Metastability of Potential Games

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Abstract

One of the main criticisms to game theory concerns the assumption of full rationality. Logit dynamics is a decentralized algorithm in which a level of irrationality (a.k.a. "noise") is introduced in players’ behavior. In this context, the solution concept of interest becomes the logit equilibrium, as opposed to Nash equilibria. Logit equilibria are distributions over strategy profiles that possess several nice properties, including existence and uniqueness. However, there are games in which their computation may take exponential time. We therefore look at an approximate version of logit equilibria, called metastable distributions, introduced by Auletta et al. [3]. These are distributions which remain stable (i.e., players do not go too far from it) for a super-polynomial number of steps (rather than forever, as for logit equilibria). The hope is that these distributions exist and can be reached quickly by logit dynamics.

We devise a sufficient condition for potential games to admit distributions which are metastable no matter the level of noise present in the system, and the starting profile of the dynamics. These distributions can be quickly reached if the rationality level is not too big when compared to the inverse of the maximum difference in potential. Our proofs build on results which may be of independent interest. Namely, we prove some spectral characterizations of the transition matrix defined by logit dynamics for generic games and relate bottleneck ratio and hitting time of Markov chains.

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1 Introduction

One of the most prominent assumptions in game theory dictates that people are rational. This is con-
trasted by many concrete instances of people making irrational choices in certain strategic situations,
such as stock markets [25]. This might be due to the incapacity of exactly determining one’s own utili-
ties: the strategic game is played with utilities perturbed by some noise.

Logit dynamics [6] incorporates this noise in players’ actions and then is advocated to be a good
model for people behavior. More in detail, logit dynamics features a rationality level \( \beta \geq 0 \) (equivalently,
a noise level \( 1/\beta \)) and each player is assumed to play a strategy with a probability which is proportional
to the corresponding utility to the player and \( \beta \). So the higher \( \beta \) is, the less noise there is and the more
rational players are. Logit dynamics can then be seen as a noisy best-response dynamics.

The natural equilibrium concept for logit dynamics is defined by a probability distribution over the
pure strategy profiles of the game. Whilst for best-response dynamics pure Nash equilibria are stable
states, in logit dynamics there is a chance, which is inversely proportional to \( \beta \), that players deviate
from such strategy profiles. Pure Nash equilibria are then not an adequate solution concept for this
dynamics. However, the random process defined by the logit dynamics can be modeled via an ergodic
Markov chain. Stability in Markov chains is represented by the concept of stationary distributions. These
distributions, dubbed logit equilibria, are suggested as a suitable solution concept in this context due to
their properties [5]. For example, from the results known in Markov chain literature, we know that any
game possesses a logit equilibrium and that this equilibrium is unique. The absence of either of these
guarantees is often considered a weakness of pure Nash equilibria. Nevertheless, as for Nash equilibria,
the computation of logit equilibria may be computationally hard depending on whether the chain mixes
rapidly or not [4].

As the hardness of computing Nash equilibria justifies approximate notions of the concept [19, 9],
so Auletta et al. [3] look at an approximation of logit equilibria that they call metastable distributions.
These distributions aim to describe regularities arising during the transient phase of the dynamics before
stationarity has been reached. Indeed, they are distributions that remain stable for a time which is long
enough for the observer (in computer science terms, this time is assumed to be super-polynomial) rather
than forever. Roughly speaking, the stability of the distributions in this concept is measured in terms
of the generations living some historical era, while stationary distributions remain stable throughout
all the generations. When the convergence to logit equilibria is too slow, then there are generations
which are outlived by the computation of the stationary distribution. For these generations, metastable
distributions appear as a reasonable equilibrium concept. (We refer the interested reader to [3] for a
complete overview of the rationale of metastability.) It is unclear whether and which strategic games
possess these distributions and if logit dynamics quickly reaches them.

The focus of this paper is the study of metastable distributions for the class of potential games
[20]. Potential games are an important and widely studied class of games modeling many strategic
settings. Each such game satisfies a number of appealing properties, the existence of pure Nash equilibria
being one of them. A study of conditions under which potential games have metastable distributions
was left open by [3] and assumes particular interest due to the known hardness results, see e.g. [12],
which suggest that the computation of pure Nash equilibria for them is an intractable problem, even for
centralized algorithms.

Our contribution. We devise a sufficient property for an \( n \)-player potential game to have a metastable
distribution for each starting profile of the logit dynamics. These distributions remain stable for a time
which is super-polynomial in \( n \), if one is content of being within a distance \( \varepsilon > 0 \) from the distributions.
(The distance is defined in this context as the total variation distance, see below.) To maintain \( n \) as our
only parameter of interest, we assume that the logarithm of the number of strategies available to players
is upper bounded by a polynomial in \( n \); this assumption can, however, be relaxed to prove bounds
asymptotic in $n$ and in the logarithm of the maximum number of strategies.

The sufficient condition is strictly related to the class of distributions we consider. We focus on stationary distributions restricted to subsets of profiles. The idea is that when the dynamics starts from a subset from which it is “hard” to leave, then the dynamics will stay for a long time close to the stationary distribution restricted to that subset. Moreover, if a subset is “easy” to leave, then the dynamics will quickly reach an “hard-to-leave” subset. Our sufficient property is intuitively a classification of subsets that are asymptotically “hard-to-leave” or “easy-to-leave”.

Once established the existence of such metastable distributions, we prove that the convergence rate, called pseudo-mixing time, is polynomial in $n$ for values of $\beta$ not too big when compared to the (inverse of the) maximum difference in potential of neighboring profiles. Note that when $\beta$ is very high then logit dynamics is “close” to best-response dynamics and therefore it is impossible to prove in general quick convergence results for potential games due to the aforementioned hardness results. We then give a picture which is, in a sense, as complete as possible relatively to the class of potential games for which an asymptotic classification of subsets is possible.

The proofs of the above results build on a number of involved technical contributions, some of which might be of independent interest. They mainly concern about properties of Markov chains. The concepts of interest are mixing time (how long the chain takes to mix), bottleneck ratio (intuitively, how hard it is for the stationary distribution to leave a subset of states), hitting time (how long the chain takes to hit a certain subset of states) and spectral properties of the transition matrix of Markov chains. To describe the metastable distributions of interest, we define a procedure which iteratively identifies in the set of pure strategy profiles the “hard-to-leave” subsets. To prove that the pseudo-mixing time is polynomial in $n$ when the starting profile belongs to the “core” of these distributions, we firstly relate the pseudo-mixing time to the mixing time of a certain family of restricted Markov chains. We then prove that the mixing time of these chains is polynomial by using a spectral characterization of the transition matrix of Markov chains. Finally, the proof that the pseudo-mixing time is polynomial when the dynamics starts outside the “core” mainly relies on a connection between bottleneck ratio and hitting time. Specifically, we prove both an upper bound and a lower bound on the hitting time of a subset of states in terms of the bottleneck ratio of its complement.

We highlight that ideas and technical contributions behind our sufficient condition are useful also in proving asymptotic results about metastability for specific classes of potential games. Indeed, we adopt them to close a problem left open by [3] about metastability of the Curie-Weiss model.

We complement the above contributions with further spectral results about the transition matrix of Markov chains defined by logit dynamics for a strategic (not necessarily potential) game. These results enhance our understanding of the dynamics and pave the way to further advancements in the area.

**Related works.** Blume [6] introduced logit dynamics for modeling a noisy-rational behavior in game dynamics. Early works about this dynamics have focused on its long-term behavior: Blume [6] showed that, for $2 \times 2$ coordination games and potential games, the long-term behavior of the system is concentrated around a specific Nash equilibrium; Alós-Ferrer and Netzer [1] gave a general characterization of long-term behavior of logit dynamics for wider classes of games. Several works gave bounds on the time that the dynamics takes to reach specific Nash equilibria of a game: Ellison [11] considered logit dynamics for graphical coordination games on cliques and rings; Peyton Young [24] and Montanari and Saberi [21] extended this work to more general families of graphs; Asadpour and Saberi [2] focused on a class of congestion games. Auletta et al. [5] were the first to propose the stationary distribution of the logit dynamics Markov chain as a new equilibrium concept in game theory and to focus on the time the dynamics takes to get close to this equilibrium [4].

In physics, chemistry, and biology, metastability is a phenomenon related to the evolution of systems under noisy dynamics. In particular, metastability concerns moves between regions of the state spaces and the existence of multiple, well separated time scales: at short time scales, the system appears to be
in a quasi-equilibrium, but really explores only a confined region of the available space state, while, at larger time scales, it undergoes transitions between such different regions. Research in physics about metastability aims at expressing typical features of a metastable state and to evaluate the transition time between metastable states. Several monographs on the subject are available in physics literature (see, for example, \[13\] \[22\] \[7\] \[14\]). Auletta et al. \[3\] applied metastability to probability distributions, introducing the concepts of metastable distribution and pseudo-mixing time for some specific potential games.

Roughly speaking, metastability is a kind of approximation for stationarity. From this point of view, metastable distributions may be likened to approximate equilibria. Two different approaches to approximated equilibria have been proposed in literature. In the multiplicative version \[9\] a profile is an approximate equilibrium as long as each player gains at least the payoff she gains by playing any other strategy minus a small additive factor \(\varepsilon>0\); for these equilibria a quasi-polynomial time approximation scheme exists \[19\] but it is impossible to have an FPTAS \[8\].

2 Preliminaries

A strategic game \(G\) is a triple \([n], S, U\), where \([n] = \{1, \ldots, n\}\) is a finite set of players, \(S = (S_1, \ldots, S_n)\) is a family of non-empty finite sets (\(S_i\) is the set of strategies available to player \(i\)), and \(U = (u_1, \ldots, u_n)\) is a family of utility functions (or payoffs), where \(u_i: S \rightarrow \mathbb{R}, S = S_1 \times \ldots \times S_n\) being the set of all strategy profiles, is the utility function of player \(i\). We let \(m\) denote an upper bound to the size of players’ strategy sets, that is, \(m \geq \max_{i=1,\ldots,n} |S_i|\). We focus on (exact) potential games, i.e., games for which there exists a function \(\Phi: S \rightarrow \mathbb{R}\) such that for any pair of \(\vec{x}, \vec{y} \in S, \vec{y} = (\vec{x}_i, y_i)\), we have:

\[
\Phi(\vec{x}) - \Phi(\vec{y}) = u_i(\vec{y}) - u_i(\vec{x}).
\]

Note that we use the standard game theoretic notation \((\vec{x}_i, s)\) to mean the vector obtained from \(\vec{x}\) by replacing the \(i\)-th entry with \(s\); i.e. \((\vec{x}_i, s) = (x_1, \ldots, x_{i-1}, s, x_{i+1}, \ldots, x_n)\). A strategy profile \(\vec{x}\) is a Nash equilibrium if, for all \(i, u_i(\vec{x}) \geq u_i(\vec{x}_i, s_i)\), for all \(s_i \in S_i\). It is fairly easy to see that local minima of the potential function correspond to the Nash equilibria of the game.

For two vectors \(\vec{x}, \vec{y}\), we denote with \(H(\vec{x}, \vec{y}) = |\{i: x_i \neq y_i\}|\) the Hamming distance between \(\vec{x}\) and \(\vec{y}\). For every \(\vec{x} \in S\), \(N(\vec{x}) = \{\vec{y} \in S: H(\vec{x}, \vec{y}) = 1\}\) denotes the set of neighbors of \(\vec{x}\) and \(N_i(\vec{x}) = \{\vec{y} \in N(\vec{x}): y_i = x_i\}\) is the set of those neighbors that differ exactly in the \(i\)-th coordinate.

In this paper, given a set of profiles \(L\) we let \(\overline{L}\) denote its complementary set, i.e., \(\overline{L} = S \setminus L\).

2.1 Logit dynamics

The logit dynamics has been introduced in \[6\] and runs as follows: at every time step (i) Select one player \(i \in [n]\) uniformly at random; (ii) Update the strategy of player \(i\) according to the Boltzmann distribution with parameter \(\beta\) over the set \(S_i\) of her strategies. That is, a strategy \(s_i \in S_i\) will be selected with probability

\[
\sigma_i(s_i \mid \vec{x}_{-i}) = \frac{1}{Z_i(\vec{x}_{-i})} e^{\beta u_i(\vec{x}_{-i}, s_i)},
\]

where \(\vec{x}_{-i} \in S_{-i} = S_1 \times \ldots \times S_{i-1} \times S_{i+1} \times \ldots \times S_n\) is the profile of strategies played at the current time step by players different from \(i\), \(Z_i(\vec{x}_{-i}) = \sum_{z_i \in S_i} e^{\beta u_i(\vec{x}_{-i}, z_i)}\) is the normalizing factor, and \(\beta \geq 0\).

