

# Multiagent resource allocation with k-additive utility functions

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# Introduction

- Society of agents negotiating over discrete resources : building (and studying) an abstract framework
- Interests of individual agents are modeled with *individual welfare*, the overall performance of the system can be measured as a *social welfare*. The concept of social welfare, as studied in welfare economics, is an attempt to characterise the well-being of a society in relation to the welfare enjoyed by its individual members.
- Discrete resource allocation problem is an ideal framework to study the link between individual behavior and the social welfare level, because: 1) the framework is formal and simple, 2) It can be well understood theoretically 3) many applications can fit this framework

# Welfarism and Welfare Engineering

Suppose a designer wants to build a multi-agent application in which agents negotiate over resources. Agents are assumed selfish and try maximizing their individual welfare.

- The social welfare ordering : which should the designer choose ?  
(e-commerce, patrolling problem, human society simulation, ...)
- Welfare engineering = to ensure that a given social welfare is maximized
  - which rationality criteria should be chosen / is admissible
  - which types of deals should be used for negotiating
  - Should there be constraints on the individual welfare ? What if there are ?

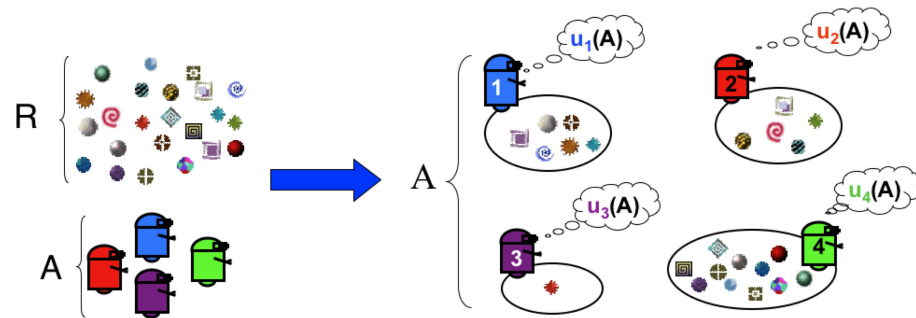
In this work, we do not focus on the details of negotiation procedures, but rather on rationality criteria. Given rationality criteria, types of deals, classes of utility function, we will show towards which global outcome the system moves.

# Outline

1. The resource allocation framework
2. Rationality criteria
  - (a) Individually rational (IR) and cooperatively rational (CR) deals
  - (b) Side-payments
  - (c) Special utility classes (additive and shared resource value)
3. Compact representation of utility function
  - (a) The bundle form
  - (b) The k-additive form
  - (c) the k-separable form

## Resource allocation framework

A resource allocation problem is a quadruple  $\langle \mathcal{A}, \mathcal{R}, \mathcal{U}, A_0 \rangle$ .



- $A_0$  is the initial allocation. An *allocation*  $A$  is a partitioning of  $\mathcal{R}$  among the agents  $\mathcal{A}$ .  
 $A(i) \cap A(j) = \emptyset$  and  $\bigcup_{i \in \mathcal{A}} A(i) = \mathcal{R}$ .
- $\mathcal{U}$  is a set of utility functions  $u_i : 2^{\mathcal{R}} \rightarrow \mathbb{Q}$

## Evaluating well being of a society

To evaluate the “well being” of the society of agents, we introduce a *social welfare* function. Social welfare induces an ordering between allocations.

- utilitarian welfare:  $sw_u(A) = \sum_i u_i(A)$
- egalitarian welfare:  $sw_e(A) = \min_i u_i(A)$
- elitist welfare, Nash product...

Other non-functionnal criteria:

- $A$  is Pareto optimal iff  $\nexists A'$  s.t.  $sw_u(A') > sw_u(A)$  and  $u_i(A') \geq u_i(A)$  for all  $i$
- $A$  is envy free iff  $\forall i, j$  having bundles of resources  $R_i, R_j, u_i(R_i) > u_i(R_j)$
- Lorenz domination...

## Example

2 agents  $a_1$  and  $a_2$ , and two resources  $r_1$  and  $r_2$ .

