An introduction to Nontransitive Decomposable Conjoint Measurement

with application to Noncompensatory Preferences

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Outline

- Introduction and Motivation
- Nontransitive Conjoint Measurement
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  - Definitions
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  - A general conjoint measurement model
  - Results
- Discussion
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Introduction: Conjoint Measurement

• Set of attributes $N = \{1, 2, \ldots, n\}$

• Set of objects evaluated on $N$: $Y \subseteq X_1 \times X_2 \times \cdots \times X_n$

• Binary relation on the set of objects: $\succsim$

Objective: Study/Build/Axiomatise numerical representations of $\succsim$

Interest of Numerical Representations

• Manipulation of $\succsim$

• Construction of numerical representations

Interest of Axiomatic Analysis

• Tests of models

• Understanding models
Introduction: Cartesian Product Structures

- **MCDM**
  - $x$ is an “alternative” evaluated on “attributes”

- **DM under uncertainty**
  - $x$ is an “act” evaluated on “states of nature”

- **Economics**
  - $x$ is a “bundle” of “commodities”

- **Dynamic DM**
  - $x$ is an “alternative” evaluated at “several moments in time”

- **Social Choice**
  - $x$ is a “distribution” between several “individuals”

$x \succeq y$ means “$x$ is at least as good as $y$”
Introduction: Additive Transitive Representation

Basic model: Additive utility

\[ x \succeq y \iff \sum_{i=1}^{n} u_i(x_i) \geq \sum_{i=1}^{n} u_i(y_i) \]

Examples:

- MCDM: Weighted sum, Additive utility, Goal programming, Compromise Programming
- DM under uncertainty: SEU
- Dynamic DM: Discounting
- Social Choice: Inequality measures à la Atkinson/Sen

Well-developed Theory (Debreu 1960, Luce & Tukey 1964)
Introduction: Problems

• Empirical problems
  – Transitivity of $\sim$ (Luce 1956)
  – Transitivity of $\succ$ (May 1954, Tversky 1969)
  – Additional conditions: Independence (EU vs. Choquet EU)

• Technical Problems
  – Asymmetry: “finite” vs. “Rich” cases
  – Asymmetry: $n = 2$ vs. $n \geq 3$ cases

Study more general models
• $X$ finite (Scott-Suppes 1958, Scott 1964)
  – Necessary and sufficient Conditions
  – Denumerable Set of “Cancellation Conditions”
  – No nice uniqueness results
  – Axioms hardly interpretable and testable

• $X$ has a “rich structure” and $\succeq$ behaves consistently in this “continuum” (Debreu 1960, Luce-Tukey 1964)
  – (Topological assumptions + continuity) or (solvability assumption + Archimedean condition)
  – A finite (and limited) set of “Cancellation Conditions” entails the representation (independence, $TC$)
  – $u_i$ define “interval scales” with common unit ($v_i = \alpha u_i + \beta_i$)
  – Asymmetry $n = 2$ vs. $n \geq 3$
  – Respective roles of necessary vs. structural conditions
Introduction: Possible extensions

• Additive utility $= \underbrace{Additive}_{1} \underbrace{Transitive}_{2}$ Conjoint Measurement

• Extensions
  1. Drop additivity
  2. Drop transitivity and/or completeness

• Other extensions: more complex additive forms (Choquet EU, Gini-like inequality measures)
Introduction: Extensions

Decomposable Transitive model (Krantz et al (1971))

\[ x \succeq y \iff F(u_i(x_i)) \geq F(u_i(y_i)) \quad F \text{ increasing} \]

**Advantages** Simple axiomatic analysis, Simple proofs

**Drawbacks** Transitivity and completeness
Introduction: Extensions

Additive Non Transitive Models


\[ x \succeq y \iff \sum_{i=1}^{n} p_i(x_i, y_i) \geq 0 \quad p_i(x_i, x_i) = 0 \text{ or } p_i \text{ skew symmetric} \]

Advantages Flexible towards transitivity and completeness, Classical results are particular cases

Drawbacks Asymmetries, Complex proofs

Particular case: Additive Difference Model (Tversky 1969)

\[ x \succsim y \iff \sum_{i=1}^{n} \Phi_i(u_i(x_i) - u_i(y_i)) \geq 0 \quad \Phi_i \text{ increasing and odd} \]
Introduction: Models

