A general model of preference aggregation

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Abstract. A general model is presented which encompasses many procedures used for aggregating preferences in multicriteria decision making (or decision aid) methods. Are covered in particular: MAUT, ELECTRE and several other outranking methods. The main interest of the model is to provide a key for understanding the differences between methods. Methods are analyzed in terms of their way of dealing with "preference differences" on each criterion/attribute. The more or less large number of equivalence classes of preference differences that can be distinguished in a method helps to situate it in a continuum going from compensatory to noncompensatory procedures, from cardinal to ordinal methods.

Keywords. Multicriteria decision analysis, aggregation of preferences, compensation, ordinality.

1 Introduction

The classical way of modelling the preferences of a Decision Maker, consists in assuming the existence of a value function u such that an alternative a is at least as good as an alternative b ($a \succeq b$) if and only if $u(a) \geq u(b)$. This leads to a model of preference in which $\succeq$ is complete and transitive. Using such a preference model to establish a recommendation in a decision-aid study is straightforward and the main task of the Analyst is to assess u. In a multicriteria/multiattribute (we will use these terms interchangeably in this paper) context (for a review, see Zionts 1992), each alternative a is usually seen as a vector $\overline{g}(a) = (g_1(a), \ldots, g_n(a))$ of evaluations of a w.r.t n points of view. Under some well-known conditions (see e.g., Krantz et al. 1978 or

Wakker 1989), u can be obtained in an additive manner, i.e. there are functions u_i such that

$$u(a) = \sum_{i=1}^{n} u_i\left(g_i(a)\right).$$

Modelling preferences therefore amounts to assess the partial value functions u_i. Several techniques have been proposed to do so (see Keeney and Raiffa 1976 or von Winterfeldt and Edwards 1986). It should be noticed that the additive model implies independence of each attribute, i.e. that the preference betwen alternatives which only differ on an attribute does not depend on their evaluations on the other attributes, and that individual (also called partial) preferences $\succeq_i$ deduced from $\succeq$ through independence are complete and transitive. In some situations, such a model might not appear to be appropriate, for instance because:

- indifference (seen as the symmetric part of $\succeq$) may not be transitive;
- $\succeq$ may not be a complete relation, i.e. some alternatives may be incomparable;
- compensation effects between criteria are more complex than with an additive model;
- criteria interact (there is no preference independence).

This calls for an extension of the additive utility framework allowing to better deal with some of these cases. Such an extension is also called for by a number of approaches developed since the early seventies. In those methods, the overall preference of a over b is usually determined by looking at the evaluation vectors $\overline{g}(a)$ and $\overline{g}(b)$ independently of the rest of the alternatives and treating the difference $g_i(a) - g_i(b)$ in rather an ordinal way by comparing the difference to a limited number of thresholds. This simple option usually leads to a global preference relation that is not a complete preorder (this being not unrelated to Arrow's theorem). This implies that the aggregation procedure results in structures from which it might not be easy to derive a recommendation (choice of an alternative, ranking of all alternatives). Elaborating a recommendation usually calls for the application of specific "exploitation techniques" (see Roy 1993). The perspective in which such methods were conceived is neither normative (what should the Decision Maker decide in order to be rational) nor descriptive (what are possibly the mechanisms at work in a Decision Maker's mind when he makes a decision); they claim to be constructive in the sense that, the resulting global preference is built or learnt through a dialog between the Decision Maker and the Analyst based on supposedly intuitive concepts. For more information on normative vs descriptive approaches, the reader is referred to Bell et al. 1988; for the constructive approach, see Roy 1993.

Among the methods alluded to, are the so-called *outranking* methods, where $\succeq$ is the outranking relation; the semantic content of a statement like "a outranks b" has been expressed by B. Roy in Roy 1985 :

> "An outranking relation is a binary relation S defined in A such that $a\ S\ b$ if, given what is known about the decision maker's preferences and given the quality of the valuations of the alternatives and the nature of the problem, there are enough arguments to decide that a is at least as good as b, while there is no essential reason to refute that statement."

There has been relatively little interest in these methods outside Europe. There are several reasons to that. Two of them might be that

- they are not well founded from a formal point of view (no axiomatization);
- they may lead to preference structures from which it is not easy to derive a recommendation.