Agent 1	Agent 2
$A_0(1) = \{r_1\}$	$A_0(2) = \{r_2\}$
$A_1(1) = \{\}$	$A_1(2) = \{r_1, r_2\}$
$u_1(\{\}) = 0$	$u_2(\{\}) = 0$
$u_1(\{r_1\}) = 3$	$u_2(\{r_1\}) = 2$
$u_1(\{r_2\}) = 2$	$u_2(\{r_2\}) = 4$
$u_1(\{r_1, r_2\}) = 10$	$u_2(\{r_1, r_2\}) = 6$

Here,  $sw_u(A_0) = 3 + 4 = 7$ ,  $sw_u(A_1) = 0 + 6 = 6$ . Also,  $sw_e(A_0) = \min\{3, 4\} = 3$ ,  $sw_e(A_1) = \min\{0, 6\} = 0$ . Thus, the deal consisting in moving from  $A_0$  to  $A_1$  decreases both utilitarian and egalitarian social welfare.

## Power of this resource allocation framework

- Ability to represent money (ex: adding resources  $R_{money} = \{r_1^{(1euro)} \dots r_{100}^{(1euro)}, r_1^{(10euros)} \dots r_{90}^{(10euros)}\}$ )
- Ability to represent flows
- Representing roles, like buyers/sellers (ex: money has same value for everyone, and some resources have less value for sellers)

## Types of deals

A deal is a pair  $\delta = (A, A')$  where  $A$  and  $A'$  are allocations of resources. Different types of deals have different complexity:

- one-resource-at-a-time (from an agent to another)
- cluster deals (one agent passes a set of resources to another)
- swap deals (two agent exchange a single resource)
- multiagent deal (many agent. Each agent passes at most one resource)
- Combined deals (no restriction)

## Rationality of agents

Agents negotiate with others according to a rationality criteria. Different criteria will lead to different outcomes.

**Definition.** A deal  $\delta(A, A')$  is Individually Rational (IR) iff  $\forall i$  involved in the deal,  $u_i(A') > u_i(A)$ .

**Definition.** A deal  $\delta(A, A')$  is cooperatively rational (CR) iff  $u_i(A') \geq u_i(A)$  for all  $i$  involved in the deal, and for at least one agent  $j$  we have  $u_j(A') > u_j(A)$ .

*Given a rationality criterion, a certain type of deals, and a set of utility functions, how will the system evolve according to social welfare orderings ?*

## IR and CR deals are not sufficient to lead to maximal $sw_u$ or $sw_e$

Within individually rational deals, reaching optimal is not always possible

Agent 1	Agent 2
$A_0(1) = \{r\}$	$A_0(2) = \{\}$
$u_1(\{\}) = 0$	$u_2(\{\}) = 0$
$u_1(\{r\}) = 4$	$u_2(\{r\}) = 7$

Passing  $r$  to agent 2 would increase utilitarian social welfare, but is not IR neither .

## Side-payments

To compensate the possible loss in utility during negotiation, we extend the framework with a payment function.

**Definition.** A payment function is a function  $p : \mathcal{A} \rightarrow \mathbb{Q}$  such that  $\sum_{i \in \mathcal{A}} p(i) = 0$

Agents for which  $p(i) > 0$  pay, whereas agents for which  $p(j) < 0$  receive compensatory money from others.

**Definition.** (Individual Rationality with payment). A deal  $\delta = (A, A')$  is IRwP iff there exists a payment function  $p$  s.t.  $u_i(A') - u_i(A) > p(i)$  for all  $i$  involved in the deal.

**Theorem.** A deal  $\delta = (A, A')$  is IRwP iff  $sw_u(A) < sw_u(A')$ .

**Proof.**

' $\Rightarrow$ ': Given an IRwP deal, for all  $i$  involved in the deal we have  $u_i(A') - u_i(A) > p(i)$ . Thus,  $\sum_{i \in \mathcal{A}} u_i(A') - u_i(A) > \sum_i p(i) = 0$ . It follows that  $sw_u(A') - sw_u(A) > 0$ .

' $\Leftarrow$ ': Let's choose the payment function  $p(i) = u_i(A') - u_i(A) - \frac{swu(A') - swu(A)}{|\mathcal{A}|}$ . Clearly,  $u_i(A') - u_i(A) > p(i)$ , and  $\sum p_i = 0$ . Thus the deal is IRwP.

