• Book: Optimization Methods in Finance, Gerard Cornuejols and Reha Tüüncü, Carnegie Mellon University, Pittsburgh, USA

• Book: Linear Programming, Vasek Chvatal, McGill University, W.H. Freeman and Company, New York, USA
Contents

1 Portfolio Selection and Asset Allocation 3
   1.1 Preliminaries and Notation ................................................. 3
   1.2 Mean-variance optimization (MVO) – Markowitz portfolio optimization ........ 6
   1.3 Karush–Kuhn–Tucker (KKT) conditions. .................................. 10
   1.4 Large-Scale Portfolio Optimization ........................................ 13
      1.4.1 Portfolio Optimization with Minimum (and/or Maximum) Transaction Levels .... 13
      1.4.2 Portfolio Optimization with Transaction Costs .......................... 14
   1.5 Portfolio Rebalancing Problem ............................................. 15
   1.6 Maximizing the Sharpe Ratio .............................................. 16

2 Value-at-Risk and Conditional Value-at-Risk 18
   2.1 Value-at-Risk – VaR ............................................................ 18
   2.2 VaR – with continuous and discrete probability distribution ................. 21
   2.3 Conditional Value-at-Risk – CVaR ........................................ 21

3 Asset Pricing and Arbitrage 24
   3.1 Pricing and Hedging of Options ........................................... 24
   3.2 Replication Problem ............................................................ 25
   3.3 The Fundamental Theorem of Asset Pricing ................................ 28
   3.4 Boolean expressions using binary variables .............................. 30
1 Portfolio Selection and Asset Allocation

- The theory of optimal selection of portfolios was developed by Harry Markowitz in the 1950s
- His work formalized the diversification principle in portfolio selection
- Key assumption: Trades off between expected returns and the perceived risk of portfolios.

1.1 Preliminaries and Notation

- Consider a set of rates of return $r^i_t$ ($t = 1, \ldots, T, i = 1, \ldots, n$)
- The value $r^i_t$ represents the rate of return of a security $i$ at time $t$
- $T$ represents the temporal horizon under consideration and $n$ is the number of securities
- Starting from the time series of the rates of return we can compute:

  **arithmetic mean**

  $$ r^a_i = \frac{1}{T} \sum_{t=1}^{T} r^i_t $$

  **geometric mean (only positive values)**

  $$ r^g_i = \left( \prod_{t=1}^{T} (1 + r^i_t) \right)^{\frac{1}{T}} - 1 $$

  Since the rates of returns are typically not smaller than 1 is it sufficient to add 1 in order to get positive values (then shift back subtracting 1).

  Example with 2 rates of returns and a specific security $i$:

<table>
<thead>
<tr>
<th>$r^a_i$</th>
<th>$r^g_i$</th>
</tr>
</thead>
<tbody>
<tr>
<td>-1</td>
<td>-1</td>
</tr>
<tr>
<td>-1</td>
<td>-1</td>
</tr>
<tr>
<td>0</td>
<td>0.5</td>
</tr>
<tr>
<td>0</td>
<td>0.0</td>
</tr>
<tr>
<td>-0.5</td>
<td>0.0</td>
</tr>
<tr>
<td>0.1</td>
<td>0.25</td>
</tr>
</tbody>
</table>

  Table 1: Arithmetic and geometric means

  **variance**

  $$ \sigma^2_i = \frac{1}{T} \sum_{t=1}^{T} (r^i_t - r^a_i)^2 $$

  **standard deviation**

  $$ \sigma_i = \sqrt{\frac{1}{T} \sum_{t=1}^{T} (r^i_t - r^a_i)^2} $$
covariance matrix (positive semidefinite)

\[ Q_{ij} = \frac{1}{T} \sum_{t=1}^{T} (r_t^i - r_t^a)(r_t^j - r_t^a) \]

correlation matrix

\[ \rho_{ij} = \frac{Q_{ij}}{\sigma_i \sigma_j} \]

equivalences

\[ Q_{ii} = \sigma_i^2 \]