Additive Transitive $\iff$ Transitive Decomposable

Additive Nontransitive $\iff$ Nontransitive decomposable models (Bouyssou and Pirlot)

$x \succeq y \iff F(p_i(x_i, y_i)_{i=1,2,...,n}) \geq 0$

with additional properties:

- $F$ increasing/nondecreasing and/or odd, $p_i$ skew symmetric
- $p_i(x_i, y_i) = \varphi_i(u_i(x_i), u_i(y_i))$ (with $\varphi_i(\uparrow, \downarrow)$)
Introduction: Analysis

Non Transitive Decomposable models:

• imply substantive requirements on \( \succcurlyeq \)

• may be axiomatized in a simple way avoiding the use of a denumerable number of conditions in the finite case and of unnecessary structural assumptions in the infinite case

• allow to study the “pure consequences” of cancellation conditions in the absence of transitivity, completeness and structural requirements on \( X \)

• are sufficiently general to include as particular cases most aggregation rules that have been proposed in the literature

• provide insights on the links and differences between methods
Noncompensatory preferences

Idea: show the usefulness of the general framework of Nontransitive decomposable Conjoint measurement to study a particular problem

Noncompensatory preferences: Preferences governed by an importance relation on the set of subsets of attributes.

Motivation

- (Weighted) majorities
- MCDM: “outranking relations”
- Experimental Psychology
Strict Noncompensatory Preferences

**Context:** Conjoint measurement (MCDM)

**Ingredients**

- an *asymmetric* binary relation on each attribute $i \in N$: $P_i$
  - $P(x, y) = \{i \in N : x_i P_i y_i\}$
  - $P(y, x) = \{i \in N : y_i P_i x_i\}$
  - Asymmetry of $P_i$ implies $P(x, y) \cap P(y, x) = \emptyset$

- an *asymmetric* importance relation $\triangleright$ between disjoint subsets of attributes *monotonic* (wrt inclusion):
  
  $$[A \triangleright B, C \supseteq A, B \supseteq D, C \cap D = \emptyset] \Rightarrow [C \triangleright D]$$
Strict Noncompensatory Preferences

**Definition.** A binary relation $\mathcal{P}$ on a set $Y \subseteq X_1 \times X_2 \times \cdots \times X_n$ of alternatives is said to be a *strict noncompensatory preference* if there are:

- an *asymmetric* binary relation $\succ$ between disjoint subsets of $N$ that is *monotonic* and
- an *asymmetric* binary relation $P_i$ on each $X_i$ ($i = 1, 2, \ldots, n$)

such that, for all $x, y \in Y$:

$$x \mathcal{P} y \iff P(x, y) \succ P(y, x)$$

where $P(x, y) = \{i \in N : x_i P_i y_i\}$

- $\mathcal{P}$ is asymmetric
- $\mathcal{P}$ may not be transitive
- $\mathcal{P}$ may have circuits

**Counterexample:** (nontrivial) additive utility model
Questions

Suppose that you observe a binary relation $\succ$ on a set $X = X_1 \times X_2 \times \cdots \times X_n$

- What distinguishes $\succ$ if it is noncompensatory?
  - Characterization of strict noncompensatory relations

- In what sense a strict noncompensatory relation is different from a relation obtained using other aggregation approaches?
  - Characterization of strict noncompensatory relations using conditions that are not entirely specific to these relations
Notation

• $N = \{1, 2, \ldots, n\}$: set of attributes

• $X = \prod_{i=1}^{n} X_i$ with $n \geq 2$: countable set of alternatives

• Abusing notations: $(x_J, y_{-J})$ and $(x_i, y_{-i}) \in X$, $X_{-J} = \prod_{i \notin J} X_i$, $X_{-i} = \prod_{j \neq i} X_j$

• $\succ$ asymmetric binary relation on $X$ interpreted as “strict preference”

• for all $J \subseteq N$, define marginal preference relation:

\[ x_J \succ_J y_J \text{ iff } (x_J, z_{-J}) \succ (y_J, z_{-J}), \text{ for all } z_{-J} \in X_{-J} \]
• attribute \( i \in N \) is essential if for some \( x_i, y_i \in X_i \) and some \( z_i \in X_{-i} \)
  \[
  (x_i, z_i) \succ (y_i, z_i)
  \]