□

## IR with side-payments are sufficient for maximizing utilitarian welfare

**Theorem.** *A sequence of individually rational deals with side payments will eventually result in an allocation with optimal utilitarian social welfare.*

**Proof.** Any sequence of IRwP deals is finite because

1. The number of different allocations is finite
2. Every IRwP deal strictly increases utilitarian welfare.

Let  $A$  be a non optimal allocation. Then there exists  $A'$  such that  $sw_u(A) < sw_u(A')$  which implies that  $\delta = (A, A')$  is IRwP.  $\square$

## Additive utilities

We saw that IRwP deals lead to optimal social welfare. However, the complexity of the deals required may be huge, as stated by the “necessary deals” theorem. Thus, we are investigating constraints on utility functions which allow the use of simpler deals.

**Definition.** A utility function  $u_i$  is additive iff  $u_i(R) = \sum_{r \in R} u_i(\{r\})$  for any  $R \subseteq \mathcal{R}$ .

**Theorem.** If all utility functions are additive, any sequence of one-resource-at-a-time deals that are IRwP will eventually result in allocation with maximal  $sw_u$ .

### Proof.

First, note that the sequence is finite because 1) the number of allocation is finite and 2) each IRwP deal increases  $sw_u$ . Because of additivity,  $sw_u(A) = \sum_{r \in \mathcal{R}} u_{\pi_A(r)}(\{r\})$  where  $\pi_A(r)$  denotes the agent owning  $r$  in  $A$ . Let  $A$  be a non optimal allocation. Then,  $\exists A'$  s.t.  $sw_u(A) < sw_u(A')$ . There must at least one resource  $r$  s.t.  $u_{\pi_A(r)}(\{r\}) < u_{\pi_{A'}(r)}(\{r\})$ . Clearly, passing this resource from agent  $\pi_A(r)$  to  $\pi_{A'}(r)$  is an IRwP deal.

□

## Additive utilities with shared resource value

**Definition.** *A set of utility functions is additive with shared resource value iff 1) all  $u_i$  are additive and 2) each resource  $r$  has a value  $\alpha_r$  s.t.  $u_i(\{r\}) = 0$  or  $u_i(\{r\}) = \alpha_r$  for all  $i$ .*

**Theorem.** *If utility functions are additive with constant resource value, any sequence of CR one-resource-at-a-time deals will eventually result with maximal  $sw_u$ .*

## Compact representation of utility functions

- **The bundle form.** List of all bundles of resources for which utility is non-zero. Corresponds to the XOR-bidding language in combinatorial auctions. There are many variants.
- **The k-additive form.** Generalizes additive utilities. Utilities are polynomials of degree  $k$ . Also called pseudo-boolean functions.

$$u_i(R) = \sum_{T \subseteq \mathcal{R}, |T| \leq k} \alpha_i^T \times I_R(T) \quad \text{with } I_R(T) = \begin{cases} 1 & \text{if } T \subseteq R \\ 0 & \text{otherwise} \end{cases}$$

That is, the utility function of agent  $i$  is characterised by the coefficients  $\alpha_i^T$  for bundles of resources  $T \subseteq \mathcal{R}$  with at most  $k$  elements. Agent  $i$  enjoys an increase in utility of  $\alpha_i^T$  when it owns all the items in  $T$  *together*. An example for a 2-additive utility function would be  $u_i(R) = 3 \times I_R(\{r_1\}) - 2 \times I_R(\{r_2, r_3\})$ . We will omit the indicator function  $I_R$ , thus it becomes  $u_i = 3 \cdot r_1 - 2 \cdot r_2 \cdot r_3$

- **The interaction index representation [Grabisch].**

## From one representation to the other

- Any utility function can be represented in both forms
- 1-additive functions = additive functions
- From bundle to k-additive representation:

Moebius transform  $\alpha_i^T = \sum_{K \subseteq T} (-1)^{|T \setminus K|} u_i(K)$ . Is exponential at worst:

$$\text{ex: } u_i(R) = \begin{cases} 1 & \text{if } R = \{r\} \\ 0 & \text{otherwise} \end{cases}$$

- From k-additive to bundle representation: all bundles  $K \subseteq \mathcal{R}$  s.t.  $u_i(K) \neq 0$ . Is exponential at worst:

$$\text{ex: } u_i(R) = |R|$$

## Connections to combinatorial auctions

In combinatorial auctions, consider the case with one seller and  $n$  bidders (potential buyers). In a combinatorial auction, bidders can put in bids for different *bundles* of items (rather than just single items). After all bids have been received, the *winner determination problem* consists in finding an allocation for the items on auction amongst the bidders in a way that maximises his revenue.