One can see parameter \(\beta\) as the inverse of the noise or, equivalently, the rationality level of the system: indeed, from \((1)\), it is easy to see that for \(\beta = 0\) player \(i\) selects her strategy uniformly at random, for

\(^1\)In this paper, we only focus on pure Nash equilibria. We avoid explicitly mentioning it throughout.
\( \beta > 0 \) the probability is biased toward strategies promising higher payoffs, and for \( \beta \) that goes to infinity player \( i \) chooses her best response strategy (if more than one best response is available, she chooses one of them uniformly at random).

The above dynamics defines a Markov chain \( \{X_t\}_{t \in \mathbb{N}} \) with the set of strategy profiles as state space, and where the transition probability from profile \( \vec{x} = (x_1, \ldots, x_n) \) to profile \( \vec{y} = (y_1, \ldots, y_n) \), denoted \( P(\vec{x}, \vec{y}) = P_x(X_1 = \vec{y}) \), is zero if \( H(\vec{x}, \vec{y}) \geq 2 \) and it is \( \frac{1}{n} \sigma_i(y_i | \vec{x}_{-i}) \) if the two profiles differ exactly at player \( i \). More formally, we can define the logit dynamics as follows.

**Definition 2.1.** Let \( \mathcal{G} = ([n], S, \mathcal{U}) \) be a strategic game and let \( \beta \geq 0 \). The logit dynamics for \( \mathcal{G} \) is the Markov chain \( \mathcal{M}_\beta = (\{X_t\}_{t \in \mathbb{N}}, S, P) \) where \( S = S_1 \times \cdots \times S_n \) and

\[
P(\vec{x}, \vec{y}) = \frac{1}{n} \cdot \begin{cases} \sigma_i(y_i | \vec{x}_{-i}), & \text{if } \vec{y}_{-i} = \vec{x}_{-i} \text{ and } y_i \neq x_i; \\ \sum_{i=1}^n \sigma_i(y_i | \vec{x}_{-i}), & \text{if } \vec{y} = \vec{x}; \\ 0, & \text{otherwise}; \end{cases}
\]

where \( \sigma_i(y_i | \vec{x}_{-i}) \) is defined in (1).

The Markov chain defined by (2) is ergodic (6). Hence, from every initial profile \( \vec{x} \) the distribution \( P^t(\vec{x}, \cdot) \) over states of \( S \) of the chain \( X_t \) starting at \( \vec{x} \) will eventually converge to a stationary distribution \( \pi \) as \( t \) tends to infinity. As in (5), we call the stationary distribution \( \pi \) of the Markov chain defined by the logit dynamics on a game \( \mathcal{G} \), the logit equilibrium of \( \mathcal{G} \). In general, a Markov chain with transition matrix \( P \) and state space \( S \) is said to be reversible with respect to a distribution \( \pi \) if, for all \( \vec{x}, \vec{y} \in S \), it holds that \( \pi(\vec{x})P(\vec{x}, \vec{y}) = \pi(\vec{y})P(\vec{y}, \vec{x}) \). If an ergodic chain is reversible with respect to \( \pi \), then \( \pi \) is its stationary distribution. Therefore when this happens, to simplify our exposition we simply say that the matrix \( P \) is reversible. For the class of potential games the stationary distribution is the well-known Gibbs measure.

**Theorem 2.2 (6).** If \( \mathcal{G} = ([n], S, \mathcal{U}) \) is a potential game with potential function \( \Phi \), then the Markov chain given by (2) is reversible with respect to the Gibbs measure \( \pi(\vec{x}) = \frac{1}{Z} e^{-\beta \Phi(\vec{x})} \), where

\[
Z = \sum_{\vec{y} \in S} e^{-\beta \Phi(\vec{y})}
\]

It is worthwhile to notice that logit dynamics for potential games and Glauber dynamics for Gibbs distributions are two ways of looking at the same Markov chain (see (6) for details). This, in particular, implies that we can write

\[
\sigma_i(s_i | \vec{x}_{-i}) = \frac{e^{-\beta \Phi(\vec{x}_{-i}, s_i)}}{\sum_{z \in S_i} e^{-\beta \Phi(\vec{x}_{-i}, z)}}.
\]

### 2.2 Convergence of Markov chains

**Mixing time.** Arguably, the principal notion to measure the rate of convergence of a Markov chain to its stationary distribution is the mixing time, which is defined as follows. Let us set

\[
d(t) = \max_{\vec{x} \in S} \| P^t(\vec{x}, \cdot) - \pi \|_{TV},
\]

where the total variation distance \( \| \mu - \nu \|_{TV} \) between two probability distributions \( \mu \) and \( \nu \) on the same state space \( S \) is defined as

\[
\| \mu - \nu \|_{TV} = \max_{A \subseteq S} | \mu(A) - \nu(A) | = \frac{1}{2} \sum_{\vec{x} \in S} | \mu(\vec{x}) - \nu(\vec{x}) |.
\]

\(^2\)Throughout this work, we denote with \( P_x(\cdot) \) the probability of an event conditioned on the starting state of the logit dynamics being \( \vec{x} \).
For $0 < \varepsilon < 1/2$, the mixing time of the logit dynamics is defined as

$$t_{\text{mix}}(\varepsilon) = \min\{t \in \mathbb{N} : d(t) \leq \varepsilon\}.$$ 

It is usual to set $\varepsilon = 1/4$ or $\varepsilon = 1/2e$. We write $t_{\text{max}}$ to mean $t_{\text{mix}}(1/4)$ and we refer generically to “mixing time” when the actual of $\varepsilon$ is immaterial. Observe that $t_{\text{mix}}(\varepsilon) \leq \lceil \log_2 \varepsilon^{-1} \rceil t_{\text{mix}}$.

**Relaxation time.** Another important measure of convergence for Markov chains is given by the relaxation time. Let $P$ be the transition matrix of a Markov chain with finite state space $S$; let us label the eigenvalues of $P$ in non-increasing order $\lambda_1 \geq \lambda_2 \geq \cdots \geq \lambda_{|S|}$. It is well-known (see, for example, Lemma 12.1 in [18]) that $\lambda_1 = 1$ and, if $P$ is irreducible and aperiodic, then $\lambda_2 < 1$ and $\lambda_2 |S| > -1$. We set $\lambda^*$ as the largest eigenvalue in absolute value other than $\lambda_1$.

$$\lambda^* = \max_{i=2,\ldots,|S|} \{|\lambda_i|\}.$$ 

The relaxation time $t_{\text{rel}}$ of a Markov chain $\mathcal{M}$ is defined as

$$t_{\text{rel}} = \frac{1}{1 - \lambda^*}.$$ 

It turns out that mixing time and relaxation time are strictly related (see results summarized in Appendix A).

**Hitting time.** In some cases, we are interested in bounding the first time that the chain hits a profile in a certain set of states, also known as its hitting time. Formally, for a set $L \subseteq S$, we denote by $\tau_L$ the random variable denoting the hitting time of $L$. Note that the hitting time, differently from mixing and relaxation time, depends on where the dynamics starts. Some useful fact about hitting time are summarized in Appendix B.

**Bottleneck ratio.** Quite central in our study is the concept of bottleneck ratio. Consider an ergodic Markov chain with finite state space $S$, transition matrix $P$, and stationary distribution $\pi$. The probability distribution $Q(\vec{x}, \vec{y}) = \pi(\vec{x})P(\vec{x}, \vec{y})$ is of particular interest and is sometimes called the edge stationary distribution. Note that if the chain is reversible then $Q(\vec{x}, \vec{y}) = Q(\vec{y}, \vec{x})$. For any $L \subseteq S$, $L \neq \emptyset$, we let $Q(L, S \setminus L) = \sum_{\vec{x} \in L, \vec{y} \in S \setminus L} Q(\vec{x}, \vec{y})$. Then the bottleneck ratio of $L$ is

$$B(L) = \frac{Q(L, S \setminus L)}{\pi(L)}.$$ 

Throughout the paper we assume that the bottleneck ratio of the entire strategy space $S$ is zero, that is, $B(S) = 0$. Useful facts about the bottleneck ratio, used in the sequel, are surveyed in Appendix C.

**2.3 Metastable distributions**

In this section we give formal definitions of metastable distributions and pseudo-mixing time. We also survey some of the tools used for our results. For a more detailed description we refer the reader to [3].

**Definition 2.3.** Let $P$ be the transition matrix of a Markov chain with finite state space $S$. A probability distribution $\mu$ over $S$ is $(\varepsilon, T)$-metastable for $P$ (or simply metastable, for short) if for every $0 \leq t \leq T$ it holds that

$$\|\mu P^t - \mu\|_{\text{TV}} \leq \varepsilon.$$
The definition of metastable distribution captures the idea of a distribution that behaves approximately like the stationary distribution: if we start from such a distribution and run the chain we stay close to it for a “long” time. Some interesting properties of metastable distributions are discussed in [3], including the following lemmata, that turn out to be useful for proving our results.

**Lemma 2.4 (3).** Let $P$ be a Markov chain with finite state space $S$ and stationary distribution $\pi$. For a subset of states $L \subseteq S$ let $\pi_L$ be the stationary distribution conditioned on $L$, i.e.

$$
\pi_L(\vec{x}) = \begin{cases}
\pi(\vec{x})/\pi(L), & \text{if } \vec{x} \in L; \\
0, & \text{otherwise.}
\end{cases}
$$

Then, $\pi_L$ is $(B(L), 1)$-metastable.

**Lemma 2.5 (3).** If $\mu$ is $(\epsilon, 1)$-metastable for $P$ then $\mu$ is $(\epsilon T, T)$-metastable for $P$.

Among all metastable distributions, we are interested in the ones that are quickly reached from a (possibly large) set of states. This motivates the following definition.

**Definition 2.6.** Let $P$ be the transition matrix of a Markov chain with state space $S$, let $L \subseteq S$ be a non-empty set of states and let $\mu$ be a probability distribution over $S$. We define the pseudo-mixing time $t^L_\mu(\epsilon)$ as

$$
t^L_\mu(\epsilon) = \inf\{t \in \mathbb{N}: \|P^t(\vec{x}, \cdot) - \mu\|_{TV} \leq \epsilon \text{ for all } \vec{x} \in L\}.
$$

Since the stationary distribution $\pi$ of an ergodic Markov chain is reached within $\epsilon$ in time $t_{\text{mix}}(\epsilon)$ from every state, according to Definition 2.6 we have that $t^S_\pi(\epsilon) = t_{\text{mix}}(\epsilon)$. The following simple lemma connects metastability and pseudo-mixing time.

**Lemma 2.7 (3).** Let $\mu$ be a $(\epsilon, T)$-metastable distribution and let $L \subseteq S$ be a set of states such that $t^L_\mu(\epsilon)$ is finite. Then for every $\vec{x} \in L$ it holds that $\|P^t(\vec{x}, \cdot) - \mu\|_{TV} \leq 2\epsilon$ for every $t^L_\mu(\epsilon) \leq t \leq t^L_\mu(\epsilon) + T$.

### 3 Spectral properties of the logit dynamics

In [4] it has been shown that all the eigenvalues of the transition matrix of logit dynamics for potential games are non-negative. The technique used in that proof can be generalized to work also for some restrictions of these matrices.

To begin, we note that the definition of reversibility can be extended in a natural way to any square matrix and probability distribution over the set of rows of the matrix. We then state a fairly standard result relating eigenvalues of matrices and certain inner products.

**Lemma 3.1.** Let $P$ be a square matrix on state space $S$ and $\pi$ be a probability distribution on $S$. If $P$ is reversible with respect to $\pi$ and has no negative eigenvalues then for any function $f : S \to \mathbb{R}$ we have

$$
\langle Pf, f \rangle_\pi := \sum_{\vec{x} \in S} \pi(\vec{x})(Pf)(\vec{x})f(\vec{x}) \geq 0.
$$

**Proof.** Let $\lambda_1, \ldots, \lambda_s = |S|$, be the eigenvalues of $P$. Moreover, let $f_1, \ldots, f_s$ denote their corresponding eigenfunctions. For any $\vec{x} \in S$, we then have $(Pf_i)(\vec{x})f_i(\vec{x}) = \lambda_i f_i(\vec{x})$. Since $P$ is reversible then we know that the eigenfunctions assume real values and that they form an orthonormal basis for the space $(\mathbb{R}^s, \langle \cdot, \cdot \rangle_\pi)$ (see, e.g., Lemma 12.2 in [18]). Then any real-valued function $f$ defined upon $S$ can be expressed as a linear combination of the $f_i$’s. Thus, there exist $\alpha_i$’s in $\mathbb{R}$ such that

$$
\sum_{\vec{x} \in S} \pi(\vec{x})(Pf)(\vec{x})f(\vec{x}) = \sum_{\vec{x} \in S} \pi(\vec{x}) \sum_{i=1}^s \alpha_i^2 (Pf_i)(\vec{x})f_i(\vec{x}) = \sum_{\vec{x} \in S} \pi(\vec{x}) \sum_{i=1}^s \alpha_i^2 \lambda_i f_i^2(\vec{x}) \geq 0. \quad \square
$$
To specify the restrictions of the transition matrix we are interested in, let \( G \) be a game with profile space \( S \) and let \( P \) be the transition matrix of the logit dynamics for \( G \); we say that a \( |A| \times |A| \) matrix \( P' \), with \( A \subseteq S \), is a nice restriction of \( P \) if there exists \( L \subseteq A \), \( L \neq \emptyset \), such that \( P'(\vec{x}, \vec{x}) \geq P(\vec{x}, \vec{x}) \) for \( \vec{x} \in L \), \( P'(\vec{x}, \vec{y}) = P(\vec{x}, \vec{y}) \) if \( \vec{x}, \vec{y} \in L \), \( \vec{x} \neq \vec{y} \), and is 0 otherwise. Note that \( P \) is a nice restriction of itself. We generalize the result given in [4] to nice restrictions of the transition matrix of logit dynamics for potential games.