• attribute \( i \in N \) is influent if for some \( x_i, y_i, z_i, w_i \in X_i \) and some
  \( x_{-i}, y_{-i} \in X_{-i} \)
  
  \[
  \begin{cases}
  (x_i, x_{-i}) \succ (y_i, y_{-i}) \\
  \text{and} \\
  \text{Not}[(z_i, x_{-i}) \succ (w_i, y_{-i})]
  \end{cases}
  \]

• essential \( \Rightarrow \) influent; influent \( \not\Rightarrow \) essential

• influence is innocuous, essentiality is not

• all attributes will be supposed influent (w(much)log)
Fishburn’s Noncompensation (1976)

• Notation: \( \succ (x, y) = \{ i : x_i \succ_i y_i \} \)
  
  \( x_i \succ_i y_i \iff (x_i, z_{-i}) \succ (y_i, z_{-i}) \), for all \( z_{-i} \in X_{-i} \)

• Asymmetry of \( \succ \Rightarrow \) asymmetry of \( \succ_i \Rightarrow \succ (x, y) \cap \succ (y, x) = \emptyset \)

**Definition:** \( \succ \) is Fishburn noncompensatory if

\[
\begin{align*}
\succ (x, y) &= \succ (z, w) \\
\succ (y, x) &= \succ (w, z)
\end{align*}
\]

\( \Rightarrow [x \succ y \iff z \succ w] \)

**Remark:** Neutrality-like condition
Properties of Fishburn noncompensatory preferences

**Proposition.** If $\succ$ is Fishburn noncompensatory then

1. $\succ$ is independent:
   $$(x_J, z_{-J}) \succ (y_J, z_{-J}) \text{ for some } z_{-J} \in X_{-J} \Rightarrow x_J \succ_J y_J$$

2. $x_i \sim_i y_i$ for all $i \in N \Rightarrow x \sim y$

3. $x_j \succ_j y_j$ for some $j \in N$ and $x_i \sim_i y_i$ for all $i \in N \setminus \{j\} \Rightarrow x \succ y$

4. all influent attributes are essential

**Important remark**

A strict noncompensatory relation may well violate *all* these conditions except independence
Example: semi-ordered weighted majorities

\[ x \mathcal{P} y \iff \sum_{i \in P(x,y)} w_i > \sum_{j \in P(y,x)} w_j + \varepsilon \]

where \( \varepsilon > 0 \) and \( w_i \geq 0 \) for all \( i \in N \)

If \( w_j < \varepsilon \) attribute \( j \) is NOT essential (but may well be influent)

Is Fishburn’s original idea useful?
Fishburn Monotonic Noncompensation

Definition: $\succ$ in *Fishburn monotonically noncompensatory* if

\[
\begin{align*}
\succ (x, y) & \subseteq \succ (z, w) \\
\succ (y, x) & \supseteq \succ (w, z)
\end{align*}
\Rightarrow [x \succ y \Rightarrow z \succ w]
\]

Theorem (adapted from Fishburn 1976). The following are equivalent:

1. $\succ$ is a strict noncompensatory relation in which all attributes are *essential*

2. $\succ$ is an asymmetric relation being Fishburn monotonically noncompensatory
Problems

Basing the analysis of noncompensation on Fishburn’s definition:

• leads to a narrow view of noncompensation excluding all relations in which attributes may be influent without being essential

• does NOT allow to point out the specific features of strict noncompensatory relations within a general framework of conjoint measurement

• amounts to using very strong conditions
An alternative approach

Nontransitive Decomposable Conjoint Measurement

Model \((M)\)

\[ x \succ y \Leftrightarrow F(p_1(x_1, y_1), p_2(x_2, y_2), \ldots, p_n(x_n, y_n)) > 0 \]

with:

- \(p_i\) skew symmetric: \(p_i(x_i, y_i) = -p_i(y_i, x_i)\)
- \(F\) odd: \(F(x) = -F(-x)\)
- \(F\) nondecreasing in all its arguments