What we aim at doing in this paper is to show a sort of continuity between the dominant "value function" model and a number of pairwise comparisons approaches. This is done through exhibiting a very general model of preference aggregation and showing that a variety of methods fits into the model. Finally we are able to situate the different aggregation procedures as more or less compensatory, the utility approach being compensatory whereas outranking methods tend to be less compensatory.

2 Models

The models presented below are built on a product space $X = \prod_{i=1}^{n} X_i$, where X_i can be viewed as the evaluations of a set $\mathcal{A}$ of alternatives with respect to criterion i. In this paper we only consider the case where $\mathcal{A}$ is finite (the analysis may easily be extended to denumerable sets of alternatives, see Bouyssou and Pirlot 1996). Working on the product space X usually amounts to extend the set of alternatives since it is implicitly considered that any combination $x = (x_1, \ldots, x_n) \in X$ of evaluations corresponds to some alternative.

Denoting by $x = (x_1, \ldots, x_n)$, $y = (y_1, \ldots, y_n)$, elements of X, the classical conjoint measurement model, alluded to above, reads;

$$(M_1) \qquad x \succeq y \text{ iff } \sum_{i=1}^{n} u_i(x_i) \geq \sum_{i=1}^{n} u_i(y_i),$$

where u_i is a real valued function defined on X_i, for all $i = 1, \ldots, n$.

Combining the u_i functions in a non necessarily additive manner yields the transitive decomposable model:

$$(M_2) \qquad x \succeq y \text{ iff } F\left((u_i(x_i))_{i=1,\ldots,n}\right) \geq F\left((u_i(y_i))_{i=1,\ldots,n}\right),$$

with u_i's as in M_1 and $F : \mathcal{R}^n \longrightarrow \mathcal{R}$, strictly increasing in each argument. As shown by Krantz et al. (1978), replacing the additivity requirement of M_1 by the more general decomposability requirement used in M_2 allows to drastically simplify the axiomatic analysis of the model while severely weakening the unicity properties of the numerical representation; note in particular, that the model does not imply that the u_i's define an interval scale whereas M_1 does imply it when the structure of X is "continuous".

Another generalization of M_1, the so called *additive difference model* (Tversky 1969), is defined as

$$(M_3) \qquad x \succeq y \text{ iff } \sum_{i=1}^{n} \phi_i(u_i(x_i) - u_i(y_i)) \geq 0,$$

with u_i as above and ϕ_i, a strictly increasing odd function $\mathcal{R} \longrightarrow \mathcal{R}$, for $i = 1, \ldots, n$. In M_3, the preference differences on each axis, $u_i(x_i) - u_i(y_i)$ are recoded and additively combined. An interesting feature is that such a model encompasses *nontransitive* global preferences. The need for nontransitive models for rational decision has been stressed by several authors (see Fishburn 1991b).

A far-reaching generalization of M_3 dropping at the same time additivity and subtractivity is

$$(M_4) \qquad x \succeq y \text{ iff } F\left(\psi_i(u_i(x_i), u_i(y_i)), i = 1, \ldots, n\right) \geq 0,$$

with u_i's as above, $F : \mathcal{R}^n \longrightarrow \mathcal{R}$, a strictly increasing function and ψ_i: $\mathcal{R}^2 \longrightarrow \mathcal{R}$, nondecreasing in its first argument and nonincreasing in the second, for $i = 1, \ldots, n$.

Model M_4, though very general, shows fundamental features. A key concept emerging from M_4 is the quaternary relation $\succeq_i^*$ defined below. Relations $\succeq_i^*$ encode the comparison of pairs of levels on each criterion; we will refer to the comparison of pairs of them as the comparison of *differences of preference*. For all $i = 1, \ldots, n$, the relation $\succeq_i^*$ is defined as follows: for all $x_i, y_i, x'_i, y'_i \in X_i$,

$$(x_i, y_i) \succeq_i^* (x'_i, y'_i)$$

iff for all $z, w \in X$,

$$(x'_i, z_{i^-}) \succeq (y'_i, w_{i^-}) \text{ implies } (x_i, z_{i^-}) \succeq (y_i, w_{i^-}),$$

where, for instance, (x_i, z_{i^-}) denotes an element of X equal to z except for its i^{th} coordinate which is equal to x_i. The relation $(x_i, y_i) \succeq_i^* (x'_i, y'_i)$ reads "the difference of preference between x_i and y_i is at least as large as that between x'_i and y'_i".

