned}
\]

**AAC3\(_i\)** if

\[
\begin{aligned}
  z \succ (x_i, a_{-i}) \\
  \text{and} \\
  (x_i, b_{-i}) \succ y
\end{aligned} \implies \begin{aligned}
  z \succ (w_i, a_{-i}) \\
  \text{or} \\
  (w_i, b_{-i}) \succ y
\end{aligned}
\]
Result

**Theorem.** $\succ$ on $X$ is a strict concordance relation having a representation in which all $P_i$ are strict semiorders iff

- $\succ$ is asymmetric
- $\succ$ satisfies $ARC_1$ and $ARC_2$
- $\succ$ satisfies $AAC_1$ and $AAC_3$
- $\succ$ satisfies $Maj_1$ and $Maj_2$

In the class of asymmetric relations, conditions $ARC_1$, $ARC_2$, $AAC_1$, $AAC_3$, $Maj_1$ and $Maj_2$ are independent
Discussion

What about strict concordance relation in which $\triangleright$ has nice properties?

Answer. Complex ... but it exists
Discussion

It is easy to generalize Arrow-like theorem to the case of MCDM using noncompensation . . .

. . . is it so with strict concordance relation (which may not be noncompensatory?)

Answer:

YES because in a strict concordance relation it is always true that

\[
\begin{align*}
P(x, y) & \subseteq P(z, w) \\
P(y, x) & \supseteq P(w, z)
\end{align*}
\Rightarrow [x \succ y \Rightarrow z \succ w]
\]
Discussion

Does the analysis generalize to reflexive concordance relations?

Answer:

YES with an alternative general model:

\[ x \succapprox y \iff F(p_1(x_1, y_1), p_2(x_2, y_2), \ldots, p_n(x_n, y_n)) \geq 0 \]

- \( p_i \) skew symmetric
- \( F(0) \geq 0 \)
- \( F \) nondecreasing in all its arguments
Related Literature (1/2)

• Bouyssou and Vansnick (1986) use Fishburn’s definition of noncompensatory preferences to characterize TACTIC. Bouyssou (1986) generalizes the analysis to reflexive relations, Bouyssou (1992) shows that Arrow-like theorems can easily be transferred to noncompensatory preferences.


All these works are based on Fishburn’s (1976) analysis of noncompensatory preferences:

• only a subset of (strict) concordance relations is studied

• conditions are quite specific to concordance relations
Greco et al. (2001) use a related approach in order to characterize a subset of concordance relations that are of ELECTRE I type ($\simeq_i^*$ has only two distinct equivalence classes, $\succ$ is independent, $(x_i, x_i)$ belong to the last equivalence class of $\simeq_i^*$)

- **Advantage:** discordance is easily captured using a very clever condition

- **Drawbacks:**
  - characterizing conditions are strong
  - they do not allow to recast concordance relations into a broader framework
Conclusion

Open problems

- discordance?
- simpler conditions for obtaining properties on $\triangleright$?

Purpose and usefulness of axiomatic analysis?

- not to characterize models
- show structures

Aggregation procedure à la ELECTRE can be analyzed using standard conjoint measurement techniques, including numerical representations.
References

• Bouyssou & Pirlot (2002) (volume in honour of B. Roy)
  http://www.lamsade.dauphine.fr/~bouyssou/