**Interpretation:** \(p_i(x_i, y_i)\) are “preference differences” adequately combined by \(F\)
Axioms

\[ ARC_1^i \text{ if} \]
\[
\begin{align*}
(x_i, a_{-i}) &\succ (y_i, b_{-i}) \quad \text{and} \quad (z_i, c_{-i}) \succ (w_i, d_{-i}) \\
\implies \quad &\left\{ \begin{array}{l}
(x_i, c_{-i}) \succ (y_i, d_{-i}) \\
\text{or} \quad (z_i, a_{-i}) \succ (w_i, b_{-i}),
\end{array} \right.
\end{align*}
\]

- \( ARC_1^i \) (Asymmetric inteR-attribute Cancellation) suggests that \( \succ \) induces on \( X_i^2 \) a relation that compares “preference differences” in a well-behaved way
- \( ARC_1 \) if \( ARC_1^i \) for all \( i \in N \)
Axioms

\( ARC_{2_i} \) if

\[
\begin{align*}
(x_i, a_{-i}) & \succ (y_i, b_{-i}) \\
\text{and} \\
(y_i, c_{-i}) & \succ (x_i, d_{-i})
\end{align*}
\]

\[ \Rightarrow \left\{ \begin{array}{l}
(z_i, a_{-i}) \succ (w_i, b_{-i}) \\
\text{or} \\
(w_i, c_{-i}) \succ (z_i, d_{-i})
\end{array} \right. \]

- \( ARC_{2_i} \) suggests that the preference difference \((x_i, y_i)\) is linked to the “opposite” preference difference \((y_i, x_i)\)

- \( ARC2 \) if \( ARC_{2_i} \) for all \( i \in N \)
Induced Comparison of Preference Differences

Two quaternary relations

\[(x_i, y_i) \succ_i^* (z_i, w_i) \iff \]

\([\text{for all } a_{-i}, b_{-i} \in X_{-i}, (z_i, a_{-i}) \succ (w_i, b_{-i}) \Rightarrow (x_i, a_{-i}) \succ (y_i, b_{-i})]\]

\[(x_i, y_i) \succ_i^{**} (z_i, w_i) \iff \]

\[[(x_i, y_i) \succ_i^* (z_i, w_i) \text{ and } (w_i, z_i) \succ_i^* (y_i, x_i)]\]

- \(\succ_i^*\) and \(\succ_i^{**}\) are transitive by construction
- \(\succ_i^*\) and \(\succ_i^{**}\) may not be complete
- \(\succ_i^{**}\) is reversible \((x_i, y_i) \succ_i^{**} (z_i, w_i) \iff (w_i, z_i) \succ_i^{**} (y_i, x_i)\)
Results

**Theorem.** Let $\succ$ be a binary relation on a finite or countably infinite set $X = \prod_{i=1}^{n} X_i$. Then $\succ$ satisfies model $(M)$ iff it is asymmetric and satisfies $ARC1$ and $ARC2$.

(can be extended to the general case using NS conditions)

**Remark.** Model $(M)$ contains as particular cases:

1. Additive utilities: $x \succ y \iff \sum_{i=1}^{n} u_i(x_i) > \sum_{i=1}^{n} u_i(y_i)$
2. Additive differences: $x \succsim y \iff \sum_{i=1}^{n} \Phi_i(u_i(x_i) - u_i(y_i)) > 0$
3. Additive Nontransitive preferences: $x \succ y \iff \sum_{i=1}^{n} p_i(x_i, y_i) > 0$
Theorem. The following are equivalent

1. $\succ$ is a strict noncompensatory relation

2. $\succ$ has a representation in model $(M)$ with all relations $\sim_i^{**}$ having three distinct equivalence classes

3. $\succ$ is asymmetric, satisfies $ARC_1$ and $ARC_2$ and all relations $\sim_i^{**}$ have three distinct equivalence classes
Remarks

• the condition that all $\succeq_i^{**}$ have three distinct equivalence classes can be expressed in terms of $\succ$ (technical, not very informative)

• full characterization of strict noncompensatory relations

• conditions $ARC_1$ and $ARC_2$ are NOT specific to strict noncompensatory relations

• asymmetry, $ARC_1$ and $ARC_2$ are independent conditions

• specific feature of strict noncompensatory relations: very rough differentiation of preference differences on each attribute (3 classes: positive, neutral, negative differences)
Discussion

Question:

Why not suppose in the definition of strict noncompensatory relations that $P_i$ have nice properties (weak orders, strict semi-orders)?