It is easy to show that in model M_4, $\succeq_i^*$ is a complete preorder even if $\succeq$ is noncomplete and/or nontransitive. The number of equivalence classes of this relation may be considered as reflecting discrimination power in the perception of degrees of difference of preference. This point will be abundantly illustrated in the sequel.

Another important characteristic of model M_4 is that it implies that individual preference relations $\succeq_i$ on each criterion defined by, for all $i = 1, \ldots, n$, for all x_i, $y_i \in X_i$,

$$x_i \succeq_i y_i \text{ iff } \forall z \in X, (x_i, z_{i^-}) \succeq (y_i, z_{i^-}),$$

are welll behaved. Though model M_4 does not necessarily imply independence of each attribute, it is not difficult to prove that (as soon as $\succeq$ is reflexive) the relations $\succeq_i$ are semiorders, i.e. complete semitransitive and Ferrers relations (see Luce 1956, Roubens and Vincke

1985). Such an ordered structure appears a particularly desirable generalization of the usual complete preorder for at least two reasons:

- it encompasses the idea that there is a threshold under which differences of performance on a point of view are not perceived as implying definite preference; it thus allows to model preferences in which indifference is not transitive;

- it actually appears in one of the oldest and most famous family of methods based on pairwise comparisons and majority, the ELECTRE family (Roy 1968, Vincke 1992, Roy and Bouyssou 1993).

3 A characterization of M_4

The axioms for model M_1 are well-known (see Krantz et al. 1978 or Wakker 1989). Model M_2 has been proposed and axiomatized in Krantz et al. (1978), Chap. 7. Axioms for model M_3 may be found in Fishburn (1992). Model M_4 is closely related to the nontransitive additive conjoint measurement model proposed in Bouyssou (1986), Fishburn (1990), Fishburn (1991a) and Vind (1991) and is fully discussed in Bouyssou and Pirlot (1996).

Although very general, model M_4 places nontrivial restrictions on $\succsim$ without imposing its completeness and/or the transitivity of $\succ$ or $\sim$. Central to many aggregation procedures is the way a "difference" on one attribute can be compensated by a "difference" of opposite sign on another attribute. Though the way of modelling these "differences" may vary, in most aggregation procedures they are computed with reference to an underlying ordering of each attribute. Model M_4 allows to capture in a simple way this idea of "differences" computed on the basis of an underlying ranking via the functions ψ_i and u_i.

A few elementary properties of preference relations in model M_4 may easily be derived. We state a few of them, which we name "weak cancellation". In the sequel we denote by K $(K', L, L', \ldots)$, elements of $X_{-i} = \prod_{j \neq i} X_j$; for all $x_i \in X_i$, $K \in X_{-i}$, we have $(x_i, K) \in X$.

(WC_i) For all $x_i, y_i, z_i, w_i \in X_i$ and for all $K, K', L, L' \in X_{-i}$,

$$\left.\begin{array}{ccc} (x_i, K) & \succsim & (y_i, L) \\ & \text{and} & \\ (z_i, K') & \succsim & (w_i, L') \end{array}\right\} \Longrightarrow \left\{\begin{array}{ccc} (x_i, K') & \succsim & (y_i, L') \\ & \text{or} & \\ (z_i, K) & \succsim & (w_i, L). \end{array}\right.$$

We say that $\succeq$ satisfies (WC) iff it satisfies (WC_i) for all $i = 1, \ldots, n$.
The (WC_i) property is linked to the fact that $\succeq_i^*$ is an ordering on the differences of preference on attribute i, i.e. that (x_i, y_i) is at least as "large" as (z_i, w_i) or the contrary.

A second kind of weak cancellation properties (WC') are in connection with the fact that the relations $\succeq_i$ are semiorders. The (WC') cancellation property splits into three conditions for each criterion i, $(WC'1)_i$, $(WC'2)_i$ and $(WC'3)_i$.