**Theorem 3.2.** Let \( G \) be a game with profile space \( S \), let \( P \) be the transition matrix of the logit dynamics for \( G \) and let \( P' \) be a nice restriction of \( P \) with state space \( A \). If \( P \) is reversible with respect to \( \pi \) then no eigenvalue of \( P' \) is negative.

**Proof.** Firstly, note that if \( P \) is reversible with respect to \( \pi \) then the nice restriction \( P' \), defined upon a subset of states \( A \), is reversible with respect to \( \pi' \) defined as \( \pi' \) restricted to \( A \), i.e., \( \pi'(\vec{x}) = \pi(\vec{x})/\pi(A) \) for \( \vec{x} \in A \).

Assume for sake of contradiction that there exists an eigenvalue \( \lambda < 0 \) of \( P' \). Let \( f_\lambda \) be an eigenfunction of \( \lambda \). Note that since \( P \) is reversible then \( f_\lambda \) is real-valued. By definition, \( f_\lambda \neq \vec{0} \); hence, since \( \lambda < 0 \) and as \( (P'f_\lambda)(\vec{x}) = \lambda f_\lambda(\vec{x}) \), then for every profile \( \vec{x} \in A \) such that \( f_\lambda(\vec{x}) \neq 0 \) we have

\[
\langle P'f_\lambda, f_\lambda \rangle' = \sum_{\vec{x} \in A} \pi'(\vec{x}) (P'f_\lambda)(\vec{x})f_\lambda(\vec{x}) < 0.
\]

Let \( L \) denote the maximal subset of \( A \) for which \( P' \) is a nice restriction of \( P \). Let us denote with \( P^L \) the transition matrix on the state space \( A \) such that \( P^L(\vec{x}, \vec{y}) = P(\vec{x}, \vec{y}) \) for every \( \vec{x}, \vec{y} \in L \) and \( P^L(\vec{x}, \vec{y}) = 0 \) otherwise. Then we can write \( P' \) as \( P^L + (P' - P^L) \); by the definition of nice restriction \( (P' - P^L) \) is a non-negative diagonal matrix. Therefore, \( (P' - P^L) \) is reversible with respect to \( \pi' \). Since the eigenvalues of a diagonal matrix are exactly the diagonal elements, we have that \( (P' - P^L) \) has non-negative eigenvalues and then, by Lemma 3.1 \( \langle (P' - P^L)f_\lambda, f_\lambda \rangle' \geq 0 \). Moreover, for every \( i \) and for every \( z_{-i} \), we denote with \( P_{i, z_{-i}} \) the matrix such that for every \( \vec{x}, \vec{y} \in A \)

\[
P_{i, z_{-i}}(\vec{x}, \vec{y}) = \frac{1}{nZ_i(z_{-i})} \left\{ \begin{array}{ll}
\frac{\beta u_i(\vec{y})}{\beta u_i(\vec{y})}, & \text{if } \vec{x}_{-i} = \vec{y}_{-i} = z_{-i} \text{ and } \vec{x}, \vec{y} \in L; \\
0, & \text{otherwise.}
\end{array} \right.
\]

Observe that \( P_{i, z_{-i}} \) has at least one non-zero row and that all non-zero rows of \( P_{i, z_{-i}} \) are the same. Thus \( P_{i, z_{-i}} \) has rank 1, and hence since it is a non-negative matrix all its eigenvalues are non-negative \(^{15}\). Moreover, since all off-diagonal entries of \( P_{i, z_{-i}} \) are either 0 or equal to the corresponding entry of \( P' \) we can conclude that \( P_{i, z_{-i}} \) is reversible with respect to \( \pi' \). Thus, Lemma 3.1 yields \( \langle P_{i, z_{-i}}f_\lambda, f_\lambda \rangle' \geq 0 \). Finally, observe that \( P^L = \sum_{i} \sum_{z_{-i}} P_{i, z_{-i}} \). Hence from the linearity of the inner product, it follows that \( \langle P'f_\lambda, f_\lambda \rangle' \geq 0 \) and thus we reach a contradiction. \( \square \)

The theorem above turns out to be very useful to prove our main result presented in the next section. We next give other interesting spectral results about the transition matrix generated by the logit dynamics. In particular, by using a matrix decomposition similar to the one adopted in the proof of Theorem 3.2, we can prove the following theorem. (We remark that the next results in this section do not need to assume that the chain is reversible and indeed apply to any strategic game.)

**Theorem 3.3.** Let \( G \) be a game with profile space \( S \) and let \( P \) be the transition matrix of the logit dynamics for \( G \). The trace of \( P \) is independent of \( \beta \).

\(^{15}\)This result about the eigenvalues of matrices with rank 1 appears as an exercise at page 61 of [15] and in [23].
Proof. For every \( i \) and for every \( \vec{x}_{-i} \) consider the transition matrices \( P_{i,\vec{x}_{-i}} \) defined in \([4]\), with \( L = S \). Let \( S_{i,\vec{x}_{-i}} = \{ (\vec{x}_{-i}, s_i) \mid s_i \in S_i \} \). Observe that for every \( \vec{x} \in S_{i,\vec{x}_{-i}} \) we have \( P_{i,\vec{x}_{-i}}(\vec{x}, \vec{x}) = 1 - \sum_{\vec{y} \in S_{i,\vec{x}_{-i}}, \vec{y} \neq \vec{x}} P(\vec{x}, \vec{y}) \). Hence, the trace of \( P_{i,\vec{x}_{-i}} \) is

\[
\sum_{\vec{x} \in S_{i,\vec{x}_{-i}}} P_{i,\vec{x}_{-i}}(\vec{x}, \vec{x}) = |S_i| - \sum_{\vec{x} \in S_{i,\vec{x}_{-i}}} \sum_{\vec{y} \in S_{i,\vec{x}_{-i}}, \vec{y} \neq \vec{x}} P(\vec{x}, \vec{y}).
\]

Since all non-zero elements in a column of \( P_{i,\vec{x}_{-i}} \) are the same we also have

\[
P_{i,\vec{x}_{-i}}(\vec{x}, \vec{x}) = \frac{1}{|S_i| - 1} \sum_{\vec{y} \in S_{i,\vec{x}_{-i}}, \vec{y} \neq \vec{x}} P(\vec{y}, \vec{x}).
\]

By setting \( C = \sum_{\vec{x} \in S_{i,\vec{x}_{-i}}} \sum_{\vec{y} \in S_{i,\vec{x}_{-i}}, \vec{y} \neq \vec{x}} P(\vec{x}, \vec{y}) = \sum_{\vec{x} \in S_{i,\vec{x}_{-i}}} \sum_{\vec{y} \in S_{i,\vec{x}_{-i}}, \vec{y} \neq \vec{x}} P(\vec{y}, \vec{x}) \), we have

\[
|S_i| - C = \frac{C}{|S_i| - 1} \implies C = |S_i| - 1,
\]

and thus, the trace of \( P_{i,\vec{x}_{-i}} \) is always 1, regardless of \( \beta \). The theorem follows since the trace of \( P \) is exactly the sum of the traces of all \( P_{i,\vec{x}_{-i}} \)'s. \( \square \)

The theorem above says that if there exists an eigenvalue of \( P \) that gets closer to 1 as \( \beta \) increases, then there are other eigenvalues that get smaller: this is very promising in the tentative to characterize the entire spectrum of eigenvalues of \( P \), necessary to use powerful tools such as the well-known random target lemma \([18]\).

In order to prove our last characterization of the transition matrix generated by the logit dynamics, we prove the following lemma which gives a lower bound on the probability that the strategy profile is not changed in one step of the logit dynamics for a generic game.

**Lemma 3.4.** Let \( \mathcal{G} \) be a game with profile space \( S \) and let \( P \) be the transition matrix of the logit dynamics for \( \mathcal{G} \). Then for every \( \vec{x} \in S \) we have that

\[
P(\vec{x}, \vec{x}) = \sum_{i} P\left((\vec{x}_{-i}, s_i^*), \vec{x}\right),
\]

where \( s_i^* \neq x_i \) is an arbitrary strategy of player \( i \).

**Proof.** Observe that

\[
P(\vec{x}, \vec{x}) = 1 - \sum_{\vec{y} \in N(\vec{x})} P(\vec{x}, \vec{y}) = \sum_{i} \left( \frac{1}{n} - \sum_{\vec{y} \in N_i(\vec{x})} P(\vec{x}, \vec{y}) \right)
\]

\[
= \sum_{i} \frac{1}{n} \left( 1 - \sum_{\vec{y} \in N_i(\vec{x})} e^{\beta u_i(\vec{y})} \right) = \sum_{i} \frac{1}{n} e^{\beta u_i(\vec{x})} + \sum_{\vec{z} \in N(\vec{x})} e^{\beta u_i(\vec{z})} + \sum_{\vec{z} \in N_i(\vec{x})} e^{\beta u_i(\vec{z})}.
\]

The proof concludes by observing that for every \( i \) and for every \( s_i^* \in S_i \), we have

\[
P\left((\vec{x}_{-i}, s_i^*), \vec{x}\right) = \frac{1}{n} e^{\beta u_i(\vec{x})} + \sum_{\vec{z} \in N_i(\vec{x})} e^{\beta u_i(\vec{z})}.
\]

**Lemma 3.4** allows us to calculate the determinant of \( P \).
\textbf{Theorem 3.5.} Let $G$ be a game with profile space $S$ and let $P$ be the transition matrix of the logit dynamics for $G$. Then the determinant of $P$ is 0.

\textbf{Proof.} It is well-known that a matrix in which one row can be expressed as a linear combination of other rows has determinant zero. In this proof, we fix a profile $\vec{x}$ and show that the row of $P$ corresponding to $\vec{x}$ can be obtained as a linear combination of other rows of the matrix. For each player $i$, fix a strategy $s^*_i \in S_i$ such that $s^*_i \neq x_i$. Let us denote with $S^j$, $j = 0, \ldots, n$, the set of profiles $\vec{y} \in S$ obtained from $\vec{x}$ by selecting $j$ players $i_1, \ldots, i_j$ and setting their strategies to $s^*_{i_1}, \ldots, s^*_{i_j}$, respectively. Notice that $\vec{x}$ belongs to $S^0$. By construction, for every profile $\vec{z} \in S^j$, $z_i \in \{x_i, s^*_i\}$. Now, for $i = 1, \ldots, n$, consider the profile obtained from $\vec{z}$ by changing $z_i = x_i$ into $s^*_i$ or vice versa. Note that there are $n$ such profiles which are neighbors of $\vec{z}$ and all contained in the sets $S^{j-1}$ and $S^{j+1}$. We claim that for every $\vec{y} \in S$

\[ P(\vec{x}, \vec{y}) = \sum_{j=1}^{n} (-1)^{j+1} \sum_{\vec{z} \in S^j} P(\vec{z}, \vec{y}). \tag{5} \]

In order to prove the claim we distinguish three cases:

1. Let $H(\vec{x}, \vec{y}) > 1$ (and thus $P(\vec{x}, \vec{y}) = 0$): if there exists $j \in \{0, \ldots, n\}$ such that $\vec{y} \in S^j$, then the r.h.s. of (5) becomes $\pm P(\vec{y}, \vec{y}) - \sum_{i \neq k} P((\vec{y}_{-i}, s^*_i), \vec{y}) = 0$, from Lemma 3.4; if $\vec{y} \notin \bigcup_{j=0}^{n} S^j$, then consider a profile $\vec{z} \in S^j$, for some $j = 1, \ldots, n$, such that $\vec{z}$ differs from $\vec{y}$ only in the strategy of player $k$: if no such profile exists, then the r.h.s. of (5) is 0; otherwise, let us assume w.l.o.g. $z_k = x_k$ (the case $z_k = s^*_k$ can be managed similarly), then the profile $\vec{z}' = (\vec{z}_{-k}, s^*_k)$ is a neighbor of $\vec{y}$, belongs to the set $S^{j+1}$ and $P(\vec{z}, \vec{y}) = P(\vec{z}', \vec{y})$: hence, this two profiles delete each other in the r.h.s. of (5), giving the aimed result.