Example

- Consider the 3 securities: \( S_1, S_2, S_3 \) (\( n = 3 \))
- Consider the price time series of one weak (\( T = 5 \))

<table>
<thead>
<tr>
<th></th>
<th>( S_1 )</th>
<th>( S_2 )</th>
<th>( S_3 )</th>
</tr>
</thead>
<tbody>
<tr>
<td>Monday</td>
<td>20.2</td>
<td>262.9</td>
<td>100.0</td>
</tr>
<tr>
<td>Tuesday</td>
<td>21.6</td>
<td>268.7</td>
<td>102.3</td>
</tr>
<tr>
<td>Wednesday</td>
<td>20.4</td>
<td>284.0</td>
<td>105.3</td>
</tr>
<tr>
<td>Thursday</td>
<td>21.7</td>
<td>283.1</td>
<td>108.8</td>
</tr>
<tr>
<td>Friday</td>
<td>23.4</td>
<td>282.8</td>
<td>113.0</td>
</tr>
</tbody>
</table>

Table 2: Market Values (Prices)

- Let \( I_t^i \) denote the price of asset \( i = 1, 2, 3 \) at time \( t = 1, \ldots, 5 \), where \( t = 0 \) corresponds to Monday and \( t = 5 \) to Friday (values in Table 2).
- For each asset \( i \), we can compute the rates of returns \( r_t^i \) using the formula:

\[ r_t^i = \frac{I_t^i - I_{t-1}^i}{I_{t-1}^i} \]

- Table 3 reports the rates of returns \( r_t^i \) (\( i = 1, 2, 3 \)) and (\( t = 2, \ldots, 5 \)). Accordingly for this example \( T = 4 \).

<table>
<thead>
<tr>
<th></th>
<th>( S_1 )</th>
<th>( S_2 )</th>
<th>( S_3 )</th>
</tr>
</thead>
<tbody>
<tr>
<td>Tuesday</td>
<td>6.93</td>
<td>2.21</td>
<td>2.30</td>
</tr>
<tr>
<td>Wednesday</td>
<td>-5.56</td>
<td>5.69</td>
<td>2.93</td>
</tr>
<tr>
<td>Thursday</td>
<td>6.37</td>
<td>-0.32</td>
<td>3.32</td>
</tr>
<tr>
<td>Friday</td>
<td>7.83</td>
<td>-0.11</td>
<td>3.86</td>
</tr>
</tbody>
</table>

Table 3: Rates of Return (in percentages): \( r_t^i \)

- From the time series of the rates of return, we can compute the arithmetic means of the rates of return for each asset:

\[ r_t^a = \frac{1}{T} \sum_{i=1}^{T} r_t^i \]
Table 4: Arithmetic means of the rates of return: $r_i^a$

- Table 4 reports the arithmetic means of the rates of return of the example:
- From the time series of the rates of return, we can also compute the geometric means of the rates of return for each asset:

$$r_i^g = \left( \prod_{t=1}^{T} (1 + r_t^i) \right)^{\frac{1}{T}} - 1$$

- Table 5 reports the geometric means of the rates of return of the example:

Table 5: Geometric means of the rates of return: $r_i^g$

<table>
<thead>
<tr>
<th></th>
<th>$S_1$</th>
<th>$S_2$</th>
<th>$S_3$</th>
</tr>
</thead>
<tbody>
<tr>
<td>$r_i^g$ (in percentages)</td>
<td>3.75</td>
<td>1.84</td>
<td>3.10</td>
</tr>
</tbody>
</table>

- We now compute the covariance matrix:

$$Q_{ij} = \frac{1}{T} \sum_{t=1}^{T} (r_t^i - r_a^i)(r_t^j - r_a^j)$$

Table 6: Covariance Matrix: $Q_{ij}$

<table>
<thead>
<tr>
<th></th>
<th>$S_1$</th>
<th>$S_2$</th>
<th>$S_3$</th>
</tr>
</thead>
<tbody>
<tr>
<td>$S_1$</td>
<td>0.0030</td>
<td>-0.0012</td>
<td>0.0001</td>
</tr>
<tr>
<td>$S_2$</td>
<td>-0.0012</td>
<td>0.0006</td>
<td>-0.0001</td>
</tr>
<tr>
<td>$S_3$</td>
<td>0.0001</td>
<td>-0.0001</td>
<td>0.0000</td>
</tr>
</tbody>
</table>