$(WC'_1)_i$ For all $x_i, y_i, z_i, w_i \in X_i$ and for all $K, K', L, L' \in X_{-i}$,

$$\left.\begin{array}{ccc}(x_i, K) & \succeq & (y_i, L)\\ & \text{and} & \\ (z_i, K') & \succeq & (w_i, L')\end{array}\right\} \Longrightarrow \left\{\begin{array}{ccc}(z_i, K) & \succeq & (y_i, L)\\ & \text{or} & \\ (x_i, K') & \succeq & (w_i, L').\end{array}\right.$$

$(WC'_2)_i$ For all $x_i, y_i, z_i, w_i \in X_i$ and for all $K, K', L, L' \in X_{-i}$,

$$\left.\begin{array}{ccc}(x_i, K) & \succeq & (y_i, L)\\ & \text{and} & \\ (z_i, K') & \succeq & (w_i, L')\end{array}\right\} \Longrightarrow \left\{\begin{array}{ccc}(x_i, K) & \succeq & (w_i, L)\\ & \text{or} & \\ (z_i, K') & \succeq & (y_i, L').\end{array}\right.$$

$(WC'_3)_i$ For all $x_i, y_i, z_i, w_i \in X_i$ and for all $K, K', L, L' \in X_{-i}$,

$$\left.\begin{array}{ccc}(x_i, K) & \succeq & (y_i, L)\\ & \text{and} & \\ (z_i, K') & \succeq & (x_i, L')\end{array}\right\} \Longrightarrow \left\{\begin{array}{ccc}(w_i, K) & \succeq & (y_i, L)\\ & \text{or} & \\ (z_i, K') & \succeq & (w_i, L').\end{array}\right.$$

We say that $\succeq$ satisfies (WC') if it satisfies $(WC'_1)_i$, $(WC'_2)_i$, $(WC'_3)_i)$ for all $i = 1, \ldots, n$. Property $(WC'_1)_i$ can be interpreted as telling that there is an ordering on the values taken by the alternatives on criterion i: x_i, is either "larger" than z_i or the converse; $(WC'_2)_i$ tells a similar thing. Considering $(WC'_1)_i$ to $(WC'_2)_i$ suggests that the orderings on X_i may differ depending on whether the i^{th} coordinate belongs to the description of an alternative which dominates another or is dominated by another. Taken together with $(WC'_1)_i$ or $(WC'_2)_i$, $(WC'_3)_i$ imply that both orderings are compatible.

The main result of this paper is a characterization of the global preferences of model M_4. The interested reader is referred to Bouyssou and Pirlot (1996) for the proof.

Theorem

A reflexive preference relation $\succeq$ *on* $X = \prod_{i=1}^{n} X_i$ *is representable as in model* M_4 *iff* $\succeq$ *satisfies the weak cancellation properties* WC *and* (WC').

This result can easily be extended to denumerable and nondenumerable infinite sets X. It should be noted replacing additivity by a mere decomposability requirement allows a simple necessary and sufficient axiomatization even in the finite case.

4 Methods

In order to illustrate how the framework of model M_4 helps to contrast aggregation procedures,

- we recall the aggregation mechanisms used in a few popular MCDA methods (for more detail, the reader is referred to Vincke (1992));
- we show how they fit into M_4;
- we interpret their differences in terms of the structure of equivalence classes of $\succeq_i^*$.

4.1 Conjoint measurement (model M_1)

We have $F(\psi_i\,(u_i(x_i), u_i(y_i)), i = 1, \ldots, n) = \sum_{i=1}^{n} (u_i(x_i) - u_i(y_i))$.

4.2 ELECTRE I (Roy 1968)

$x \succeq y$ if and only if (x, y) belongs to the *concordance* relation, i.e. there is a majority of viewpoints on which x is at least as good as y, and there is no *veto* against declaring x at least as good as y. More precisely,

- **veto** against $x \succeq y$ occurs if for at least one i, $u_i(y_i) - u_i(x_i)$ is too large, i.e. is at least equal to some *veto threshold* Q_i; then one may not have $x \succeq y$;

- (x, y) belongs to the **concordance** relation if

$$\frac{1}{\sum_{i=1}^{n} w_i} \sum_{i: x_i \succeq y_i} w_i \geq s,$$

  where w_i denotes a nonnegative weight associated to criterion i, s is the so-called *concordance threshold* ($\frac{1}{2} \leq s \leq 1$) and $x_i \succeq_i y_i$ if $u_i(x_i) - u_i(y_i) \geq -q_i$, q_i, a non-negative threshold ($q_i << Q_i$).

