On some ordinal models for decision making under uncertainty

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Outline

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I. Introduction and Motivation

II. Notation

III. Definitions

IV. A general framework for decision making under uncertainty

V. Putting the framework to work

VI. Discussion

Introduction and motivation

Context: Decision making under uncertainty

- Mainstream Decision Theory (Economics, Psychology)
  - Subjective Expected Utility (SEU)
  - Variants: Choquet Expected utility (CEU)

  These models require a detailed analysis of preferences for acts in order to derive appropriate numerical representations of preference for outcomes (utility u) and likelihood of events (probability θ, capacity v)

- Artificial Intelligence

  Artificial agents, Real agents distributed on a network. A detailed analysis of preferences for acts is often impossible

  - Less refined models

Models used in AI

Qualitative Decision Theory

- Classical models for decision making under "complete ignorance" (Max Min, Min Max Regret): Brafman & Tennenholtz (2000)
- Possiblistic refinements of these criteria (Pessimistic Expected Utility): Dubois et al. (2001)
- Most plausible state: Boutiller (1994), Tan & Pearl (1994)

→ Ordinal approaches

Likely Dominance: model LD

Dubois, Fargier & Prade (1997)

- Act a is preferred to act b as soon as the subset of states for which a gives a better outcome than b is "more likely" than the subset of states for which b gives a better outcome than a

  This model:

  - can be applied as soon as there is a preference relation on the set of consequences and a relation comparing the likelihood of events
  - is apparently quite distinct from model SEU
  - has a definite "ordinal" flavor (voting analogy)
  - does not lead to complete and/or transitive preference relations on the set of acts

Aims

Propose a framework for decision under uncertainty that

- is simple and intuitive
- is nontrivial
- has a numerical representation
- tolerates incomplete and/or intransitive preferences

Within this framework:

- characterize model SEU
- characterize model LD

Better understanding of the similarities and differences of these two approaches

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Setting

Savagean framework with a finite number of states: *acts* as functions from *states* to *outcomes*

- $\Gamma = \{\alpha, \beta, \dots\}$: set of outcomes
- $N = \{1, 2, \dots, n\}$: set of states (one and only one will turn out to be true)
- $\mathcal{A} = \Gamma^N = \{a, b, \dots\}$: set of acts
- $\succsim$: a binary relation on $\mathcal{A}$ "at least as good as"
- $a \in \mathcal{A}$ associates with each state $i \in N$ an outcome $a(i) \in \Gamma$
  $a(i)$ is often denoted a_i

Notation

- $\overline{\alpha}$: constant act giving outcome $\alpha \in \Gamma$ in all states
- preference relation on outcomes defined via constant acts: $\alpha \succsim_\Gamma \beta \Leftrightarrow \overline{\alpha} \succsim \overline{\beta}$
- $i \in N$, $a, b \in \mathcal{A}$, $\alpha \in \Gamma$: $a_i b$ and $\alpha_i b$ are acts

  $a_i b = (b_1 b_2 \ldots a_i \ldots b_n)$, $\alpha_i b = (b_1 b_2 \ldots \alpha_i \ldots b_n)$
- $\succsim$ is *independent* if $a_i c \succsim a_i d,$ for some $a \in \mathcal{A} \Rightarrow b_i c \succsim b_i d,$ for all $b \in \mathcal{A}$
- state $i \in N$ is influent if $\alpha_i a \succsim \beta_i b$ and $Not[\gamma_i a \succsim \delta_i b]$

  We suppose (wmlog) that all states are influent (does not forbids null states)

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Model SEU

Savage (1954), Wakker (1989)

$$a \succsim b \Leftrightarrow \sum_{i=1}^{n} \theta_i u(a_i) \geq \sum_{i=1}^{n} \theta_i u(b_i) \tag{SEU}$$

- θ_i are nonnegative real numbers that add up to one
- u is a real-valued function on Γ

Likely Dominance: model LD

Dubois, Fargier & Prade, (1997)

A reflexive binary relation $\succsim$ has a representation in model LD if there are:

- a *complete* binary relation $\mathcal{S}$ on Γ
- a binary relation $\trianglerighteq$ between subsets of N having N for union that is *monotonic* w.r.t. inclusion

  $[A \trianglerighteq B, C \supseteq A, B \supseteq D, C \cup D = N] \Rightarrow C \trianglerighteq D$

such that, for all $a, b \in \mathcal{A}$,

$$a \succsim b \Leftrightarrow \mathcal{S}(a, b) \trianglerighteq \mathcal{S}(b, a) \tag{LD}$$

where $\mathcal{S}(a, b) = \{i \in N : a_i \; \mathcal{S} \; b_i\}$

$\langle \trianglerighteq, \mathcal{S} \rangle$ is a representation of $\succsim$ in model LD

Properties of model LD

If $\succsim$ has a representation in model LD then

- $\succsim$ is independent
- $\mathcal{S} = \succsim_\Gamma$
- exactly one of $A \rhd B$, $B \rhd A$, $A \triangleq B$ and $A \bowtie B$ holds
- $N \triangleq N$, $N \rhd \varnothing$ and $N \trianglerighteq A$
- $\succsim$ has a *unique* representation $\langle \trianglerighteq, \mathcal{S} \rangle$