- We can now compute the standard deviation (also called Volatility):

$$\sigma_i = \sqrt{\frac{1}{T} \sum_{t=1}^{T} (r_t^i - r_a^i)^2}$$

Table 7: Standard deviation: $\sigma_i$

<table>
<thead>
<tr>
<th></th>
<th>$S_1$</th>
<th>$S_2$</th>
<th>$S_3$</th>
</tr>
</thead>
<tbody>
<tr>
<td>$\sigma_i$</td>
<td>0.0548</td>
<td>0.0242</td>
<td>0.0057</td>
</tr>
</tbody>
</table>

- We now compute the correlation matrix:

$$\rho_{ij} = \frac{Q_{ij}}{\sigma_i \sigma_j}$$
<table>
<thead>
<tr>
<th></th>
<th>$S_1$</th>
<th>$S_2$</th>
<th>$S_3$</th>
</tr>
</thead>
<tbody>
<tr>
<td>$S_1$</td>
<td>1.0000</td>
<td>-0.9107</td>
<td>0.2166</td>
</tr>
<tr>
<td>$S_2$</td>
<td>-0.9107</td>
<td>1.0000</td>
<td>-0.5265</td>
</tr>
<tr>
<td>$S_3$</td>
<td>0.2166</td>
<td>-0.5265</td>
<td>1.0000</td>
</tr>
</tbody>
</table>

Table 8: Correlation Matrix: $\rho_{ij}$

1.2 Mean-variance optimization (MVO) – Markowitz portfolio optimization

- Consider now an investor who wants to invest a certain amount of money in a number of different securities (stocks, bonds, etc.) with random expected returns
- For each security the price time series is available
- For each security $i = 1, \ldots, n$, we estimates:
  - the expected return $\mu_i$ (typically the geometric mean of the rates of return $r_i^g$)
  - the variance $\sigma_i^2$ or the standard deviation $\sigma_i$
- For any couple of securities $i$ and $j$, we estimates:
  - covariance $Q_{ij}$ or the correlation coefficient $\rho_{ij}$
- If we represent the proportion of the total funds invested in security $i$ by $x_i$, a portfolio is a vector:
  
  $x = (x_1, \ldots, x_n)$

expected return of a portfolio

$E[x] = x_1 \mu_1 + \cdots + x_n \mu_n = \mu^\top x$

where:

$\mu = \begin{bmatrix} \mu_1 \\ \mu_2 \\ \vdots \\ \mu_n \end{bmatrix}$

variance of a portfolio

$\text{Var}[x] = \sum_{i=1}^{n} \sum_{j=1}^{n} Q_{ij} x_i x_j = \sum_{i=1}^{n} \sum_{j=1}^{n} \rho_{ij} \sigma_i \sigma_j x_i x_j = x^\top Q x$

where:

$Q = \begin{bmatrix} Q_{11} & Q_{12} & \cdots & Q_{1n} \\ Q_{21} & Q_{22} & \cdots & Q_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ Q_{n1} & Q_{n2} & \cdots & Q_{nn} \end{bmatrix}$

- In linear algebra, a symmetric $n \cdot n$ real matrix $M$ is said to be positive semidefinite:
– if the scalar $x^T M x$ is non-negative for every non-zero column vector $x$ of $n$ real numbers.
– All the eigenvalues of $M$ are non-negative

• The covariance matrix $Q$ is positive semidefinite:

$$\sum_i \sum_j x_i x_j Q_{ij} = \sum_i \sum_j x_i x_j \frac{1}{T} \sum_{t=1}^{T} (r_t^i - \bar{r}_t^i)(r_t^j - \bar{r}_t^j) =$$

$$= \frac{1}{T} \sum_{t=1}^{T} \left( \sum_i x_i (r_t^i - \bar{r}_t^i) \right) \left( \sum_j x_j (r_t^j - \bar{r}_t^j) \right) = \frac{1}{T} \sum_{t=1}^{T} \left( \sum_i x_i (r_t^i - \bar{r}_t^i) \right)^2 \geq 0$$

• then for any $x$, $x^T Q x \geq 0$.