From the definition of $\succeq_i$, by means of a numerical representation u_i with constant threshold q_i, it is clear that $\succeq_i$ is a semi-order.

The procedure just described is covered by model M_4 ; with M denoting an arbitrarily large positive number, take

$$\psi_i(x_i, y_i) = \begin{cases} (1-s)w_i & \text{if } \; u_i(x_i) - u_i(y_i) \geq -q_i, \\ -sw_i & \text{if } \; -Q_i < u_i(x_i) - u_i(y_i) < -q_i, \\ -M & \text{if } \; u_i(x_i) - u_i(y_i) \leq -Q_i \end{cases}$$

and for F, the summation operator.

4.3 TACTIC (Vansnick 1986)

We present here a variant of TACTIC, itself a variant of ELECTRE I. Using the formalism of ELECTRE I, we have $x \succeq y$ according to TACTIC if there is no veto against this assertion (as in ELECTRE) and (x, y) belongs to the TACTIC concordance relation defined by

$$\sum_{i: x_i \succ_i y_i} w_i \geq \frac{1}{s} \sum_{j: y_j \succ_j x_j} w_j,$$

with $s \geq 1$, the concordance threshold. Of course, one has $x_i \succ_i y_i$ if not $(y_i \succeq_i x_i)$, i.e. iff $u_i(x_i) - u_i(y_i) > q_i$; $\succ_i$ is the asymmetric part of a semiorder.

The TACTIC method enters into M_4 formalism if one considers F as the summation operator and

$$\psi_i(x_i, y_i) = \begin{cases} w_i & \text{if } \; u_i(x_i) - u_i(y_i) > q_i, \\ 0 & \text{if } \; |u_i(x_i) - u_i(y_i)| \leq q_i, \\ -(1/s)w_i & \text{if } \; -Q_i < u_i(x_i) - u_i(y_i) < -q_i, \\ -M & \text{if } \; u_i(x_i) - u_i(y_i) \leq -Q_i. \end{cases}$$

4.4 Valued global preferences

An interesting extension of the M_4 model consists in considering the global preference as a valued relation; in the *valued* M_4 model, the global preference attached to any pair $(x, y) \in X^2$ is computed as

$$p(x, y) = F(\psi_i(u_i(x_i), u_i(y_i)), i = 1, \ldots, n),$$

with F and ψ as in M_4. An example of a method of that type is PROMETHEE (Brans and Vincke 1985); we have, for all $x, y \in X$,

$$p(x, y) = \sum_{i=1}^{n} w_i \phi_i(u_i(x_i) - u_i(y_i))$$

where ϕ_i can take the forms shown in Figure 1.

Obviously, $\psi_i(x_i, y_i) = \phi_i(u_i(x_i) - u_i(y_i))$ and F can be interpreted as a weighted sum operator.

5 Compensation versus non-compensation

Since in all examples above, F is the summation operator, the procedures formally differ only in the manner they code "preference differences", i.e. through the ψ_i functions. In all considered examples $\psi_i(x_i, y_i)$ is a function of the difference $u_i(x_i) - u_i(y_i)$ which is graphed in Figure 2 (and in Figure 1 for the valued preference relation of PROMETHEE). Each particular coding ψ_i induces a complete preorder on the pairs (x_i, y_i); in Figure 3, the hierarchy of equivalence classes of pairs (x_i, y_i) is shown on the same three examples.

ELECTRE I is characterized by a very rough structure on preference differences; in the absence of veto threshold ($Q_i = +\infty$), only two classes can be distinguished. TACTIC relies on essentially the same perception but explicitly distinguishes strict preference from indifference in its aggregation procedure. TACTIC (without veto) is the prototype of noncompensatory aggregation procedures. Intuitively, a method is compensatory when a difference on some attribute may be compensated by a "sufficiently large" difference in the opposite direction on another attribute.

