On the Price of Anarchy of Restricted Job Scheduling Games

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Model

1
2
3
4
5

A
B
C

$n = 5, m = 3$

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Model

Restricted Job Scheduling

$n = 5, m = 3$
Model

Restricted Unrelated Job Scheduling

1

2

3

4

5

$w_{2A} = 31$

$w_{2B} = 23$

$w_{3B} = 55$

$w_{3C} = 1.5$

$w_{4C} = 24$

$w_{5C} = 7$

$w_{4A} = 5$

$n = 5, m = 3$

$w_{min} = 1.5$

$w_{max} = 55$

$s = \frac{w_{max}}{w_{min}} = \frac{55}{1.5}$
Model

Restricted Related Job Scheduling

Model

Previous Results

Our Results

$\mathbf{Model}$

$\mathbf{Previous\ Results}$

$\mathbf{Our\ Results}$

$w_1 = 4$

$w_2 = 6$

$w_3 = 9$

$w_4 = 5$

$w_5 = 1$

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$n = 5, m = 3$

$w_{ij} = \frac{w_i}{v_j}$

$r = \frac{\max_j v_j}{\min_j v_j} = \frac{7}{2}$

$t = \frac{\max_i w_i}{\min_i w_i} = \frac{9}{1}$
Model

Previous Results

Our Results

\[ j_1 = A \]
\[ j_2 = B \]
\[ j_3 = C \]
\[ j_4 = C \]
\[ j_5 = C \]

\[ w_{1A} = 5 \]
\[ w_{2A} = 31 \]
\[ w_{3B} = 55 \]
\[ w_{4C} = 24 \]
\[ w_{5C} = 7 \]

\[ n_A = 1 \]
\[ n_B = 1 \]
\[ n_C = 3 \]

\[ c_A = 5 \]
\[ c_B = 23 \]
\[ c_C = 32.5 \]
An **Optimal Solution** \( (OPT) \) is an assignment that minimize the objective function selected by the designer. (Ex: \( \sum_i c_i \))
A Nash Equilibrium Solution (NASH) is an assignment where no job can unilaterally improve its latency moving to another machine.

\[ \forall \text{ job } i, \forall \text{ machine } j, \quad c_{ji} \leq c_j + w_{ij}. \]
Price of Anarchy

Definition

The Price of Anarchy ($PoA$) [KP99] is the ratio between the cost of the worst Nash Equilibrium Solution and the cost of the Optimal Solution.

$$PoA = \frac{C(\text{NASH})}{C(\text{OPT})}$$

It measures how much the lack of central coordination can affect the system.
Previous Results

Existence
A Nash Equilibrium Solution always exists. [EdKM03]

Maximum Latency \((\min \max_j c_j)\)
- \(PoA = \Theta \left( s + \frac{\log m}{\log(1+\log m)} \right)\) in the Unrelated Setting; [AART03]
- \(PoA = \Theta \left( \frac{\log m}{\log \log m} \right)\) in Related Setting. [AART03]
Our Results

Bounds on PoA for
- Server Latency;
- Total Latency;
- Weighted Total Latency.
Server Latency

**Definition**

This objective function minimizes the total load suffered by the machines.

\[
\min \sum_j c_j = \min \sum_i w_{ij};
\]
Our Result - Unrelated Setting

\[ PoA = \Theta(s) \]

Our Result - Related Setting

\[ PoA = \Theta(\min(r, n)) \]
### Model

**Previous Results**

<table>
<thead>
<tr>
<th>Server Latency</th>
<th>Total Latency</th>
<th>Weighted Total Latency</th>
</tr>
</thead>
</table>

**Our Results**

<table>
<thead>
<tr>
<th>Server Latency - 3</th>
</tr>
</thead>
</table>

### Unrelated Setting - Upper Bound

\[ w_{ij} \leq sw_{ij}^* \Rightarrow \sum_j c_j \leq s \sum_j c_j^* \]

### Unrelated Setting - Lower Bound

- **Node A:** 
  - \( w_{1A} = 1 \)
  - \( w_{1B} = s \)
  - \( w_{3A} = s \)

- **Node B:** 
  - \( w_{2B} = 1 \)
  - \( w_{2C} = s \)

- **Node C:** 
  - \( w_{3C} = 1 \)
Unrelated Setting - Upper Bound

\[ w_{ij} \leq s w_{ij}^* \Rightarrow \sum_j c_j \leq s \sum_j c_j^* \]