Res Alloc Framework (bundle form utilities)	Comb Auctions
Each agent $i$ assigns utility values to bundles	Each agent $i$ puts bid on bundles
Goal: Find alloc. maximizing $\sum u_i$ .	Goal: Find alloc. maximizing revenue ( $=\sum_i \text{paiements}$ )

Both problems are equivalent !!

This equivalence holds, at least, in cases where the optimal allocation of items in an auction is such that *all* of the items on auction are in fact being sold (so-called *free disposal*).

The problem is known to be NP-hard.

## ex of bundle form

agent 1	agent 2
$u_1(\{r_1\}) = 2$	$u_2(\{r_2\}) = 2$
$u_1(\{r_1, r_2\}) = 21$	$u_2(\{r_1, r_2\}) = 5$
	$u_2(\{r_1, r_2, r_3\})$

The winner determination problem : how to distribute resources from seller to ag1,ag2 ?

## The k-additive form

Previously, we saw that with 1-additive utilities, one-resource-at-a-time IRwP deals were sufficient to reach optimal  $sw_u$ . For the k-additive case, are deals much more complex? ..yes!

Let us build an example with 2-additive utility functions in which n-resource-at-a-time deals are needed. Consider 2 agents sharing  $n$  resources  $\{r_1, r_2, \dots, r_n\}$ , with the following 2-additive utility functions:  $u_1 = 0$  and  $u_2 = r_1 - r_1.r_2 - r_1.r_3 - r_1.r_4 - \dots - r_1.r_n$ . Let  $A_{init}$  be the initial allocation describing which agent owns which resource at time 0, and let  $A_{opt}$  be the allocation maximizing the utilitarian social welfare.

	$A_{init}$	$A_{opt}$
$a_1$	$\{r_1\}$	$\{r_2, r_3, \dots, r_n\}$
$a_2$	$\{r_2, r_3, \dots, r_n\}$	$\{r_1\}$

Here,  $sw(A_{init}) = 0$  and  $sw(A_{opt}) = 1$ . In fact, the *only* allocation which has a social welfare greater than  $sw(A_{init})$  is  $A_{opt}$ . Thus, the only rational deal here is  $\delta(A_{init}, A_{opt})$ , which is a bilateral deal of  $n$  resources at a time.

## Complexity of the centralized problem with $k$ -additive utilities

**Theorem.** *Reaching optimal allocation with  $k$ -additive utilities using IRwP deals is NP-complete.*

### Proof.

The problem can be reduced to Maximum-Independent-Set. Given  $G = (V, E)$  we will introduce an agent for every vertex in  $V$  and a resource for every edge in  $E$ . We define the utility coefficients in the  $k$ -additive form as follows: Let  $T$  be the set of resources corresponding to edges in  $E$  that are adjacent to the vertex corresponding to  $i$ . We define  $\alpha_i^T = 1$  and there are no other utility coefficient for agent  $i$ . Now every allocation  $A$  corresponds to an independent set  $V'$  and the utilitarian social welfare to the cardinality of  $V'$ .

□

**Theorem.** *Solving the optimal allocation problem with 2-additive utilities is as hard as with  $k$ -additive utilities for any given  $k \geq 2$ .*

### Proof.

As any pseudo-boolean function, a  $k$ -additive function can be reformulated into a 2-additive function without changing its optimal value. Thus, even for  $k = 2$ , finding optimal social welfare with  $k$ -additive utilities is NP-hard

□

## k-separability

As k-additivity is not enough to ensure convergence with low-complexity deals, we add a new constraint on the set of utility functions. This constraint implies k-additivity.

A domain is called k-separable iff there exists a partition  $(R_1 \dots R_q)$  of  $\mathcal{R}$  s.t.  $|R_i| \leq k$  and for every utility function satisfies :  $u_i(R) = \sum_{j=1}^q u_i(R \cap R_j)$ .

**Theorem.** *In k-separable domains, any sequence of rational k-deals with payments will eventually result in an allocation of resources with maximal  $sw_u$ .*

**Proof.** Termination is as before. Let  $A$  be a non optimal allocation. There exists  $A'$  s.t.  $sw(A) < sw(A')$ . We have :  $sw(A) = \sum_{i \in \mathcal{A}} u_i(A(i)) = \sum_{j=1}^q \sum_{i \in \mathcal{A}} u_i(A(i) \cap R_j)$ . Thus,  $\sum_{j=1}^q \sum_{i \in \mathcal{A}} u_i(A(i) \cap R_j) < \sum_{j=1}^q \sum_{i \in \mathcal{A}} u_i(A'(i) \cap R_j)$ . There exists a  $j^*$  s.t.  $\sum_{i \in \mathcal{A}} u_i(A(i) \cap R_{j^*}) < \sum_{i \in \mathcal{A}} u_i(A'(i) \cap R_{j^*})$ . We can define the deal  $\delta = (A, A^*)$  as a k-deal IRwP:  $A^*(i) = [A(i) \setminus R_{j^*}] \cup [A'(i) \cap R_{j^*}]$  for all  $i \in \mathcal{A}$ .  $\square$