In other terms, in a noncompensatory procedure, the only things that matter in the comparison of x and y are the lists of criteria $P(x, y)$ (resp. $P(y, x)$) on which x (resp. y) is better than y (resp. x). The notion of noncompensation, introduced and studied in Fishburn (1976)

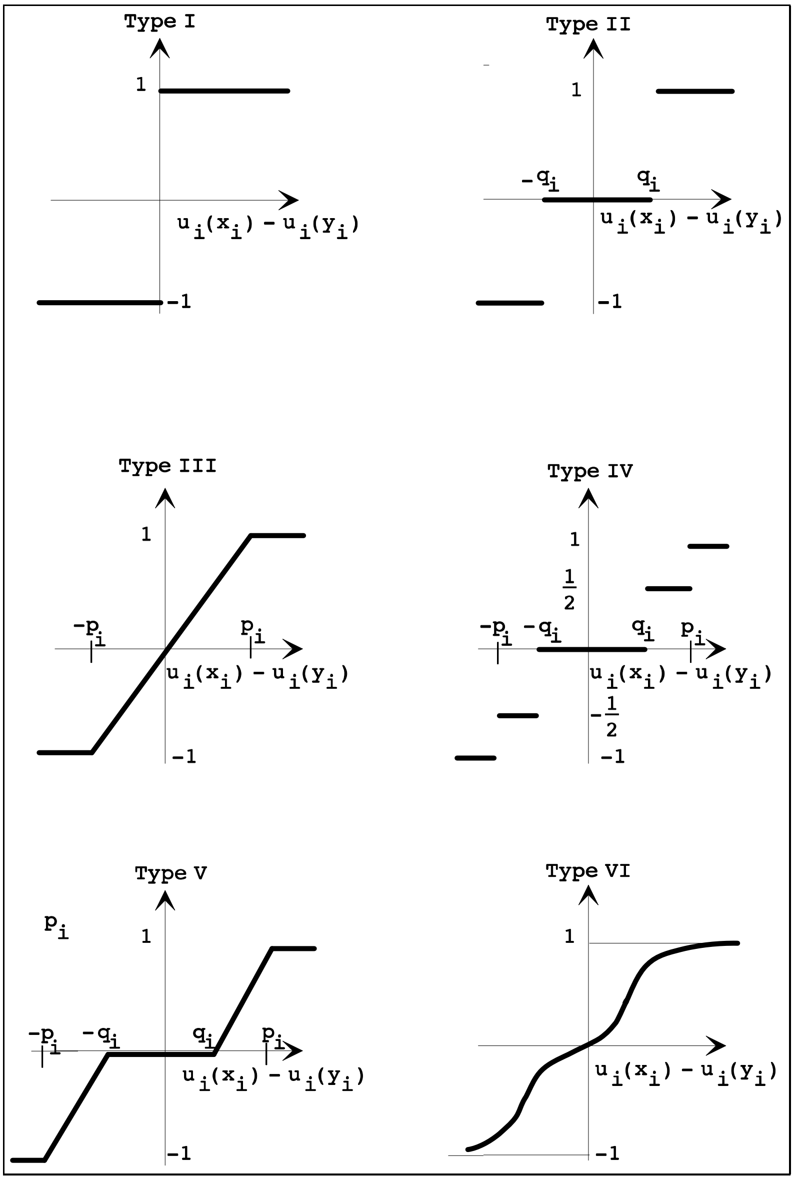

Fig. 1: The six types of recoding of the difference of evaluations used in PROMETHEE