Portfolio features:
• The portfolio vector $x$ must satisfy $\sum_i x_i = 1$
• There may or may not be additional feasibility constraints
• A feasible portfolio $x$ is called efficient if:
  – it has the minimum variance among all portfolios that have at least a certain expected return
  – it has the maximal expected return among all portfolios with less than a certain variance level

• The collection of efficient portfolios forms the efficient frontier of the portfolio universe

• The efficient frontier is often represented as a curve in a two-dimensional graph where the coordinates of a plotted point corresponds to:
  – the expected return
  – the variance of the efficient portfolio

The MVO problem can be formulated in 2 but equivalent ways:

**First formulation**

• the first formulation results in the problem of finding a minimum variance portfolio of the securities 1 to $n$ that yields at least a target value $R$ of expected return (target return).

Convex quadratic programming problem:

$$\min x^T Q x \quad \text{(1)}$$

$$e^T x = 1 \quad \text{(2)}$$

$$\mu^T x \geq R \quad \text{(3)}$$

$$x \geq 0 \quad \text{(4)}$$
where $\mathbf{e}$ is an $n$-dimensional vector all of which components are equal to 1

\[ \mathbf{e} = \begin{bmatrix} 1 \\ 1 \\ \vdots \\ 1 \end{bmatrix} \]

- The first constraint indicates that the proportions $x_i$ should sum up to 1
- The second constraint indicates that the expected return is no less than the target value $R$
- the objective function corresponds to the total variance of the portfolio
- Nonnegativity constraints on $x_i$ are introduced to avoid short sales

Solving this problem for values of $R$ ranging between $R_{\text{min}}$ and $R_{\text{max}}$ one obtains all efficient portfolios

- $R_{\text{min}}$ can be set to 0 since portfolios with a negative target return can be discarded.
  Attention: if
  \[ \max_{i=1,\ldots,n} r_i^g < 0 \]
  the model become infeasible
- $R_{\text{max}}$ can be set to:
  \[ \max_{i=1,\ldots,n} r_i^g \]
  since in this case the portfolio will consists only of the security $i^*$ with the largest geometric mean of the rates of return. Accordingly the variance of this portfolio will be the $\sigma_i^2$.

Second formulation

- the second formulation results in the problem of finding a portfolio of maximum expected return of the securities 1 to $n$ that respects a maximum level of variance $\bar{R}$.

  Convex quadratically constrained programming problem:

\[
\begin{align*}
\text{max} & \quad \mu^\top \mathbf{x} \\
\text{subject to} & \quad \mathbf{e}^\top \mathbf{x} = 1 \\
& \quad \mathbf{x}^\top \mathbf{Q} \mathbf{x} \leq \bar{R} \\
& \quad \mathbf{x} \geq 0
\end{align*}
\]

- this formulation is less used and accordingly in the rest of the section we will focus on the first formulation

Possible additional constraints

- the set of admissible portfolios can be a nonempty polyhedral set represented as
  \[ \chi = \{ \mathbf{x} : \mathbf{A} \mathbf{x} = \mathbf{b}, \mathbf{C} \mathbf{x} \geq \mathbf{d} \} \]
  where $\mathbf{A}$ is an $m \cdot n$ matrix, $\mathbf{b}$ is an $m$-dimensional vector, $\mathbf{C}$ is a $p \cdot n$ matrix and $\mathbf{d}$ is a $p$-dimensional vector.
- In this case it is necessary to add the following constraints to the models

\[
\begin{align*}
\mathbf{A} \mathbf{x} & = \mathbf{b} \\
\mathbf{C} \mathbf{x} & \geq \mathbf{b}
\end{align*}
\]
For example, if regulations or investor preferences limit the amount of investment in a subset of the securities, the model can be augmented with a linear constraint to reflect such a limit.