2. Let $\vec{x}, \vec{y}$ differ in the strategy adopted by the player $k$: if there exists $j \in \{0, \ldots, n\}$ such that $\vec{y} \in S^j$, then the r.h.s. of (5) becomes $P(\vec{y}, \vec{y}) - \sum_{i \neq k} P((\vec{y}_{-i}, s^*_i), \vec{y}) = P(\vec{x}, \vec{y})$, from Lemma 3.4; if $\vec{y} \notin \bigcup_{j=0}^{n} S^j$, then, as above, all profiles in $\bigcup_{j=0}^{n} S^j$ that differ from $\vec{y}$ only in one player $i \neq k$ delete each other in the r.h.s. of (5); thus, the only element that survives in the r.h.s. of (5) is $P((\vec{x}_{-k}, x_k), \vec{y}) = P(\vec{x}, \vec{y})$.

3. If $\vec{x} = \vec{y}$, then the r.h.s. of (5) becomes $\sum_{i \neq k} P((\vec{y}_{-i}, s^*_i), \vec{y}) = P(\vec{x}, \vec{x})$, from Lemma 3.4.

Since, as observed above, logit dynamics for potential games defines a reversible Markov chain, Theorems 3.2 and 3.3 imply that the last eigenvalue of the logit dynamics for these games is exactly 0. (Note that in [4] is only stated the last eigenvalue is non-negative.) Moreover, from the proof above, it turns out that an eigenvector of such zero eigenvalue is given by the function $f: S \rightarrow \mathbb{R}$ defined as

\[ f(\vec{w}) = \begin{cases} -1, & \text{if } \vec{w} \in S^j \text{ and } j \text{ is even;} \\ 1, & \text{if } \vec{w} \in S^j \text{ and } j \text{ is odd;} \\ 0, & \text{otherwise;} \end{cases} \]

where the sets $S^j$'s are defined as in the above proof from some fixed profile $\vec{x}$.

\section{A sufficient condition for metastability of potential games}

In this section we will show a condition on $n$-player potential games sufficient to prove that for each starting profile $\vec{x}$ there is a distribution $\mu$ metastable for super-polynomial time and whose pseudo-mixing time from $\vec{x}$ is polynomial. However, we feel it is necessary to formalize what we mean when we talk about asymptotic results about metastability (and then about the behavior of the logit dynamics in the transient phase).
4.1 Formalization of the asymptotic behavior of the transient phase

Given an \( n \)-player potential game for which the logit dynamics takes super-polynomial time for mixing, we ideally would like to prove that for each starting profile \( \vec{x} \) there is a distribution \( \mu \) metastable for super-polynomial time and whose pseudo-mixing time from \( \vec{x} \) is polynomial. Such a statement is meaningful only when the transient phase of the logit dynamics is asymptotically well-defined. In other words, it should not be the case that moving from \( n \) to \( n + 1 \) players the game (and then the behavior of the dynamics) changes completely. We now give our formalization of this requirement.

We define a \( n \)-player potential game as a sequence \( \mathcal{G} = \{ \mathcal{G}[n] \} \), where \( \mathcal{G}[n] \) is the instance of the potential game with \( n \) players.\(^\text{4}\) Given, an \( n \)-player potential game we can define two other sequences \( \Phi = \{ \Phi[n] \} \) and \( S = \{ S[n] \} \) where \( \Phi[n] \) and \( S[n] \) are the potential function and the profile space of \( \mathcal{G}[n] \), respectively. More generally, if we have an object \( A \) defined for infinitely many instances of the \( n \)-player potential game \( \mathcal{G} \), then we adopt the bold symbol \( \mathbf{A} \) to denote the sequence of these object and \( A[n] \) to identify the \( n \)-th entry in the sequence.

Note that we assume that in a \( n \)-player potential game, there is only one instance of the game for each \( n \). This assumption is without loss of generality since we can always represent a potential game with multiple instances for a fixed \( n \) as a set of \( n \)-player potential games as intended by our definition (that is, with a single potential for every \( n \)). Roughly speaking, instead of seeing the game “horizontally”, that is as a sequence of sets of instances, where each set contains all instances defined for some specified number of players, we see the game “vertically”, that is as a set of sequences of instances, where each sequence contains at most one instance for each number of players. Since we will prove results that hold for any of these “vertical point-of-view” potential games, then they hold also for any “horizontal point-of-view” game.

In order to study the metastability of an \( n \)-player potential game, we need to analyze the asymptotic behavior of the dynamics with respect to subsets of profiles. Clearly, subsets of states may not be identified asymptotically in \( n \). Indeed, as an instance for new \( n \) is considered new subsets are created. Nevertheless, we can say that a game has an asymptotically well-defined transient phase if each subset of profiles of the game with \( n \) players “resembles” subsets of profiles of the game with \( n' \neq n \) number of players, where “resembles” intuitively means that the behavior of the logit dynamics on these subsets can be described as a unique function of the number of players. Indeed, if this is not the case, that is, there are infinitely many subsets that do not resemble any other subset, then it is impossible to asymptotically describe the behavior of the transient phase of the logit dynamics for these subsets and, hence, to give asymptotic bounds on metastability and pseudo-mixing time.

In this work we adopt a specific definition of resemblance that allows us to focus on the distributions \( \pi_L \) defined in (3) whose stability is measured by the bottleneck \( B(L) \), cf. Lemma 2.4. We assume indeed that we can asymptotically classify the bottleneck ratio of each subset of profiles as either polynomial or super-polynomial. More specifically, we assume there are two functions \( p \) at most polynomial in the input and \( q \) at least super-polynomial in the input such that for each \( \beta \) and each subset \( A \) there is a sequence \( \mathbf{A} \) to which \( A \) belongs such that the bottleneck ratio of \( A[n] \) can be bounded by functions that depends on either \( p \) or \( q \). (The kind of functions will be clarified in the next sections.) An equivalent viewpoint would be to see \( n \)-player potential games as a class to which a kind of oracle is attached that distinguishes between polynomial and super-polynomial bottleneck ratios for any fixed \( \beta \). Formally, given a \( n \)-player potential game \( \mathcal{G} \) and fixed \( \beta > 0 \), this oracle can be described as follows: when it is queried about the bottleneck ratio of a subset \( A \subseteq S[n] \) its answer states that the bottleneck ratio is either i) at most polynomial if it is lower-bounded by \( 1/p(n) \); or ii) at least super-polynomial if it is upper-bounded by \( 1/q(n) \).

\(^4\)Note that we can also allow that the game is not defined for some number of players \( n \). That is, any of our results holds even if \( n \) is restricted to a subset \( J \) of natural numbers, as long as \( J \) is infinitely large (or otherwise there is no need for asymptotic results).
Technically, we do not need the behavior of each subset to be classified as above. That is, we can allow that some subsets are not linked with any other subsets or that the behavior is not defined by functions depending on $p$ and $q$. In this case, we will say that the subset of profiles is *unclassified*. Below, we will describe which class of subsets is sufficient to classify in order to prove that for each starting profile $\bar{x}$ there is a distribution $\mu$ metastable for super-polynomial time and whose pseudo-mixing time from $\bar{x}$ is polynomial.

### 4.2 The sufficient condition

**Assumptions.** Henceforth, we will assume that the logarithm of the maximal number of strategies available to a player is at most a polynomial in $n$. Specifically, we denote as $m(\cdot)$ the function such that $m(n)$ is the maximum number of strategies available to a player in the instance $G[n]$. Then, we will assume that the function $\log m(\cdot)$ is at most polynomial in its input. We can easily drop this assumption by asking for results that are asymptotic in $\log |S|$, where $|S|$ denotes the function returning the number of profiles of the game: each one of our proof can be rewritten according to this requirement with very small changes. Note that having results asymptotic in the logarithm of the number of states is a common requirement in Markov chain literature. Moreover, since $|S[n]| \leq m(n)^n$, this requirement is equivalent to asking for results asymptotic in $n$ and in the logarithm of the function $m$.

Note we focus only on $n$-player potential games whose mixing time is at least super-polynomial in $n$, otherwise the stationary distribution enjoys all the desired properties of stability and convergence.

**Preliminary definitions.** Let $\Phi$ be a potential function on profile space $S$. Let $P$ be the transition matrix of the logit dynamics on $\Phi$ and let $\pi$ be the corresponding stationary distribution. For $L \subseteq S$ non-empty, we define a Markov chain with state space $L$ and transition matrix $\tilde{P}_L$ defined as follows.

$$\tilde{P}_L(\bar{x}, \bar{y}) = \begin{cases} P(\bar{x}, \bar{y}) & \text{if } \bar{x} \neq \bar{y}; \\ 1 - \sum_{\bar{z} \in L, \bar{z} \neq \bar{x}} P(\bar{x}, \bar{z}) & \text{otherwise}. \end{cases} \quad (6)$$

It easy to check that the stationary distribution of this Markov chain is given by the distribution $\pi_L(\bar{x}) = \frac{\pi(\bar{x})}{\pi(L)}$, for every $\bar{x} \in L$. Note also that the Markov chain defined upon $\tilde{P}_L$ is reversible and aperiodic, since the Markov chain defined upon $P$ is, and it will be irreducible if $L$ is a connected set. Moreover, it is immediate to see that $\tilde{P}_L$ is also a nice restriction of $P$ and hence all its eigenvalues are non-negative by Theorem 3.2. For a fixed $\varepsilon > 0$, we will denote with $t_{\text{mix}}^L(\varepsilon)$ the mixing time of the chain described in (6). We also denote with $B_L(A)$ the bottleneck ratio of $A \subset L$ in the Markov chain with state space $L$ and transition matrix $\tilde{P}_L$.

**The sufficient condition.** We first introduce an algorithm that defines some subsets $R_1, \ldots, R_k$ of the strategy profile set $S[n]$ of $G[n]$, with $k \geq 1$. Moreover, this algorithm partitions $S$ in $k + 1$ subsets $T_1, \ldots, T_k$ and $N$, where $T_1, \ldots, T_k$ represent the core of the sets $R_1, \ldots, R_k$ and the last subset $N$ simply contains the remaining profiles of $S$. The procedure works its way by finding subsets of profiles that act as super-polynomial bottlenecks for the Markov chain. The algorithm $\mathcal{A}_{p,q}$ is parametrized by two functions $p$ at most polynomial and $q$ at least super-polynomial. It takes in input an $n$-player potential game $G$, a rationality level $\beta$, a constant $\varepsilon > 0$ and $n$.

**Algorithm 4.1 ($\mathcal{A}_{p,q}$).** Set $N = S = S[n]$ and $i = 1$. While there is a set $L \subseteq N$ with $\pi(L) \leq 1/2$ such that $B(L) \leq 1/q(n)$, do:

1. Denote with $R_i$ one such subset with the smallest stationary probability;
2. Denote with $T_i$ the largest subset of $R_i$ such that for every $\vec{y} \in T_i$,

$$P_{\vec{y}} \left( \tau_{S \setminus R_i} \leq t_{\text{mix}}^{R_i}(\varepsilon) \right) \leq \varepsilon;$$

3. If $T_i$ is not empty, return $R_i$ and $T_i$, delete from $N$ all profiles contained in $T_i$ and increase $i$. Otherwise, terminate the algorithm.

Observe that if there is a disconnected set $L$ such that $B(L) \leq 1/q(n)$, then each connected component $L'$ of $L$ will have $B(L') \leq 1/q(n)$ and smaller stationary probability: hence, the set $R_i$ returned by the algorithm will be connected.

We finally define the class of asymptotically well-defined games.

**Definition 4.2 (Asymptotically well-defined games).** An $n$-player potential games $G$ is asymptotically well-defined (AWD) if for every $\beta, \varepsilon > 0$ there exist a pair of functions $p$ at most polynomial and $q$ at least super-polynomial, that for each $n$ sufficiently large, satisfy the following conditions:

1. $q(n) \leq \max_{L: \pi(L) \leq 1/2} B^{-1}(L)$;
2. for each $R_i$ computed by $A_{p,q}$ and for any $L \subset R_i$ such that $\pi_{R_i}(L) \leq 1/2$, if $B(R_i)(L) < 1/p(n)$, then both $B(L)$ and $B(R_i \setminus L)$ are not unclassified;
3. for each subset $L \subseteq N$, $N$ being as at the end of the algorithm $A_{p,q}$, such that $\pi(L) \leq 1/2$, $B(L)$ is not unclassified.

Note that the assumption on $q$ in the definition of AWD $n$-player potential games implies that, for each $n$ sufficiently large, at least one subset $L \subseteq S[n]$ has bottleneck ratio at least super-polynomial. Then, by Theorem 4.1 and the assumption of super-polynomial mixing time, the algorithm above enters at least once in the loop (and thus at least a subset $R_i$ is computed).

Let us also denote with $\Delta(\cdot)$ the function that, for every $n$, gives the Lipschitz constant of $\Phi = \Phi[n]$, i.e.,

$$\Delta(n) := \max \{ \Phi(\vec{x}) - \Phi(\vec{y}) : H(\vec{x}, \vec{y}) = 1 \}.$$

We will show that being AWD is sufficient for proving asymptotic results on metastability and pseudo-mixing time. Specifically we have the following theorem.