Unrelated Setting - Lower Bound

\[ w_{1A} = 1 \]
\[ w_{1B} = s \]
\[ w_{1C} = s \]
\[ w_{2A} = s \]
\[ w_{2B} = 1 \]
\[ w_{2C} = s \]
\[ w_{3A} = s \]
\[ w_{3B} = 1 \]
\[ w_{3C} = 1 \]

\[ C(OPT) = 3 \]
Server Latency - 3

Unrelated Setting - Upper Bound

\[ w_{ij} \leq sw_{ij}^* \Rightarrow \sum_j c_j \leq s \sum_j c_j^* \]

Unrelated Setting - Lower Bound

\[ C(NASH) = 3s \]

\[ w_{1A} = 1 \]
\[ w_{1B} = s \]
\[ w_{1C} = s \]
\[ w_{2A} = s \]
\[ w_{2B} = 1 \]
\[ w_{2C} = s \]
\[ w_{3A} = s \]
\[ w_{3B} = 1 \]
\[ w_{3C} = 1 \]
∀ \varepsilon > 0 \; \text{PoA} \geq \begin{cases} r - \varepsilon & \text{with } \min(r, n) = r \\ n - \varepsilon & \text{with } \min(r, n) = n \end{cases}
$\forall \varepsilon > 0 \text{ PoA} \geq \begin{cases} r - \varepsilon & \text{with min}(r, n) = r \\ n - \varepsilon & \text{with min}(r, n) = n \end{cases}$

$w_{a_1} = r$

$w_{a_2} = 1$

$w_{a_n} = 1$

$C(OPT) = \frac{r + n - 1}{r}$
 ∀ \varepsilon > 0 \quad \text{PoA} \geq \begin{cases} r - \varepsilon & \text{with } \min(r, n) = r \\ n - \varepsilon & \text{with } \min(r, n) = n \end{cases}

C(\text{NASH}) = n
Related Setting - Lower Bound

\[ \forall \, \varepsilon > 0 \quad \text{PoA} \geq \begin{cases} r - \varepsilon & \text{with } \min(r, n) = r \\ n - \varepsilon & \text{with } \min(r, n) = n \end{cases} \]

- Let \( n \geq \frac{r^2}{\varepsilon} \), and so \( \min(n, r) = r \), then \( \text{PoA} \geq r - \varepsilon \);
- Let \( r \geq \frac{n^2}{\varepsilon} \), and so \( \min(n, r) = n \), then \( \text{PoA} \geq n - \varepsilon \).
Total Latency

Definition

This objective function minimizes the total latency suffered by jobs.

\[
\min \sum_i c_{j_i} = \min \sum_j n_j c_j
\]
## Total Latency - 2

### Previous Results - Unrestricted Related Setting
- $PoA \leq 4s$; [BGGM06]
- $\frac{n}{2w} \leq PoA \leq \frac{n}{w} + \frac{m^2 + m}{w^2}$. [HS07]

### Previous Results - Restricted Unweighted Related Setting
- $2.5 - \varepsilon \leq PoA \leq 2.5$. [STZ04][CFK+06]

### Our Results - Unrelated Setting
- $PoA = O(ms)$;
- $PoA = \Omega(s)$.

### Our Results - Related Setting
- $PoA = \Omega \left( \min \left( \frac{m + \sqrt{t}}{m + 3}, \frac{n}{m + 3} \right) \right)$. 

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Total Latency - 3

Upper Bound

∀ job \( i \), \( c_{ji} \leq c_{ji}^* + w_{ij}^* \)

\[
\sum_i c_{ji} \leq \sum_i (c_{ji}^* + w_{ij}^*) \leq \sum_i (c_{ji}^* + c_{ji}^*)
\]

\[
C(NASH) = \sum_j n_j c_j \leq \sum_j n_j^* c_j + \sum_j n_j^* c_j^* = \sum_{j: c_j^* \neq 0} n_j^* c_j + C(OPT) \quad (1)
\]
Total Latency - 4

Upper Bound

- Now we can upper-bound $\sum_j n_j^* c_j$ as follows:

$$\sum_j n_j^* c_j \leq \sum_j n_j^* n_j w_{\text{max}} \leq w_{\text{max}} (\sum_j n_j^*) (\sum_j n_j) = n^2 w_{\text{max}} \quad (2)$$

- We can easily lower-bound $C(OPT)$ as follows:

$$\sum_j n_j^* c_j^* \geq \sum_j (n_j^*)^2 w_{\text{min}} \geq w_{\text{min}} \frac{(\sum_j n_j^*)^2}{m} = \frac{n^2}{m} w_{\text{min}} \quad (3)$$