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A general framework for decision making under uncertainty tolerating intransitivity

$$a \succsim b \Leftrightarrow F(p(a_1, b_1), p(a_2, b_2), \dots, p(a_n, b_n)) \geq 0 \tag{M}$$

with

- p *skew symmetric* $(p(x, y) = -p(y, x))$
- F *nondecreasing* in all its arguments and such that $F(\mathbf{0}) \geq 0$

Interpretation

- p measures *preference differences* between outcomes
- F synthesizes the preference differences measured in each state

Variants of model (M)

$$a \succsim b \Leftrightarrow F(p(a_1, b_1), p(a_2, b_2), \ldots, p(a_n, b_n)) \geq 0 \qquad \text{(M)}$$

with F nondecreasing and p skew symmetric

Strengthening model (M)

- (M) with F is *odd* ($F(\mathbf{x}) = -F(-\mathbf{x})$)
- (M) with F is odd and *increasing* in all its arguments

Weakening model (M)

- Not studied here

Properties of model (M)

If $\succsim$ has a representation in model (M) then:

- $\succsim$ is reflexive, independent and marginally complete (two acts that only differ in one state are always comparable)
- $a_i \succ_\Gamma b_i$, for all $i \in N \Rightarrow a \succ b$
- $\succsim_\Gamma$ is complete

Relations comparing preference differences between outcomes

$$(\alpha, \beta) \succsim^* (\gamma, \delta) \Leftrightarrow [\text{for all } a, b \in \mathcal{A} \text{ and all } i \in N, \gamma_i a \succsim \delta_i b \Rightarrow \alpha_i a \succsim \beta_i b]$$

$$(\alpha, \beta) \succsim^{**} (\gamma, \delta) \Leftrightarrow [(\alpha, \beta) \succsim^* (\gamma, \delta) \text{ and } (\delta, \gamma) \succsim^* (\beta, \alpha)]$$

- $\succsim^*$ and $\succsim^{**}$ are reflexive and transitive (traces on preference differences)
- $\succsim^{**}$ is reversible
- $a \succsim b$ and $(c_i, d_i) \succsim^* (a_i, b_i) \Rightarrow c_i a \succsim d_i b$
- $a \succ b$ and $(c_i, d_i) \succsim^{**} (a_i, b_i) \Rightarrow c_i a \succ d_i b$

Axioms

$$\left.\begin{array}{c} \alpha_i a \succsim \beta_i b \\ \text{and} \\ \gamma_j c \succsim \delta_j d \end{array}\right\} \Rightarrow \left\{\begin{array}{c} \gamma_i a \succsim \delta_i b \\ \text{or} \\ \alpha_j c \succsim \beta_j d \end{array}\right. \qquad \text{URC1}$$

Independently of the state, either (α, β) is larger than (γ, δ) or vice versa

$$\left.\begin{array}{c} \alpha_i a \succsim \beta_i b \\ \text{and} \\ \beta_j c \succsim \alpha_j d \end{array}\right\} \Rightarrow \left\{\begin{array}{c} \gamma_i a \succsim \delta_i b \\ \text{or} \\ \delta_j c \succsim \gamma_j d \end{array}\right. \qquad \text{URC2}$$

Independently of the state, either

(α, β) is larger than (γ, δ) or

(δ, γ) is larger than (β, α)

Interpretation

- URC1 $\Leftrightarrow \succsim^*$ is complete
- URC1 and URC2 $\Leftrightarrow \succsim^{**}$ is complete
- URC1 and URC2 are independent conditions
- URC2 implies independence

Remarks

- Preference difference comparisons between outcomes are consistent across states
- Independence holds

Cardinal Coordinate Independence

Wakker (1984, 1989)

$$\left.\begin{array}{c} \alpha_i a \succsim \beta_i b \\ \text{and} \\ \gamma_i b \succsim \delta_i a \\ \text{and} \\ \delta_j c \succsim \gamma_j d \end{array}\right\} \Rightarrow \alpha_j c \succsim \beta_j d \qquad \text{CCI}$$

When $\succsim$ is *complete*:

- CCI implies both URC1 and URC2
- CCI holds iff $\succsim^{**}$ is complete and $\succsim$ is strictly responsive to $\succsim^{**}$

Results

Theorem. If Γ is finite or countably infinite, model (M) holds *iff* $\succsim$ is reflexive and satisfies URC1 and URC2

- Can be generalized to sets of arbitrary cardinality (order denseness condition to be added)
- Model (M) with F *odd* iff $\succsim$ is *complete* and satisfies URC1 and URC2
- Model (M) with F *odd and increasing* iff $\succsim$ is *complete* and satisfies *CCI*

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Model SEU

Theorem. (Bouyssou & Pirlot 2004, MSS) Model (M) with F odd and increasing holds iff $\succsim$ is complete and satisfies CCI plus an order denseness condition

Theorem. (Wakker, 1989) Suppose that $n \geq 2$, that Γ is a connected topological space and endow $\mathcal{A}$ with the product topology. Then model SEU holds (with u continuous) iff

- $\succsim$ is complete
- $\succsim$ satisfies CCI
- $\succsim$ is transitive
- $\succsim$ is continuous (the sets $\{a \in \mathcal{A} : a \succ b\}$ and $\{a \in \mathcal{A} : b \succ a\}$ are open)

The function u is an interval scale and the probabilities θ_i are unique.