Answer:

We could have done so. However this would not have allowed to improve the characterization.

New conditions: AAC1, AAC2 and AAC3 (traces of $\succ_i^{**}$)
Discussion

Question:
It is easy to generalize Arrow-like theorems to the case of MCDM using Fishburn’s noncompensation or monotonic noncompensation. Is it so with strict noncompensatory relations?

Answer:
YES because in a strict noncompensatory relation it is always true that

\[
\begin{align*}
P(x, y) & \subseteq P(z, w) \\
P(y, x) & \supseteq P(w, z)
\end{align*}
\Rightarrow [x \succ y \Rightarrow z \succ w]
\]
Sample result

**Theorem.** Let $\succ$ be a *nonempty* strict noncompensatory relation on a finite set $X = \prod_{i=1}^{n} X_i$. Suppose that $\succ$ has been obtained using, on each $i \in N$, a relation $P_i$ for which there are $a_i, b_i, c_i \in X_i$ such that $a_i P_i b_i$, $b_i P_i c_i$ and $a_i P_i c_i$.

Then, if $\succ$ is *transitive*, it has an *oligarchy*, i.e. there is a unique nonempty $O \subseteq N$ such that, for all $x, y \in X$:

- $x_i P_i y_i$ for all $i \in O \Rightarrow x \succ y$,
- $x_i P_i y_i$ for some $i \in O \Rightarrow Not[y \succ x]$.
Discussion

Question:
Does the analysis generalize to “large” preference relations?

Answer:
YES with an alternative general model:

\[ x \succcurlyeq y \Leftrightarrow F(p_1(x_1, y_1), p_2(x_2, y_2), \ldots, p_n(x_n, y_n)) \geq 0 \]

- \( p_i \) skew symmetric: \( p_i(x_i, y_i) = -p_i(y_i, x_i) \)
- \( F(0) \geq 0 \)
- \( F \) nondecreasing in all its arguments

More difficult however because \( \succcurlyeq \) may not be complete
Discussion

Question:
How to define “degrees” of noncompensation?

Answer: the analysis provides a mean to define the “degree of compensatoriness” of a binary relation using the number $c_i^{**}$ of equivalence classes of $\succsim_i^{**}$

Degree of compensatoriness of $\succ$

$$c^{**} = \max_{i=1,2,\ldots,n} c_i^{**}$$

$c^{**} = 3$ iff $\succ$ is a strict noncompensatory relation
Discussion

Question:

What about Decision under Uncertainty?

Finite number of states ⇒ Homogeneous Cartesian product: $X = C^n$

Counterpart of strict noncompensatory relations = Strict Lifting Rules (Dubois et al. 1997)

$$x \mathcal{P} y \Leftrightarrow P(x, y) \triangleright P(y, x) \quad \text{with } P(x, y) = \{ i \in N : x_i P y_i \}$$

($\triangleright$ model likelihood)

Examples: Probabilistic lifting, Possibilist Lifting

A full characterization of strict lifting rules is at hand using a variant of model $(M)$ taking into account the homogeneity of the Cartesian product
Open Problems

- There are “intuitively” noncompensatory preference relations that do not enter our framework
  - *Min, Max* (particular cases of Choquet or Sugeno), Conjunctive, Disjunctive

- All these relations violate independence (and even weak independence).

- The present framework should be enlarged in order to encompass non-independent relations
Conjunctive rule

\[ X_i = A_i \cup U_i \text{ with } A_i \cap U_i = \emptyset \]

\[ x \in A \iff x_i \in A_i \text{ for all } i \in N \]

\[ x \succ y \iff x \in A \text{ and } y \in U \]

**Example**

\( x_i \in A_i, y_i \in U_i, a_{-i} \in A_{-i}, b_{-i} \in U_{-i} \)

\( (x_i, a_{-i}) \in A, (y_i, a_{-i}) \in U \Rightarrow (x_i, a_{-i}) \succ (y_i, a_{-i}) \)

\( (x_i, b_{-i}) \in U, (y_i, b_{-i}) \in U \Rightarrow (x_i, b_{-i}) \sim (y_i, a_{-i}) \)

**Weak separability**

\( (x_i, a_{-i}) \succ (y_i, a_{-i}) \text{ and } (y_i, b_{-i}) \succ (y_i, b_{-i}) \) is impossible