**Theorem 4.3 (Sufficient condition).** Let $G$ be an AWD $n$-player potential game. Let $\Delta(j)$ be the function returning the Lipschitz constant of the potential function of the game when the number of players is $j$. Fix constant $\varepsilon > 0$ and a function $\rho$ at most polynomial in its input. Then, there is a function $T$ at least super-polynomial in the input and a function $p_*$ at most polynomial in the input such that for every $n$ large enough, if $0 < \beta \leq \frac{\Delta(n)}{\Delta(n)}$, then for each profile $\vec{x}$ there is a distribution $\mu$ such that $\mu$ is $(\varepsilon, T(n))$-metastable and the pseudo-mixing time of $\mu$ from $\vec{x}$ is $t_{\mu}^{(\vec{x})}(3\varepsilon) = O(p_*(n))$.

The proof then starts by describing the metastable distributions in Section 4.3. We then bound the pseudo-mixing time when the starting profile is in the “core” of a metastable distribution (Section 4.4) and when the starting profile is out of the “core” (Section 4.5). Our analysis for the pseudo-mixing time assumes that $\beta$ is “small” according to the maximum difference in potential.

For every piece of our proof, we introduce the necessary technical tools first: in particular, the main tools adopted in our proofs are represented by Corollary 4.10 that relates the pseudo-mixing time with the mixing time of the restricted chains described in (6), and by Lemma 4.13 and Lemma 4.14 that, instead, relate the hitting time to the bottleneck ratio.
4.3 Metastable distributions

The following lemmata prove that some distributions defined on the sets \( R_i \) described by the definition of AWD game are metastable for super-polynomial time.

**Lemma 4.4.** Let \( G \) be an AWD \( n \)-player potential game and consider the stationary distribution \( \pi \) of the logit dynamics for \( G \). Fix \( \beta \geq 0 \). There exists a function \( T \) at least super-polynomial in the input such that for each \( i \), and \( n \) large enough the distribution \( \mu_i \) that sets \( \mu_i(\bar{x}) = \pi(\bar{x})/\pi(R_i) \) is \((\epsilon, T(n))\)-metastable for every \( \epsilon > 0 \).

**Proof.** Fix \( i \). Given \( \epsilon > 0 \), consider the function \( T \) such that \( T(n) = \sum_{E(R_i)} \geq \epsilon q(n) \), where \( R_i \) is the support of \( \mu_i \). By the definition of \( q \), \( T \) is at least super-polynomial in the input.

Now, fix \( n \). By Lemma 2.4, \( \mu_i \) is \((B(R_i), 1)\)-metastable. By Lemma 2.5, \( \mu_i \) is also \((B(R_i) \cdot T(n), T(n))\)-metastable. The lemma follows since \( B(R_i) \cdot T(n) = \epsilon \).

Finally, the following lemma shows that a combination of metastable distributions is metastable.

**Lemma 4.5.** Let \( P \) the transition matrix of a Markov chain with state space \( S \) and let \( \mu_i \) be a distribution \((\epsilon_i, T_i)\)-metastable for \( P \), for \( i = 1, 2, \ldots \). Set \( \epsilon = \max_i \epsilon_i \) and \( T = \min_i \{T_i\} \). Then, the distribution \( \mu = \sum_i \alpha_i \mu_i \), with \( \sum_i \alpha_i = 1 \) and \( \alpha_i \geq 0 \), is \((\epsilon, T)\)-metastable.

**Proof.** For every \( t \leq T \) we have

\[
\|\mu P^t - \mu\|_{TV} = \max_{A \subseteq S} \left| (\mu P^t)(A) - \mu(A) \right| \\
= \max_{A \subseteq S} \left| \sum_i \alpha_i \left( (\mu_i P^t)(A) - \mu_i(A) \right) \right| \\
\leq \sum_i \alpha_i \max_{A \subseteq S} \left| (\mu_i P^t)(A) - \mu_i(A) \right| \leq \epsilon. \]

4.4 Pseudo-mixing time starting from \( T_i \)

In this section, we will prove that the logit dynamics for an AWD \( n \)-player potential game and \( \beta \) small enough converges in polynomial time to the metastable distribution \( \mu_i \) defined above, whenever the starting point is selected from the core \( T_i \) of this distribution. Specifically, we prove the following lemma.

**Lemma 4.6.** Let \( G \) be an AWD \( n \)-player potential game and fix \( \epsilon > 0 \) and a function \( \rho \) at most polynomial in its input. There exists a function \( \rho_\star \) at most polynomial in the input such that for each \( i \), \( n \) large enough and \( 0 < \beta \leq \frac{\rho(n)}{\sum_{k(n)} k(n)} \), the pseudo-mixing time of \( \mu_i \) from \( T_i \) is \( t_{\mu_i}^{R_i}(2\epsilon) = O(\rho_\star(n)) \).

We first prove in Section 4.4.1 that the mixing time \( t_{\mu_i}^{R_i}(\epsilon) \) of the restriction (6) of the original dynamics to the set \( R_i \) is polynomial. Finally, in Section 4.4.2 we show that the previous result is sufficient to prove the lemma.

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5There can be values of \( n \) for which the algorithm does not run the \( i \)-th iteration and thus \( R_i \) and \( \mu_i \) are not well defined. However, as long as there are infinite values of \( n \) for which \( R_i \) is computed then asymptotic bounds on the pseudo-mixing time of \( \mu_i \) from \( T_i \) are well defined. Since the algorithm executes at least one iteration for any input, we have that there exists \( n_0 \) such that for \( i \leq \max_{n \geq n_0} k(n) \), \( R_i \) is computed infinite times.
4.4.1 Mixing time of the restricted chain

We start by proving some preliminary lemmata.

**Lemma 4.7.** Let $G$ be an AWD $n$-player potential game and fix $\beta, \varepsilon > 0$. Let $p, q$ the functions for which $G$ is AWD. Consider the sequence $R_i$ of sets $R_i$ returned by Algorithm 4.1. Then, for $n$ sufficiently large, we have

$$B_{R_i}(A) \geq \frac{1}{p(n)} - \frac{1}{\ell(n)},$$

where $\ell$ is at least super-polynomial.

**Proof.** Let us postpone the exact definition of $\ell$ and suppose, by contradiction, that there are infinitely many $n$ for which there is $A \subseteq R_i$ such that $B_{R_i}(A) < \frac{1}{p(n)} - \frac{1}{\ell(n)}$. Then, by definition of AWD potential game, both $B(A)$ and $B(\overline{A})$ are classified for each of these $n$, where $\overline{A} = R_i \setminus A$.

We will show that for $n$ sufficiently large either $B(A) \leq 1/q(n)$ or $B(\overline{A}) \leq 1/q(n)$. Then, since they are contained in $R_i$ and hence their stationary probability is less than $\pi(R_i)$, one of these set must be chosen before $R_i$ by Algorithm 4.1. But since in the third step of the algorithm either at least one element of such sets should be deleted from $N$ or the algorithm terminates, as a consequence, we have that $R_i$ cannot be returned by the algorithm, thus a contradiction.

Consider the function $v(\cdot)$ that sets $v(n) = \frac{\pi(A)}{Q(A, S \setminus R_i)}$. We distinguish two cases depending on how $v$ evolves as $n$ grows.

If $v(\cdot)$ is at least super-polynomial in the input: We have

$$B(A) = \frac{Q(A, S \setminus A)}{\pi(A)} = \frac{Q(A, R_i \setminus A)}{\pi(A)} + \frac{Q(A, S \setminus R_i)}{\pi(A)} = \sum_{\vec{x} \in A} \sum_{\vec{y} \in R_i \setminus A} \pi(\vec{x}) p(\vec{x}, \vec{y}) \frac{Q(A, S \setminus R_i)}{\pi(A)} = B_{R_i}(A) + \frac{Q(A, S \setminus R_i)}{\pi(A)} < \frac{1}{p(n)} + \frac{1}{v(n)} - \frac{1}{\ell(n)}.$$

By taking $\ell(n) \leq v(n)$ for each $n$ sufficiently large, we have that $B(A) < \frac{1}{p(n)}$. Hence, since $G$ is AWD, $B(A) \leq \frac{1}{q(n)}$.

If $v(\cdot)$ is polynomial in the input: Note that $\frac{Q(A, S \setminus R_i)}{\pi(R_i)} + \frac{Q(\overline{A}, S \setminus R_i)}{\pi(R_i)} = B(R_i) \leq \frac{1}{q(n)}$, otherwise $R_i$ was not returned by the algorithm. Hence, we obtain

$$Q(A, S \setminus R_i) \leq \frac{1}{q(n)} \cdot \pi(R_i) \quad \text{and} \quad Q(\overline{A}, S \setminus R_i) \leq \frac{1}{q(n)} \cdot \pi(R_i).$$

From the first of these inequalities, we have $\pi(A) \leq \frac{v(n)}{q(n)} \cdot \pi(R_i)$. Hence

$$\frac{Q(A, \overline{A})}{\pi(R_i)} \leq \frac{v(n)}{q(n)} \cdot \frac{Q(A, \overline{A})}{\pi(A)} = \frac{v(n)}{q(n)} \cdot B_{R_i}(A) < \frac{v(n)}{q(n)} \left( \frac{1}{p(n)} - \frac{1}{\ell(n)} \right).$$

Then we obtain

$$B(\overline{A}) = \frac{Q(\overline{A}, S \setminus \overline{A})}{\pi(\overline{A})} = \frac{Q(\overline{A}, A)}{\pi(R_i) - \pi(A)} + \frac{Q(\overline{A}, S \setminus R_i)}{\pi(R_i) - \pi(A)}.$$
(by reversibility of $P$)
\[
\frac{Q(A, \overline{A})}{\pi(R_i) - \pi(A)} + \frac{Q(\overline{A}, S \setminus R_i)}{\pi(R_i) - \pi(A)} \leq \frac{v(n)}{q(n)^2} \left( \frac{1}{p(n)} - \frac{1}{\ell(n)} \right) \left( 1 - \frac{v(n)}{q(n)} \right)^{-1} \left( 1 - \frac{1}{q(n)} \right)^{-1} = O \left( \frac{1}{q(n) - v(n)} \right),
\]

where the upper bounds hold for each choice of super-polynomial function $\ell$. Since $q(n) - v(n)$ evolves at least as a super-polynomial, if $n$ is sufficiently large, $B(\overline{A}) < \frac{1}{p(n)}$. Hence, since $G$ is AWD, $B(\overline{A}) \leq \frac{1}{q(n)}$.

Now we are ready to prove the mixing time of the chain restricted to $R_i$ is polynomial.

**Lemma 4.8.** Let $G$ be an AWD $n$-player potential game and fix $\varepsilon > 0$ and a function $\rho$ at most polynomial in its input. Let $p, q$ the functions for which $G$ is AWD. Consider the sequence $R_i$ of sets $R_i$ returned by Algorithm 4.1. For any $n$ sufficiently large, if $0 < \beta \leq \frac{\rho(n)}{\Delta(n)}$, then $t_{\text{mix}}^{R_i}(\varepsilon)$ is at most polynomial.

**Proof.** Fix $n$. Consider the set of profiles $A_\varepsilon \subset R_i = R_i[n]$ that minimizes $B_{R_i}(A)$ among all $A \subset R_i$ such that $\pi_{R_i}(A) \leq 1/2$. By Lemma 4.7, $B_{R_i}(A_\varepsilon) \geq 1/p(n) - 1/\ell(n)$ for each $n$ sufficiently large.

Moreover, for each $n$ and each $\vec{x} \in R_i$, since $|S| = |S[n]| \leq m(n)^n$, it follows that
\[
\log \frac{1}{\pi_{R_i}(\vec{x})} \leq \log |S| e^{-\beta \Phi_{\text{min}}} \leq \log \frac{n \log m(n) e^{-\beta \Phi_{\text{min}}}}{e^{-\beta \Phi_{\text{max}}}} = n \log m(n) + \beta (\Phi_{\text{max}} - \Phi_{\text{min}}),
\]
where $\Phi_{\text{max}}$ and $\Phi_{\text{min}}$ denote the maximum and minimum of the potential $\Phi[n]$ over all possible strategy profiles. Since $\Phi_{\text{max}} - \Phi_{\text{min}} \leq n \cdot \Delta(n)$ and $\beta \leq \rho(n)/\Delta(n)$, then
\[
\log \frac{1}{\pi_{R_i}(\vec{x})} \leq n \cdot (\log m(n) + \rho(n)).
\]

Then, since $\left( \frac{1}{p} - \frac{1}{\ell} \right) = \Theta \left( \frac{1}{p} \right)$, from Theorem 3.2 and the properties of the relaxation time (see Theorems C.2 and A.1) it follows that the mixing time is
\[
t_{\text{mix}}^{R_i}(1/4) \leq \left( \frac{1}{p(n)} - \frac{1}{\ell(n)} \right)^{-2} \cdot (n \log m(n) + \rho(n)) \cdot 2 \log \frac{4}{\varepsilon} = O(p_*(n)).
\]

Since $p, \log m$ and $\rho$ are at most polynomial, then $p_*$ is at most polynomial in its input and the lemma follows.