In principle, any linear constraint can be added to the model without making it significantly harder to solve.

**Markowitz's MVO model example**

- Consider the 3 securities: $S_1, S_2, S_3, S_4$ ($n = 4$)
- Consider the following price time series of 4 days ($T = 4$)

<table>
<thead>
<tr>
<th></th>
<th>$S_1$</th>
<th>$S_2$</th>
<th>$S_3$</th>
<th>$S_4$</th>
</tr>
</thead>
<tbody>
<tr>
<td>Monday</td>
<td>100.0</td>
<td>100.0</td>
<td>1.0</td>
<td>2.0</td>
</tr>
<tr>
<td>Tuesday</td>
<td>110.0</td>
<td>90.0</td>
<td>1.1</td>
<td>1.5</td>
</tr>
<tr>
<td>Wednesday</td>
<td>132.0</td>
<td>99.0</td>
<td>1.1</td>
<td>1.5</td>
</tr>
<tr>
<td>Thursday</td>
<td>145.2</td>
<td>108.9</td>
<td>1.1</td>
<td>2.0</td>
</tr>
</tbody>
</table>

- The rates of returns $r_t^i$ ($i = 1, 2, 3, 4$) and ($t = 2, 3, 4$) are: (this example $T = 3$).

<table>
<thead>
<tr>
<th></th>
<th>$S_1$</th>
<th>$S_2$</th>
<th>$S_3$</th>
<th>$S_4$</th>
</tr>
</thead>
<tbody>
<tr>
<td>Tuesday</td>
<td>10.00</td>
<td>-10.00</td>
<td>10.00</td>
<td>-25.00</td>
</tr>
<tr>
<td>Wednesday</td>
<td>20.00</td>
<td>10.00</td>
<td>0.00</td>
<td>0.00</td>
</tr>
<tr>
<td>Thursday</td>
<td>10.00</td>
<td>10.00</td>
<td>0.00</td>
<td>33.33</td>
</tr>
</tbody>
</table>

- The arithmetic means of the rates of return of the example are:

<table>
<thead>
<tr>
<th></th>
<th>$S_1$</th>
<th>$S_2$</th>
<th>$S_3$</th>
<th>$S_4$</th>
</tr>
</thead>
<tbody>
<tr>
<td>$r_t^a$</td>
<td>13.33</td>
<td>3.33</td>
<td>3.33</td>
<td>2.78</td>
</tr>
</tbody>
</table>

- The geometric means of the rates of return of the example are:

<table>
<thead>
<tr>
<th></th>
<th>$S_1$</th>
<th>$S_2$</th>
<th>$S_3$</th>
<th>$S_4$</th>
</tr>
</thead>
<tbody>
<tr>
<td>$r_t^g$</td>
<td>13.24</td>
<td>2.88</td>
<td>3.23</td>
<td>0.00</td>
</tr>
</tbody>
</table>

- We now compute the covariance matrix:

<table>
<thead>
<tr>
<th></th>
<th>$S_1$</th>
<th>$S_2$</th>
<th>$S_3$</th>
<th>$S_4$</th>
</tr>
</thead>
<tbody>
<tr>
<td>$S_1$</td>
<td>0.0022</td>
<td>0.0022</td>
<td>-0.0011</td>
<td>-0.0009</td>
</tr>
<tr>
<td>$S_2$</td>
<td>0.0022</td>
<td>0.0089</td>
<td>-0.0044</td>
<td>0.0185</td>
</tr>
<tr>
<td>$S_3$</td>
<td>-0.0011</td>
<td>-0.0044</td>
<td>0.0022</td>
<td>-0.0093</td>
</tr>
<tr>
<td>$S_4$</td>
<td>-0.0009</td>
<td>0.0185</td>
<td>-0.0093</td>
<td>0.0571</td>
</tr>
</tbody>
</table>