- From (1), (2) and (3) we obtain

$$PoA = \frac{C(NASH)}{C(OPT)} \leq 1 + \frac{\sum_j n_j^* c_j}{C(OPT)} \leq 1 + m \frac{n^2 w_{\text{max}}}{n^2 w_{\text{min}}} = O(ms).$$
Lower Bound - Related Setting

\[ |A_1| = \lceil r \rceil w, \ w_{A_1} = 1 \]

\[ |A_2| = \lceil r \rceil w, \ w_{A_2} = 1 \]

\[ w_{C_i} = t \geq \left( 2 \lceil r \rceil w + lw \right)^2 \]

\[ |B_i| = w, \ w_{B_i} = 1 \]

\[ l = m - 2, \ w_{d_i} = \frac{t}{r} \]

\[ v_{f_1} = v_{f_2} = r \]

\[ v_{f_i} = 1 \]

\[ w = \frac{(k-1)(m+3)+3}{2r+m-2} \]
Total Latency - 5

Lower Bound - Related Setting

\[ |A_1| = \lceil r \rceil w, \quad w_{A_1} = 1 \]
\[ |A_2| = \lceil r \rceil w, \quad w_{A_2} = 1 \]
\[ w_{c_i} = t \geq (2 \lceil r \rceil w + lw)^2 \]
\[ |B_i| = w, \quad w_{B_i} = 1 \]
\[ l = m - 2, \quad w_{d_i} = \frac{t}{r} \]

\[ v_{f_1} = v_{f_2} = r \]
\[ v_{f_i} = 1 \]

\[ w = \left(\frac{(k-1)(m+3)+3}{2r+m-2}\right) \]

\[ C(OPT) \leq (m + 3) \frac{t}{r} \]
Lower Bound - Related Setting

\[ |A_1| = \lceil r \rceil w, \ w_{A_1} = 1 \]

\[ |A_2| = \lceil r \rceil w, \ w_{A_2} = 1 \]

\[ w_{c_i} = t \geq (2 \lceil r \rceil w + lw)^2 \]

\[ |B_i| = w, \ w_{B_i} = 1 \]

\[ l = m - 2, \ w_{d_i} = \frac{t}{r} \]

\[ v_{f_i} = 1 \]

\[ w = \lceil \frac{(k - 1)(m + 3) + 3}{2r + m - 2} \rceil \]

\[ k = \min \left( \frac{m + \sqrt{t}}{m + 3}, \frac{n}{m + 3} \right) \]

\[ C(\text{NASH}) \geq k(m + 3) \frac{t}{r} \]
Weighted Total Latency

**Definition**

This objective function minimizes the latency suffered by any unit of jobs’ weight.

\[
\min \sum_i w_{ij} c_i = \min \sum_j c_j^2
\]
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### Previous Result - Related Setting

\[ 2.5 \leq PoA \leq \frac{3+\sqrt{5}}{2} \approx 2.618 \] [AAE05] [CFK+06]

### Our Result - Unrelated Setting

\[ PoA = \Theta(s^2) \]
Upper Bound

\[ \forall \text{ job } i, \quad c_{ji} \leq c_{j_i}^* + w_{ij_i}^* \Rightarrow w_{ij_i} c_{ji} \leq w_{ij_i} (c_{j_i}^* + w_{ij_i}^*) \]

\[ \sum_i w_{ij_i} c_{ji} \leq \sum_i w_{ij_i} (c_{j_i}^* + w_{ij_i}^*) \leq \sum_i w_{ij_i} (c_{j_i}^* + c_{j_i}^*) \leq s \sum_i w_{ij_i} (c_{j_i}^* + c_{j_i}^*) \]

\[ C(NASH) = \sum_j c_j^2 \leq s(\sum_j c_j^* c_j + \sum c_j^*)^2 = s(\sum_j c_j^* c_j + C(OPT)) \]
Using the “Cauchy-Schwartz inequality”:

\[ \sum_j c_j c_j^* \leq \sqrt{C(NASH)} \cdot \sqrt{C(OPT)} \]

Merging this result in previous statement

\[ C(NASH) \leq s \left[ \left( \sqrt{C(NASH)} \sqrt{C(OPT)} \right) + C(OPT) \right] \Rightarrow \]

\[ \Rightarrow PoA - s \sqrt{PoA} - s \leq 0 \]

Solving this disequation

\[ PoA = O(s^2). \]
Conclusions

### Server Latency
- $PoA = \Theta(s)$ in Unrelated Setting;
- $PoA = \Theta(\min(r, n))$ in Related Setting.

### Total Latency
- $PoA = O(ms)$;
- $PoA = \Omega(s)$ in Unrelated Setting;
- $PoA = \Omega\left(\min\left(\frac{m+\sqrt{t}}{m+3}, \frac{n}{m+3}\right)\right)$ in Related Setting.
- Open problem: Close the gap.

### Weighted Total Latency
- $PoA = \Theta(s^2)$. 
Bibliography I