### 4.4.2 Pseudo-mixing time

For $L \subset S$ non-empty, consider the Markov chain defined in $P_L$. Let us abuse the notation and denote with $P_L$ and $\pi_L$ also the Markov chain and the distribution defined on the entire state space $S$, assuming $P_L(\vec{x}, \vec{y}) = 0$ if $\vec{x} \notin L$ or $\vec{y} \notin L$, and similarly $\pi_L(\vec{x}) = 0$ when $\vec{x} \notin L$: reversibility and non-negativity of eigenvalues continue to hold also in this case.

For $L \subset S$ we set $\partial L$ as the border of $L$, that is the set of profiles in $L$ with at least a neighbor in $S \setminus L$. Recall that $\tau_{\partial L}$ is the random variable denoting the first time the Markov chain with transition matrix $P$ hits a profile $\vec{x} \in S \setminus L$. The following lemma formally proves the intuitive fact that, by starting from a profile in $L$ the chain $P_L$ are the same up to the time in which the former chain hits a profile in $S \setminus L$. The proof uses the well-known coupling technique (cf., e.g., [13]) which is summarized in Appendix D.
Lemma 4.9. Let P be the transition matrix of a Markov chain with state space S and let \( \hat{P}_L \) be the restriction of P to \( L \subseteq S, L \neq \emptyset \), as given in (6). Then, for every \( \vec{x} \in L \) and for every \( t > 0 \),

\[
\left\| P^t(\vec{x}, \cdot) - \hat{P}_L^t(\vec{x}, \cdot) \right\|_{TV} \leq P_\vec{x} (\tau_{S \setminus L} \leq t).
\]

Proof. Consider the following coupling \((X_t, Y_t)_{t>0}\) of the Markov chains with transition matrix P and \( \hat{P}_L \), respectively:

- If \( X_t = Y_t \in L \setminus \partial L \), then we update the first chain according to P and obtain \( X_{i+1} \); we then set \( Y_{i+1} = X_{i+1} \);
- If \( X_t = Y_t \in \partial L \), then we update the first chain according to P: if \( X_{i+1} \in L \), then we set \( Y_{i+1} = X_{i+1} \), otherwise we set \( Y_{i+1} = Y_i \);
- If \( X_t \neq Y_t \), then we update the chains independently.

Since \( X_0 = Y_0 = \vec{x} \in L \), we have that \( X_t \neq Y_t \) only if \( \tau_{S \setminus L} \leq t \). Thus, by the properties of couplings (see Theorem 8.1), we have

\[
\left\| P^t(\vec{x}, \cdot) - \hat{P}_L^t(\vec{x}, \cdot) \right\|_{TV} \leq P_\vec{x} (X_t \neq Y_t) \leq P_\vec{x} (\tau_{S \setminus L} \leq t) .
\]

The following corollary follows from the Lemma 4.9 and the triangle inequality property of the total variation distance.

Corollary 4.10. Let P be the transition matrix of a Markov chain with state space S and let \( \hat{P}_L \) be the restriction of P to a non-empty \( L \subseteq S \) as given in (6). Then, for every \( \vec{x} \in L \) and for every \( t > 0 \),

\[
\left\| P^t(\vec{x}, \cdot) - \pi_L \right\|_{TV} \leq \left\| \hat{P}_L^t(\vec{x}, \cdot) - \pi_L \right\|_{TV} + P_\vec{x} (\tau_{S \setminus L} \leq t).
\]

Using Corollary 4.10 we can prove Lemma 4.6.

Proof of Lemma 4.6 Fix \( n \). For each \( \vec{x} \in T_i \), by Corollary 4.10 and since \( P_\vec{x} \left( \tau_{S \setminus R_i} \leq t_{\max}^{R_i}(\varepsilon) \right) \leq \varepsilon \), we obtain

\[
\left\| P_{\max}(\varepsilon)(\vec{x}, \cdot) - \mu_i \right\|_{TV} \leq \varepsilon + \varepsilon.
\]

The lemma follows from \( t_{\max}^{R_i}(\varepsilon) \) being at most a polynomial by Lemma 4.8.

4.5 Pseudo-mixing time starting from any profile

In order to bound the pseudo-mixing time from any remaining profile and \( \beta \) not too large we first introduce some useful technical tools.

4.5.1 Technical tools

For a game \( G \) with potential function \( \Phi \) and profile space \( S \), and a rationality level \( \beta \), let \( P \) be the transition matrix of the Markov chain defined by the logit dynamics on \( G \). For a non-empty \( L \subseteq S \), we denote with \( P_L \) the matrix

\[
P_L(\vec{x}, \vec{y}) = \begin{cases} P(\vec{x}, \vec{y}) & \text{if } \vec{x}, \vec{y} \in L; \\ 0 & \text{otherwise.} \end{cases}
\]

Let \( \lambda_1^T \geq \lambda_2^T \geq \ldots \geq \lambda_{|S|}^T \) be the eigenvalues of \( P_L \); notice that \( \lambda_1^T \) can be different from 1 since the matrix \( P_L \) is not stochastic. Theorem 3.2 implies that \( \lambda_1^T \geq \lambda_2^T \geq \ldots \geq \lambda_{|S|}^T \geq 0 \), and thus for \( \lambda_{\max}^T \), the largest eigenvalue of \( P_L \) in absolute value we have: \( \lambda_{\max}^T = \max_i |\lambda_i^T| = \lambda_1^T \).
We start with two characterizations in terms of bottleneck ratio of $1 - \lambda_{\text{max}}^L$, the largest eigenvalue in absolute value of the matrix $P^L$ defined in (7). The first one is an easy extension of the similar characterization of the spectral gap of stochastic matrices.

**Lemma 4.11.** For finite $\beta$ and any $\emptyset \neq L \subseteq S$, $1 - \lambda_{\text{max}}^L \leq B(L)$.

**Proof.** Define the function $\varphi_L : S \to [0, 1]$ to be such that $\varphi_L(\vec{x}) = \pi(\vec{x})$ if $\vec{x} \in L$, and $\varphi_L(\vec{x}) = 0$ otherwise. Consider now the function

$$E_P(\varphi_L) := \frac{1}{2} \sum_{\vec{x}, \vec{y} \in S} \pi(\vec{x})P(\vec{x}, \vec{y})(\varphi_L(\vec{x}) - \varphi_L(\vec{y}))^2. \quad (8)$$

By Theorem 2.2, $\pi(L) \neq 0$ and then $E_P[\varphi_L^2] = \pi(L)^2 \neq 0$. Moreover, by denoting with $\partial L$ the set of profiles $\vec{x} \in L$ that have at least one neighbor profile in $S \setminus L$ and with $E(A_1, A_2)$ the pairs of neighbor profiles $(\vec{x}, \vec{y})$ such that $\vec{x} \in A_1$ and $\vec{y} \in A_2$. We have:

$$E_P(\varphi_L) = \frac{\pi(L)^2}{2} \left( \sum_{(\vec{x}, \vec{y}) \in E(L,S \setminus L)} \pi(\vec{x})P(\vec{x}, \vec{y}) + \sum_{(\vec{x}, \vec{y}) \in E(S \setminus L, L)} \pi(\vec{x})P(\vec{x}, \vec{y}) \right)$$

$$= \pi(L)^2 \sum_{\vec{x} \in \partial L} \pi(\vec{x}) \sum_{\vec{y} \in S \setminus L : H(\vec{x}, \vec{y}) = 1} P(\vec{x}, \vec{y}) = \pi(L)^2 Q(L, S \setminus L),$$

where we used the reversibility of $P$ in the penultimate equality. Hence, we have $E_P(\varphi_L) = B(L)$. The Lemma follows since $1 - \lambda_{\text{max}}^L \leq \frac{E_P(\varphi_L)}{E_P[\varphi_L^2]}$ (see Lemma B.1 in Appendix).

As for the second one, it may be proved in exactly the same way as a similar well-known characterization for the spectral gap of stochastic matrices (see, for example, Section 13.3.3 in [18]).

**Lemma 4.12.** For any $\emptyset \neq L \subseteq S$,

$$1 - \lambda_{\text{max}}^L \geq \frac{(B^L)^2}{2}. \quad (9)$$

The above lemmata represent the main ingredients to prove the following relations between bottleneck ratio and hitting time.

**Lemma 4.13.** Let $G$ be a potential game with profile space $S$ and let $P$ be the transition matrix of the logit dynamics for $G$. Then for finite $\beta$ and $L \subseteq S$, $L \neq \emptyset$, we have

$$\min_{\vec{x} \in L} P_{\vec{x}}(\tau_{S \setminus L} \leq t) \leq t \cdot \frac{B(L)}{1 - B(L)}.$$

**Proof.** We observe:

$$\min_{\vec{x} \in L} P_{\vec{x}}(\tau_{S \setminus L} \leq t) = 1 - \max_{\vec{x} \in L} P_{\vec{x}}(\tau_{S \setminus L} > t)$$

(see Theorem B.2) \quad \leq 1 - \exp \left( t \log \lambda_{\text{max}}^L \right)$$

$$= 1 - \exp \left( t \log(1 - (1 - \lambda_{\text{max}}^L)) \right)$$

(since $1 - a \geq e^{-a}$) \quad \leq 1 - \exp \left( -t \cdot \frac{1 - \lambda_{\text{max}}^L}{\lambda_{\text{max}}^L} \right)$$

(by Lemma 4.11) \quad \leq 1 - \exp \left( -t \cdot \frac{B(L)}{1 - B(L)} \right)$$

(since $1 - e^{-a} \leq a$) \quad \leq t \cdot \frac{B(L)}{1 - B(L)}. \quad \Box
Moreover, for $\bar{x} \in L$ and $0 < \varepsilon < 1$, we also let $T_{S \setminus L}^\varepsilon(\bar{x})$ be the first time step $t$ in which $P_{\bar{x}}(\tau_{S \setminus L} > t) \leq \varepsilon$. Then we have the following lemma.

**Lemma 4.14.** Let $G$ be a potential function with profile space $S$ and $P$ be the transition matrix of the logit dynamics for $G$. For $\beta > 0$, $\emptyset \neq L \subset S$, $\bar{x} \in L$ and $0 < \varepsilon < 1$, let $T_{S \setminus L}^\varepsilon(\bar{x})$ be defined as above (with respect to $P$). Then

$$T_{S \setminus L}^\varepsilon(\bar{x}) \leq (B_L^\varepsilon)^{-2} \left( \frac{2(1 - \varepsilon)}{\varepsilon} + \log \frac{1}{\pi_L(\bar{x})} \right),$$

where $\pi_L(\bar{x}) = \frac{\pi(\bar{x})}{\pi(L)}$ and $B_L^\varepsilon = \min_{A \subseteq L: \pi(A) \leq 1/2} B(A)$.

**Proof.** It is known that the hitting time of $S \setminus L$ can be expressed as a function of the eigenvalues of the matrix $P_L$ (see Theorem B.3). In particular, we have

$$P_{\bar{x}}(\tau_{S \setminus L} > t) \leq \exp \left( t \log \lambda_{\text{max}}^T + \frac{1}{2} \log \frac{1}{\pi_L(\bar{x})} \right)$$

(since $1 - a \leq e^{-a}$) \leq \exp \left( -t \left( 1 - \lambda_{\text{max}}^T \right) + \frac{1}{2} \log \frac{1}{\pi_L(\bar{x})} \right)

(by Lemma 4.12) \leq \exp \left[ -\frac{1}{2} \left( t(B_L^\varepsilon)^2 - \log \frac{1}{\pi_L(\bar{x})} \right) \right]

(since $e^{-a} \leq (1 + a)^{-1}$) \leq \left( 1 + \frac{1}{2} \left( t(B_L^\varepsilon)^2 - \log \frac{1}{\pi_L(\bar{x})} \right) \right)^{-1}.