Model LD

Observations

- if $\succsim$ has a representation in model LD, it satisfies URC1 and URC2
- if $\succsim$ has a representation in model (M) in which function p takes at most *three* distinct values ($-k$, 0, $+k$), it has a representation in model LD

Consequences

- model (M) provide an adequate framework for characterizing model LD
- common grounds for quite different models

Axioms

$$\left.\begin{array}{c}\alpha_i a \succsim \beta_i b \\ \text{and} \\ \gamma_j c \succsim \delta_j d\end{array}\right\} \Rightarrow \left\{\begin{array}{l}\beta_i a \succsim \alpha_i b \text{ or} \\ \delta_i a \succsim \gamma_i b \text{ or} \\ \alpha_j c \succsim \beta_j d\end{array}\right. \qquad \text{UM1}$$

$$\left.\begin{array}{c}\alpha_i a \succsim \beta_i b \\ \text{and} \\ \beta_j c \succsim \alpha_j d\end{array}\right\} \Rightarrow \left\{\begin{array}{c}\beta_i a \succsim \alpha_i b \text{ or} \\ \gamma_i a \succsim \delta_i b \text{ or} \\ \gamma_j c \succsim \delta_j d\end{array}\right. \qquad \text{UM2}$$

Interpretation

- URC2 and UM1 $\Rightarrow [Not[(\beta,\alpha) \succsim^* (\alpha,\beta)] \Rightarrow (\alpha,\beta) \succsim^* (\gamma,\delta)]$

  If a preference difference is not larger than it opposite, its opposite is the largest possible difference

- URC1 and UM2 $\Rightarrow [Not[(\beta,\alpha) \succsim^* (\alpha,\beta)] \Rightarrow (\gamma,\delta) \succsim^* (\beta,\alpha)]$

  If a preference difference is not larger than it opposite, it is the smallest possible difference

- URC1, URC2, UM2 and UM2 are independent conditions

Result

Theorem. Model LD holds *iff*

- $\succsim$ is reflexive
- $\succsim$ satisfies URC1 and URC2
- $\succsim$ satisfies UM1 and UM2

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Summary

Model (M) provides a framework that:

- is quite flexible while being nontrivial and having a simple interpretation in terms of preference differences
- can be characterized using simple conditions
- provides an adequate basis to characterize models SEU and LD (the same can be done with Fishburn's model SSA)

→ The extension of the analysis in Bouyssou & Pirlot (2002, JMP) to the case of decision under uncertainty seems to work well

The message remains the same: *follow the traces*!

Comparison with Fargier & Perny (1999)

Fargier & Perny (1999) and Dubois et al. (2003) proposed an alternative characterization of model LD

Monotonic Qualitative Independence

$$\left.\begin{array}{c} R(a,b) \supseteq R(c,d) \\ \text{and} \\ R(b,a) \subseteq R(d,c) \end{array}\right\} \Rightarrow [c \succsim d \Rightarrow a \succsim b] \qquad \text{MQI}$$

where $R(a,b) = \{i \in N : a_i \succsim_\Gamma b_i\}$

Theorem. (Fargier & Perny 1999) Model LD holds *iff*

- $\succsim$ is reflexive
- $\succsim_\Gamma$ is complete
- $\succsim$ satisfies MQI

Comparison of the two approaches

MQI is a simple condition inspired from "neutrality" conditions in Social Choice Theory. May appear simpler than URC1, URC2, UM1 and UM2 but…

- This simplicity is only apparent. MQI is not directly phrased in terms of $\succsim$
- MQI is quite strong and nearly characterizes *on its own* model LD
- Using MQI does not allow to characterize model LD *within* a broader framework

MQI exploits a "voting analogy". Dubois et al. (2003) have shown that Arrow-like theorems hold in this context:

If $\succsim$ *has nice transitivity properties and has a representation in model LD then the repartition of likelihood between states is quite uneven*

Discussion

What about LD relations in which $\mathcal{S}$ has nice transitivity properties?

Answer

- Add appropriate axioms
- These new axioms are independent from the previous ones
- Saari (1992): "ordinal aggregation" does not take into account the transitivity properties of what is aggregated
- Translation: in order to characterize model LD, supposing that $\mathcal{S}$ has nice transitivity properties is neither necessary nor helpful

Underlying model

$$a \succsim b \Leftrightarrow F(\varphi(u(a_1), u(b_1)), \ldots, \varphi(u(a_n), u(b_n))) > 0 \qquad (\text{M}^*)$$

with F as in model (M) and φ skew symmetric and nondecreasing in its first argument