## Conclusion

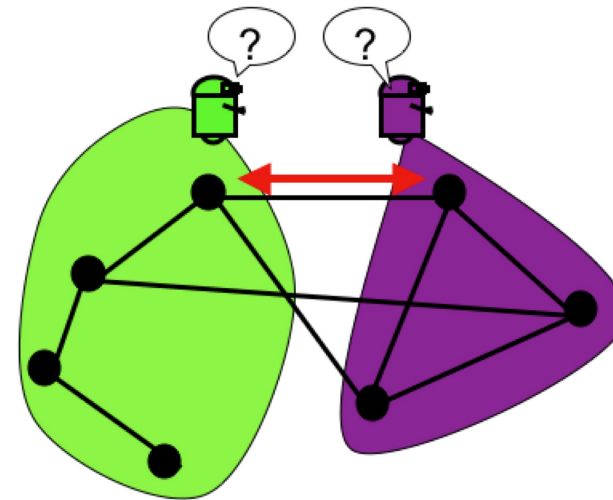
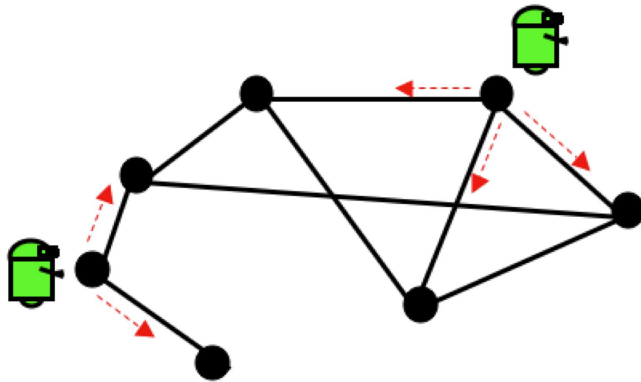
- We have shown links between rationality criterion, classes of utility functions and reaching optimal social welfare
- We have described properties of k-additive and k-separable representation in this framework
- We have shown with combinatorial optimization

## Future work

- Mechanism design : can we define rules for interaction, to help converge towards a desired state ?
- Creating an experimentation platform
- Testing several algorithm (“relaxation” based)

## Applications : the patrolling problem

- Make agents (robots) move over a territory.
- Find the sequence of moves which minimize time between 2 visits on each node



## CR deals (but not IR) are sufficient for Pareto Optimality

**Theorem.** *A sequence of cooperatively rational deals will eventually result in a Pareto optimal allocation.*

**Proof.** Any sequence of cooperatively rational deals is finite because

1. The number of different allocations is finite
2. Every CR deal strictly increases utilitarian welfare.

For the sake of contradiction, suppose negotiation ends with allocation  $A$  not pareto optimal. Then there exists  $A'$  pareto optimal s.t.  $sw_u(A) < sw_u(A')$  and  $u_i(A) \leq u_i(A')$  for all  $i$ , and thus  $\delta(A, A')$  is CR.  $\square$

*Note that sequences of IR deals do not lead to pareto optimality !!*

## Additive utilities with shared resource value

**Definition.** *A set of utility functions is additive with shared resource value iff 1) all  $u_i$  are additive and 2) each resource  $r$  has a value  $\alpha_r$  s.t.  $u_i(\{r\}) = 0$  or  $u_i(\{r\}) = \alpha_r$  for all  $i$ .*

**Theorem.** *If utility functions are additive with constant resource value, any sequence of CR one-resource-at-a-time deals will eventually result with maximal  $sw_u$ .*

**Proof.** Termination was shown earlier. If  $A$  is not optimal then there exists a resource  $r$  owned by an agent  $i$  for which  $u_i(\{r\}) = 0$  and such that there exists an agent  $j$  having  $u_j(\{r\}) > 0$ . The deal passing  $r$  from  $i$  to  $j$  is cooperatively rational.  $\square$