Thus, by setting $t = (B_L^\varepsilon)^{-2} \left( \frac{2(1 - \varepsilon)}{\varepsilon} + \log \frac{1}{\pi_L(\bar{x})} \right)$, we have $P_{\bar{x}}(\tau_{S \setminus L} > t) \leq \varepsilon$ and then $T_{S \setminus L}^\varepsilon(\bar{x})$ is upper bounded by this value of $t$. \qed

### 4.5.2 Pseudo-mixing time

Fix $n$ and consider the distributions $\mu_i$ defined above (i.e., the stationary distribution restricted to $R_i$). We focus here on distributions of the form

$$\nu(\bar{y}) = \sum_i \alpha_i \mu_i(\bar{y}),$$

for $\alpha_i \geq 0$ and $\sum_i \alpha_i = 1$. Specifically, for every profile $\bar{y} \in N$, we then define the distribution

$$\nu_{\bar{x}}(\bar{y}) = \sum_i \mu_i(\bar{y}) \cdot P_{\bar{x}} \left( X_{\tau_{S \setminus N}} \in T_i \mid \tau_{S \setminus N} \leq T_{S \setminus N}^\varepsilon(\bar{x}) \right). \quad (9)$$

Observe that by definition of $\tau_{S \setminus N}$, since the $T_i$’s and $N$ are a partition of $S$, $X_{\tau_{S \setminus N}} \in \bigcup_i T_i$ is a certain event for all values of $\tau_{S \setminus N}$. Moreover, we show below that we can condition on the event $\tau_{S \setminus N} \leq T_{S \setminus N}^\varepsilon(\bar{x})$. Thus, $\sum_i P_{\bar{x}} \left( X_{\tau_{S \setminus N}} \in T_i \mid \tau_{S \setminus N} \leq T_{S \setminus N}^\varepsilon(\bar{x}) \right) = 1$. The above is then a valid definition of the $\alpha_i$’s.

Then, we prove the following lemma.

**Lemma 4.15.** Let $G$ be an AWD $n$-player potential game and fix $\varepsilon > 0$ and a function $p$ at most polynomial in its input. There is a function $T$ at least super-polynomial in the input and a function $p_\varepsilon$ at most polynomial in the input such that for every $n$ large enough if $0 < \beta \leq \frac{p(n)}{\Delta(n)}$, then for each $\bar{x} \in N$ the corresponding distribution $\nu_{\bar{x}}$ is $(\varepsilon, T(n))$-metastable and the pseudo-mixing time of $\nu_{\bar{x}}$ from the profile $\bar{x}$ is $t_{\bar{x}}^\varepsilon(3\varepsilon) = O(p_\varepsilon(n))$. 

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The following lemma turns out to be useful for proving fast convergence.

**Lemma 4.16.** Let $\mathcal{G}$ be an AWD $n$-player potential game and fix $\beta, \varepsilon > 0$. Let $p, q$ the functions for which $\mathcal{G}$ is AWD. Then, for each $n$ sufficiently large, at the end of algorithm $\mathcal{A}_{p,q}$ on input $\mathcal{G}$, $\beta, \varepsilon$ and $n$ it holds that for each subset $L \subseteq N$ such that $\pi(L) \leq 1/2$, $B(L) \geq 1/p(n)$.

**Proof.** It is sufficient to prove that for each $R_i$ chosen by $\mathcal{A}_{p,q}$, its core $T_i$ is non-empty. Indeed, in this case, the algorithm ends only if no subset $L \subseteq N$ such that $\pi(L) \leq 1/2$ has $B(L) > 1/q(n)$. Since $\mathcal{G}$ is AWD, then the last condition is equivalent to $B(L) \geq 1/p(n)$.

As for the non-emptiness of the core, Lemma 4.13 implies that there exists at least one $\vec{x} \in R_i$ such that

$$\mathbb{P}_{\vec{x}} \left( \tau_{S^i R_i} \leq t_{\max}^R(\varepsilon) \right) \leq \frac{t_{\max}^R(\varepsilon) \cdot B(R_i)}{1 - B(R_i)} \leq \varepsilon,$$

where the last step holds for $n$ sufficiently large since $t_{\max}^R$ is at most polynomial by Lemma 4.8 and $B(R_i)$ is at most the inverse of a super-polynomial by hypothesis. $\square$

We are now ready to prove Lemma 4.15

**Proof of Lemma 4.15** Notice that, the distribution $\nu_{\vec{x}}$ is a convex combination of distributions that are metastable for super-polynomial time: thus, from Lemma 4.5 there exist a function $\tau$ at least super-polynomial in the input such that each such $\nu_{\vec{x}}$ is $(\varepsilon, \tau(n))$-metastable.

Consider now the set of profiles $A_x \subseteq N$ that minimizes $B(A)$ among all $A \subseteq N$ such that $\pi(A) \leq 1/2$. By Lemma 4.16, $B(A_x) \geq 1/p(n)$. Moreover, for each $n$ and each $\vec{x} \in N$, observe that

$$\log \frac{1}{\pi_N(\vec{x})} \leq \log \frac{1}{B(A_x)} \leq \log \frac{e^{-\beta \phi_{\min}}}{e^{-\beta \phi_{\max}}} = n \log m(n) + \beta (\phi_{\max} - \phi_{\min}),$$

where $\phi_{\max}$ and $\phi_{\min}$ denote the maximum and minimum of the potential $\phi[n]$ overall possible strategy profiles. Since $\phi_{\max} - \phi_{\min} \leq n \cdot \Delta(n)$ and $\beta \leq \rho(n)/\Delta(n)$, then

$$\log \frac{1}{\pi_N(\vec{x})} \leq n \cdot (\log m(n) + \rho(n)) = \rho'(n),$$

where, by assumption on $m$ and $\rho$, $\rho'$ is a function at most polynomial in its input. Then, for every $\vec{x} \in N$, from Lemma 4.14 it follows

$$\mathcal{T}_{S\setminus N}(\vec{x}) \leq \left( \frac{1}{B(A_x)} \right)^2 \cdot \left( \frac{2(1 - \varepsilon)}{\varepsilon} + \log \frac{1}{\pi_N(\vec{x})} \right) \leq p(n)^2 \cdot \left( \frac{2(1 - \varepsilon)}{\varepsilon} + \rho'(n) \right) = \rho_*(n),$$

where $\rho_*$ is a function at most polynomial in its input, since $p$ and $\rho'$ are.

Consider now the function $p_*(\cdot)$ such that $p_*(n) = \rho_*(n) + \max_{i} t_{\max}^R(\varepsilon)$. From the definition of AWD game and the fact that $\rho_*$ is at most polynomial, it turns out that $p_*(\cdot)$ is at most a polynomial function in the input.

We complete the proof by showing that, for any sufficiently large $n$ and any $\vec{x} \in N$, $p_*(n)$ upper bounds the pseudo-mixing time $t_{\nu_{\vec{x}}}(3\varepsilon)$ of $\nu_{\vec{x}}$ from the profile $\vec{x}$.

Fix $n$. We denote $t^* = p^*(n)$, with $E$ the event “$\tau_{F,N} \leq \mathcal{T}_{S\setminus N}(\vec{x})$” and with $\overline{E}$ its complement. Recall from Definition 2.7 that $X_t$ denotes the state of the Markov chain defined by logit dynamics at step $t$ and observe that

$$\left\| P^{t^*}(\vec{x}, \cdot) - \nu_{\vec{x}} \right\|_{TV} = \max_{A \subseteq S} \left| \mathbb{P}_{\vec{x}}(X_{t^*} \in A) - \nu_{\vec{x}}(A) \right|$$

$$= \max_{A \subseteq S} \left| \mathbb{P}_{\vec{x}}(X_{t^*} \in A \cap E) - \nu_{\vec{x}}(A) + \mathbb{P}_{\vec{x}}(X_{t^*} \in A \cap \overline{E}) \right|$$

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Observe that for every player \( i \) and \( i + 1 \) equivalently the magnetization of \( H \) where on the complete graph), that we will call \( \beta > c \) \( \beta < 1 \).

Moreover, \([3]\) describes metastable distributions for specific games. We will show in this section that our ideas can be used to this aim: specifically, we apply these ideas to solve a problem left open in \([3]\). Hence, we have for every sufficiently large \( n \) and every \( \bar{x} \in N \), \( t_{\mu_{\bar{x}}} (3\varepsilon) \leq t^* = p_* (n) \).

\[\text{where the definition of } T_{S \setminus N} (\bar{x}) \text{ implies that } \mathbb{P}_{\bar{x}} (E) \geq 1 - \varepsilon > 0 \text{ and then yields the third equality and last inequality. The penultimate inequality, instead, simply follows from the subadditivity of the absolute value and the fact that the difference between two probabilities is upper bounded by 1. As every } \mu_i \text{ is metastable for at least a super-polynomial number of steps, we have, by using } \tau^* \text{ as a shorthand for } T_{S \setminus N}, \]

\[\| \mathbb{P}_{\bar{x}} (X_{t^*} \mid E) - \nu_{\bar{x}} \|_{TV} = \left\| \sum_i \sum_{\bar{y} \in T_i} \mathbb{P}_{\bar{x}} (X_{t^*} = \bar{y} \mid E) \cdot \mathbb{P}_{\bar{x}} (X_{t^*} = \bar{y} \wedge E) - \nu_{\bar{x}} \right\|_{TV} \leq \left\| \sum_i \sum_{\bar{y} \in T_i} \mathbb{P}_{\bar{x}} (X_{t^*} = \bar{y} \mid E) \left( \mathbb{P}^{t^* - \tau^*} (\bar{y}, \cdot) - \mu_i \right) \right\|_{TV} \leq 2\varepsilon,\]

where the definition of \( \tau^* \) yields \( X_{t^*} \in T_i \), for some \( i \), which in turns yields the first equality by the law of total probability. In the first inequality above, instead, we use the definition of \( \nu_{\bar{x}} \) and the fact that by definition of \( t^* \), \( E \) implies \( t^* - \tau^* \geq t^* - T_{S \setminus N} (\bar{x}) \geq \max_i t_{\mu_i} (\varepsilon) \); the second inequality follows from a simple union bound; and the last inequality follows from Lemma \([2, 7]\) (note that \( t^* - \tau^* \) satisfies the hypothesis of the lemma: the lower bound is showed above, while the upper bound follows from the fact that the \( \mu_i \)'s are metastable for at least super-polynomial time). Hence, we have for every sufficiently large \( n \) and every \( \bar{x} \in N \), \( t_{\mu_{\bar{x}}} (3\varepsilon) \leq t^* = p_* (n) \).

### 5 An application: the Curie-Weiss model

In the previous section we showed that for any AWD game there is for each starting point a distribution metastable for super-polynomial time and reached in polynomial time. These distributions can be described through the sets returned by Algorithm \([4, 1]\). However, the algorithm is unpractical and does not allow to explicitly define the metastable distributions. Hence, since we know that such distributions exist it is natural to ask how we can find a more explicit description of metastable distributions for specific games. We will show in this section that our ideas can be used to this aim: specifically, we apply these ideas to solve a problem left open in \([3]\).

Consider the following game-theoretic formulation of the well-studied Curie-Weiss model (the Ising model on the complete graph), that we will call CW-game: each one of \( n \) players has two strategies, \(-1 \) and \(+1 \), and the utility of player \( i \) at profile \( \bar{x} = (x_1, \ldots, x_n) \in \{-1, +1\}^n \) is \( u_i (\bar{x}) = x_i \sum_j x_j \). Observe that for every player \( i \) it holds that

\[u_i (\bar{x} - x_i) - u_i (\bar{x} - x_i) = 1,\]

where \( \mathcal{H} (\bar{x}) = - \sum_{j \neq k} x_j x_k \), hence the CW-game is a potential game with potential function \( \mathcal{H} \). The magnetization of \( \bar{x} \) is defined as \( S(\bar{x}) = \sum_i x_i \).

It is well-known (see, for example, Chapter 15 in \([18]\)) that the logit dynamics for this game (or equivalently the Glauber dynamics for the Curie-Weiss model) has mixing time polynomial in \( n \) for \( \beta < 1/n \) and exponential as long as \( \beta > 1/n \). Moreover, \([3]\) describes metastable distributions for \( \beta > c \log n/n \) and shows that such distributions are quickly reached from profiles where the number

\[\text{is defined as } S(\bar{x}) = \sum_i x_i.\]
of +1 (respectively −1) is a sufficiently large majority, namely if the magnetization $k$ is such that $k^2 > c \log n / \beta$. Thus it is left open what happens when $\beta$ lies in the interval $(1/n, \log n/n)$ and if a metastable distribution is quickly reached when in the starting point the number of +1 is close to the number of −1. The following theorem closes such an open problem.

**Theorem 5.1.** Let $G$ be the $n$-player CW-game and consider the logit dynamics for $G$. Let $S_+$ (resp., $S_-$) be the set of profiles with positive (resp., negative) magnetization and $\pi_+$ (resp., $\pi_-)$ be the restriction of the stationary distribution to $S_+$ (resp., $S_-$). If $\beta > 1/n$, then $\pi_+$ and $\pi_-$ are $(\varepsilon, T)$-metastable, with $\varepsilon > 0$ and $T$ exponential in $n$. Moreover, for every starting profile the logit dynamics reaches a convex combination of these distributions in polynomial time.