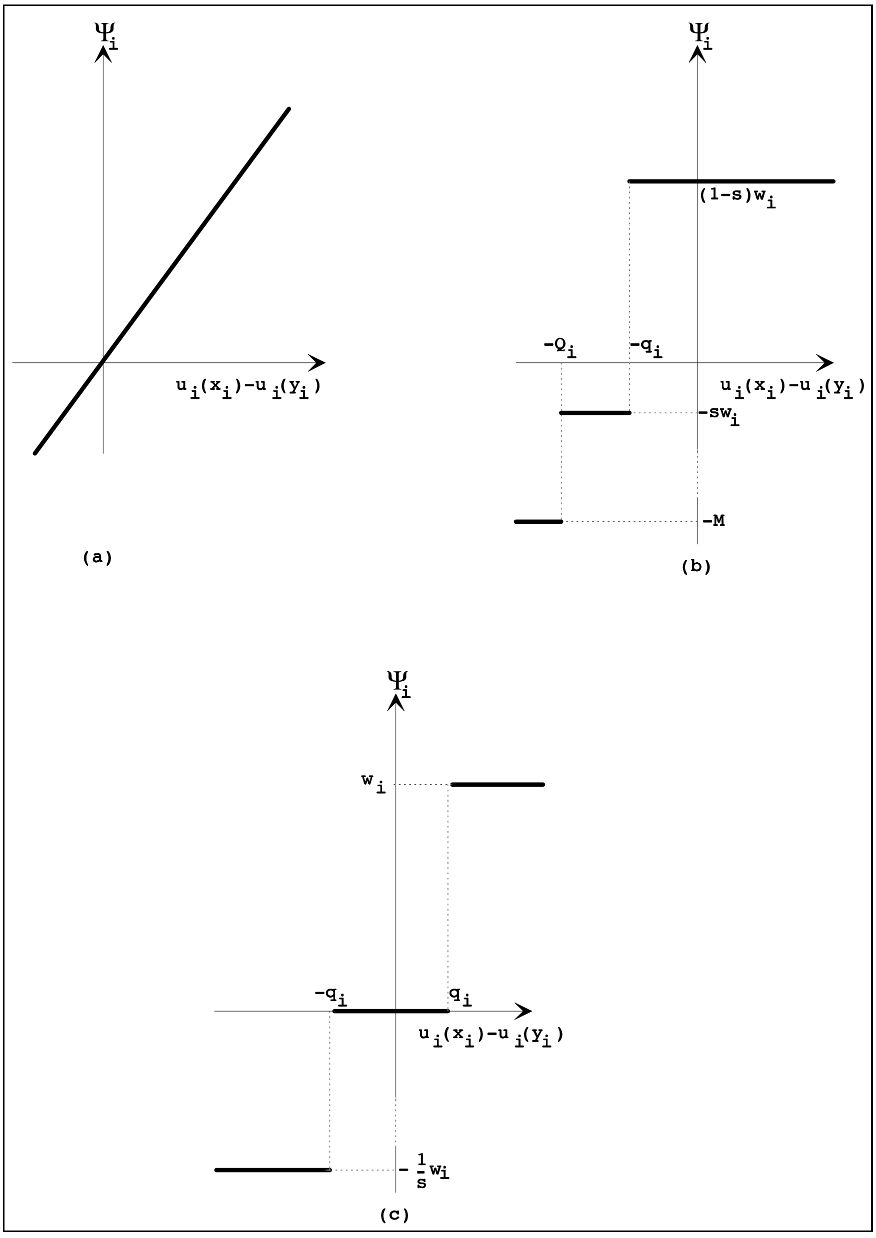

Fig. 2: The function ψ_i in conjoint measurement (a), ELECTRE I (b) and TACTIC without veto (c)