**Sketch of proof.** The proof consists in defining sets $R_+, R_-$ and $N$ that satisfy the conditions of Definition 4.2. The result will then follow from Theorem 4.3. However, we do not use Algorithm 4.1 to define these sets but we will take $S_+$ and $S_-$ in place of $R_i$ and we will characterize $T_i$ and $N$ accordingly. In particular, we first show that the bottleneck ratio of $S_+$ and $S_-$ is super-polynomial in $n$. Then, we characterize the “core” of these sets. However, $S_+$ and $S_-$ may not be the smallest subsets with super-polynomial bottleneck ratio (as instead are the sets returned by Algorithm 4.1). Hence, we cannot invoke Lemma 4.8 and, in order to establish the characterization for the cores, we need first to prove that the mixing time of the chains restricted to $S_+$ and $S_-$ is polynomial. Finally, we will show that the bottleneck ratio of profiles “out-of-core” is polynomial.

More specifically, the metastability of $\pi_+$ and $\pi_-$ quickly follows from the fact that, for $\beta > 1/n$ the bottleneck ratio of $S_+$ and $S_-$ is exponential in $n$ (see, for example, Theorem 15.3 in [18]) and from Lemma 2.4.

As for the mixing time of the chains restricted to $S^+$ and $S^-$, we distinguish two cases: for $\beta = O(\log n/n)$ we achieve a polynomial bound following the approach adopted by [17] for bounding the mixing time of censored chain, for larger $\beta$, a polynomial bound follows since the extremal profiles are hit in polynomial time, as showed by Lemma 4.7 in [3].

The “core” of these distribution, i.e., profiles from which the dynamics leaves $S^+$ or $S^-$ in polynomial time with probability at most $\varepsilon$, is given by states with magnetization $k$ such that $|k| \geq c/\beta$, where $c = c(\varepsilon)$ is constant. This can be proved by considering the magnetization chain, i.e., the birth and death chain on the space $\{-n, 2-n, \ldots, n-2, n\}$, and asking for the hitting time of $l \leq 0$ when the starting point is $k$: it is well known that the hitting time in a birth and death chain depends only on the ratio between the probability to go back and to go ahead (see, for example, Section 2.5 in [18]). The characterization then follows by showing that this ratio is at least constant for any starting profile (see, for example, Lemma 4.5 in [3]).

From above characterization of profiles in the “core”, it turns out that the number of remaining profiles are at most polynomial; profiles on the boundary are the one that maximizes the stationary probability among “out-of-core” profiles; moreover, the probability to leave the subset is greater than $1 / \text{poly}(n)$ since there are always neighbors with a lower potential value. This proves a polynomial bottleneck for “out-of-core” profiles and completes the proof of the theorem.

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6 Conclusions and open problems

In this work we study sufficient conditions for the metastability of potential games. Our property is game-independent and related to the asymptotic behavior of certain distributions. It is not clear, and a very interesting open problem, what property is instead necessary in order to have metastable distributions for potential games. It seems that in order to answer this question, new ideas are needed (either

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6The censored chain of [17] is exactly our restricted chain, except that the probability that the original chain from a profile $\vec{x}$ goes out from $L$ is “reflected” to some profile in $L$ different from $\vec{x}$, instead than to be “added” to the probability to do not leave $\vec{x}$.}
new distributions have to be considered or new mathematical tools used) or game-specific arguments ought to be used.

Our convergence rate results hold if $\beta$ is small enough. As we mention above, an assumption on $\beta$ is in general necessary because when $\beta$ is high enough logit dynamics roughly behaves as best-response dynamics. Moreover, in this case, the only metastable distributions have to be concentrated around the set of Nash equilibria. This is because for $\beta$ very high, it is extremely unlikely that a player leaves a Nash equilibrium. Then, the hardness results about the convergence of best-response dynamics for potential games, cf. e.g. [12], imply that the convergence to metastable distributions for high $\beta$ is similarly computationally hard. Interestingly, this difference in the behavior of the logit dynamics for different values of $\beta$ suggests that “the more noisy the system is, the more (meta)stable it is.”

Our result is in a sense existential, since we are not able to explicitly describe the distributions. It is an interesting open problem to characterize the sets $R_i$’s and $T_i$’s returned by Algorithm 4.1 for some specific class of games in order to understand better the stability guarantee of the distributions. Alternatively, our ideas may turn out to be useful to find new different distributions which can be explicitly defined and then allow to make predictions about the future. (We give a first example of this approach in Section 5) A better understanding of spectra of the transition matrix along the lines of the results we prove may help in answering some of the questions above.

Naturally, there are other questions of general interest about metastability that we do not consider. For example, akin to price of anarchy and price of stability, one may ask what is the performance of a system in a metastable distribution? One might also want to investigate metastable behavior of different dynamics, such as best-response dynamics. However, in the latter case, no matter what selection rule is used to choose which player has to move next, a profile is never visited twice in time since at each step the potential goes down. Therefore, the “metastable” behavior of best-response dynamics would roughly correspond to an (exponentially long) sequence of profiles visited. This, however, would not add much to our understanding of the transient phase of best-response dynamics.

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References


[5] Vincenzo Auletta, Diodato Ferraioli, Francesco Pasquale, and Giuseppe Persiano. Mixing time and
stationary expected social welfare of logit dynamics. Theory of Computing Systems, pages 1–38,
2013.


1517, 2011.


[14] F. Hollander. Three lectures on metastability under stochastic dynamics. In Methods of Contempo-


Society, 2008.


143, 1996.


The usefulness of relaxation time

The relaxation time is related to the mixing time by the following theorem (see, for example, Theorems 12.3 and 12.4 in [18]).

**Theorem A.1 (Relaxation time)**. Let \( P \) be the transition matrix of a reversible, irreducible, and aperiodic Markov chain with state space \( S \) and stationary distribution \( \pi \). Then

\[
(t_{\text{rel}} - 1) \log 2 \leq t_{\text{mix}} \leq \log \left( \frac{4}{\pi_{\min}} \right) t_{\text{rel}},
\]

where \( \pi_{\min} = \min_{\vec{x} \in S} \pi(\vec{x}) \).

**B Hitting time bounds**

Consider a reversible Markov chain with state space \( S \) and transition matrix \( P \). For \( L \subseteq S \) let \( P_L, \lambda_L^T \) and \( \lambda_{\max}^T \) as defined in Section 4.5.1. Here we give a well known (see, e.g., [21]) variational characterization of \( \lambda_{\max}^T \) as expressed by the following lemma.

**Lemma B.1.** Consider a reversible Markov chain with state space \( S \), transition matrix \( P \) and stationary distribution \( \pi \). For any \( L \subseteq S \) we have

\[
1 - \lambda_{\max}^T = \inf_{\varphi} \frac{\mathcal{E}_P(\varphi)}{\mathbb{E}_\pi[\varphi^2]},
\]

where \( \mathcal{E}_P(\varphi) \) is defined as in (8), \( \mathbb{E}_\pi[\varphi^2] = \sum_{\vec{x}} \pi(\vec{x}) \varphi^2(\vec{x}) \) and the inf is taken over functions \( \varphi \) such that \( \varphi(\vec{x}) = 0 \) for \( \vec{x} \in S \setminus L \) and \( \mathbb{E}_\pi[\varphi^2] \neq 0 \).

The following theorems relate \( \tau_{S \setminus L} \) and \( \lambda_{\max}^T \) and are already stated in e.g. [21].

**Theorem B.2.** For a reversible Markov chain with state space \( S \), any \( L \subseteq S \) and any \( t \) it holds that

\[
\max_{\vec{x} \in L} P_{\vec{x}}(\tau_{S \setminus L} > t) \geq \exp \left( t \log \lambda_{\max}^T \right).
\]

**Theorem B.3.** For a reversible Markov chain with state space \( S \), any \( L \subseteq S \) and any \( t \) it holds that

\[
\mathbb{P}_\vec{x}(\tau_{S \setminus L} > t) \leq \exp \left( t \log \lambda_{\max}^T + \frac{1}{2} \log \frac{1}{\pi_L(\vec{x})} \right),
\]

where \( \pi_L(\vec{x}) \) has been defined in (3).

Since the statement of Theorem B.3 is slightly different from the ones found in previous literature, we attach a proof for sake of completeness.

**Proof.** Let \( \varphi_L \) be the characteristic function on \( L \), that is \( \varphi_L(\vec{x}) = 1 \) if \( \vec{x} \in L \) and 0 otherwise. Then

\[
\mathbb{P}_\vec{x}(\tau_{S \setminus L} > t) = \sum_{\vec{y} \in S} P_L^t(\vec{x}, \vec{y}) = \sum_{\vec{y} \in S} P_L^t(\vec{x}, \vec{y}) \varphi_L(\vec{y}) = (P_L^t \varphi_L)(\vec{x}).
\]  \( \tag{10} \)

Since \( P_L \) is reversible with respect to \( \pi_L \), we have that its eigenvectors, \( \psi_1, \ldots, \psi_{|S|} \), form an orthonormal basis with respect to the inner product \( \langle \cdot, \cdot \rangle_{\pi_L} \); in particular we can write \( \varphi_L = \sum_i \alpha_i \psi_i \), where

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\[ \sum_i \alpha_i = 1 \text{ and each } \alpha_i > 0. \] Hence and from the linearity of the inner product we have
\[ \langle P_t^L \varphi_L, P_t^L \varphi_L \rangle_{\pi_L} = \sum_i \sum_j (\alpha_i (\lambda^L_i)^t \psi_i, \alpha_j (\lambda^L_j)^t \psi_j)_{\pi_L} \]
(by orthogonality)
\[ \leq \left( \lambda^T_{\max} \right)^{2t} \langle \varphi_L, \varphi_L \rangle_{\pi_L} = \left( \lambda^T_{\max} \right)^{2t}, \]
where the last equality follow from the definition of \( \varphi_L \). Moreover,
\[ \pi_L(\bar{x})(P_t^L \varphi_L)_{\pi_L}(\bar{x}) \leq \sum_{y \in S} \pi_L(y) \langle (P_t^L \varphi_L)(y) \rangle^2 = \langle P_t^L \varphi_L, P_t^L \varphi_L \rangle_{\pi_L}. \] (12)
The theorem follows from (10), (11), (12).

**C Bottleneck ratio bounds**

We use the following theorem to derive lower bounds to the mixing time (see, for example, Theorem 7.3 in [18]).

**Theorem C.1** (Bottleneck ratio). Let \( \mathcal{M} = \{ X_t : t \in \mathbb{N} \} \) be an irreducible and aperiodic Markov chain with finite state space \( S \), transition matrix \( P \), and stationary distribution \( \pi \). Let \( L \subseteq S \) be any set with \( \pi(L) \leq 1/2 \). Then the mixing time is
\[ t_{\text{mix}} \geq \frac{1}{4B(L)}. \]

The bottleneck ratio is also strictly related to the relaxation time. Indeed, let
\[ B_* = \min_{L: \pi(L) \leq 1/2} B(L), \]
then the following theorem holds (see, for example, Theorem 13.14 in [18]).

**Theorem C.2.** Let \( P \) be the transition matrix of a reversible, irreducible, and aperiodic Markov chain with state space \( S \). Let \( \lambda_2 \) be the second largest eigenvalue of \( P \). Then
\[ \frac{B_*^2}{2} \leq 1 - \lambda_2 \leq 2B_* \]

**D Markov chain coupling**

A **coupling** of two probability distributions \( \mu \) and \( \nu \) on a state space \( S \) is a pair of random variables \( (X, Y) \) defined on \( S \times S \) such that the marginal distribution of \( X \) is \( \mu \) and the marginal distribution of \( Y \) is \( \nu \). A **coupling of a Markov chain** \( \mathcal{M} \) on \( S \) with transition matrix \( P \) is a process \( (X_t, Y_t)_{t=0}^{\infty} \) with the property that \( X_t \) and \( Y_t \) are both Markov chains with transition matrix \( P \). Similarly, a **coupling of Markov chains** \( \mathcal{M}, \bar{\mathcal{M}} \) both defined on \( S \) with transition matrices \( P \) and \( \bar{P} \), respectively, is a process \( (X_t, Y_t)_{t=0}^{\infty} \) with the property that \( X_t \) is a Markov chain with transition matrix \( P \) and \( Y_t \) is a Markov chain with transition matrix \( \bar{P} \).

When the two coupled chains start at \( (X_0, Y_0) = (\bar{x}, \bar{y}) \), we write \( P_{\bar{x},\bar{y}}(\cdot) \) for the probability of an event on the space \( S \times S \). The following theorem, which follows from Proposition 4.7 and Theorem 5.2 in [18] establishes the importance of this tool.
Theorem D.1 (Coupling). Let $\mathcal{M}$, $\mathcal{M}'$ be two Markov chains with finite state space $S$ and transition matrices $P$ and $\bar{P}$, respectively. For each pair of states $\vec{x}, \vec{y} \in S$ consider a coupling $(X_t, Y_t)$ of $\mathcal{M}$ and $\mathcal{M}'$ with starting states $X_0 = \vec{x}$ and $Y_0 = \vec{y}$. Then
\[
\|P^t(\vec{x}, \cdot) - \bar{P}^t(\vec{y}, \cdot)\|_{\text{TV}} \leq P_{\vec{x}, \vec{y}}(X_t \neq Y_t).
\]