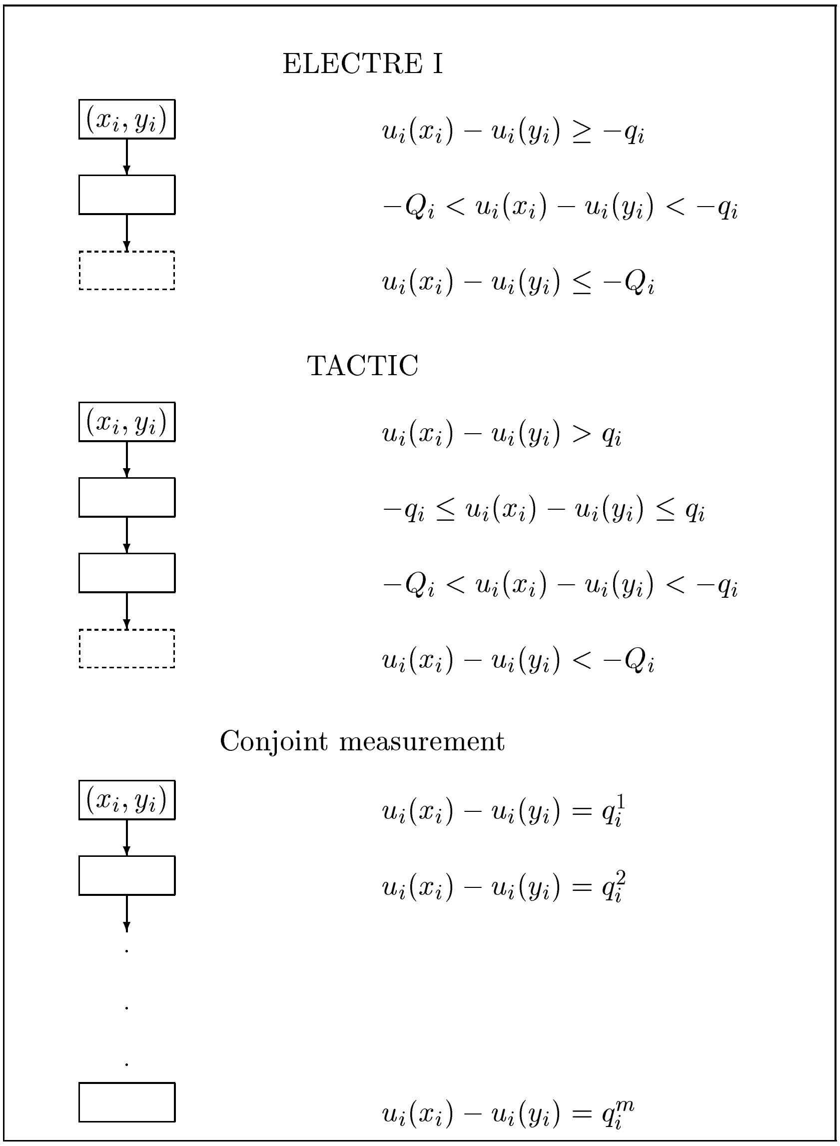

Fig. 3: Equivalence classes of pairs (x_i, y_i) in ELECTRE I, TACTIC and Conjoint measurement

was generalized by Bouyssou and Vansnick (Bouyssou and Vansnick 1986; see also Bouyssou 1986) for taking vetoes into account.

The precise definition reads as follows. A preference relation is *non compensatory* if for all $x, y, z, w \in X$,

$$\left.\begin{array}{rcl} P(x,y) & = & P(z,w) \\ P(y,x) & = & P(w,z), \end{array}\right\} \Rightarrow [x \succeq y \text{ iff } z \succeq w]\,,$$

with $P(x,y) = \{i : x_i \succ y_i\}$.

Formally very similar is the following consequence of the weak cancellation axiom WC: for all $x, y, z, w \in X$,

$$[\forall\ i, (x_i, y_i) \sim_i^* (z_i, w_i)] \Rightarrow (x \succeq y \text{ iff } z \succeq w)\,.$$

The latter condition can be interpreted as a generalization of the former: if on each criterion, the difference of preference between x and y belongs to the same class as the difference of preference between z and w, then the two pairs (x, y) and (z, w) must compare in the same manner in the global preference $\succeq$. In case there are only three classes, strict preference, indifference and the opposite of strict preference (as in TACTIC without veto), the noncompensatory property is satisfied. So, the more or less compensatory character of a method can be viewed as the possibility of actually taking into account a larger or a smaller number of differences of preference classes on each criterion in the aggregation procedure.

Note that in our model, preference differences do not need necessarily to be reversible. We may have $(x_i, y_i) \sim_i^* (z_i, w_i)$ without having $(y_i, x_i) \sim_i^* (w_i, z_i)$; this is in contrast with Fishburn's noncompensation axiom in which "differences" appear to be reversible.

6 Conclusion

With the M_4 model, we have a flexible aggregation scheme that admits a simple axiomatic foundation and encompasses many aggregation models; moreover, we believe that the comparison of preference differences is a key concept for analysing the similarities and dissimilarities of aggregation models (the importance of the concept of preference differences in conjoint measurement has been clearly stressed in Wakker (1989)). A particularly appealing feature of our scheme is that it shows the "continuity" between full compensation and noncompensation.

The present paper emphasizes an interpretation of the technical results obtained in Bouyssou and Pirlot (1996). It opens some research perspectives both on axiomatic and experimental grounds. In the latter, particular models and conditions compatible with observed intuitive preferences could be searched for. On the theoretical side, it would be of interest

- to characterize special models where, e.g., F is additive, the ψ_i's are differences,...;
- to characterize, in a more precise manner, well-known aggregation procedures within our general framework;
- to examine in depth the interconnections of the complete preorder structures on preference differences and the semiorder structure $\succeq_i$ modelling individual preferences on each criterion. Note that the semiordinal character of individual preferences was already stressed as an essential feature of ELECTRE methods in Pirlot (1996);
- to further investigate valued preference relations in the framework of the valued M_4 model.

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