- We construct now the Markowitz’s MVO model to construct a portfolio of four securities: $S_1, S_2, S_3, S_4$ of the previous example.
• $R_{\text{min}}$ can be set to: 0.0

$R_{\text{max}}$ can be set to: 0.1324, i.e., $\mu_1$

• For an expected target return $R$ the QP model reads as follows:

\[
\begin{align*}
\min & \ [ 0.0044 \ x_1^2 + 0.0088 \ x_1 \cdot x_2 + \\
& -0.0044 \ x_1 \cdot x_3 - 0.0037 \ x_1 \cdot x_4 + \\
& +0.0177 \ x_2^2 - 0.0177 \ x_2 \cdot x_3 + \\
& +0.0740 \ x_2 \cdot x_4 + 0.0044 \ x_3^2 + \\
& -0.0370 \ x_3 \cdot x_4 + 0.1141 \ x_4^2 ] / 2 \\
& x_1 + x_2 + x_3 + x_4 = 1 \\
0.1324 \ x_1 + 0.0288 \ x_2 + 0.0323 \ x_3 + 0.0 \ x_4 \geq R \\
x_1, x_2, x_3, x_4 \geq 0
\end{align*}
\]

• Solving the model for $R = 0.06$ to $R = 0.01324$ with increments of 0.01 we get the following optimal portfolios and the corresponding variance.

<table>
<thead>
<tr>
<th>Target Return R</th>
<th>Variance</th>
<th>$x_1$</th>
<th>$x_2$</th>
<th>$x_3$</th>
<th>$x_4$</th>
</tr>
</thead>
<tbody>
<tr>
<td>0.06</td>
<td>0.00000</td>
<td>0.3279</td>
<td>0.0000</td>
<td>0.5738</td>
<td>0.0984</td>
</tr>
<tr>
<td>0.07</td>
<td>0.00003</td>
<td>0.4044</td>
<td>0.0000</td>
<td>0.5102</td>
<td>0.0854</td>
</tr>
<tr>
<td>0.08</td>
<td>0.00015</td>
<td>0.4991</td>
<td>0.0000</td>
<td>0.4315</td>
<td>0.0694</td>
</tr>
<tr>
<td>0.09</td>
<td>0.00035</td>
<td>0.5939</td>
<td>0.0000</td>
<td>0.3528</td>
<td>0.0533</td>
</tr>
<tr>
<td>0.10</td>
<td>0.00064</td>
<td>0.6886</td>
<td>0.0000</td>
<td>0.2741</td>
<td>0.0373</td>
</tr>
<tr>
<td>0.11</td>
<td>0.00103</td>
<td>0.7833</td>
<td>0.0000</td>
<td>0.1954</td>
<td>0.0212</td>
</tr>
<tr>
<td>0.12</td>
<td>0.00150</td>
<td>0.8781</td>
<td>0.0000</td>
<td>0.1167</td>
<td>0.0052</td>
</tr>
<tr>
<td>0.13</td>
<td>0.00207</td>
<td>0.9763</td>
<td>0.0000</td>
<td>0.0237</td>
<td>0.0000</td>
</tr>
<tr>
<td>0.1324</td>
<td>0.00222</td>
<td>1.0000</td>
<td>0.0000</td>
<td>0.0000</td>
<td>0.0000</td>
</tr>
</tbody>
</table>

Table 9: Efficient Portfolios

• Efficient frontier

**Asset allocation**

• Asset allocation problems have the same mathematical structure as portfolio selection problems.

• In these problems the objective is not to choose a portfolio of stocks (or other securities) but to determine the optimal investment among a set of asset classes.

• Examples of asset classes are large capitalization stocks, small capitalization stocks, foreign stocks, government bonds, corporate bonds, etc.

• After estimating the expected returns, variances, and covariances for different asset classes, one can formulate a Markowitz model and obtain efficient portfolios of these asset classes.

**1.3 Large-Scale Portfolio Optimization**

• We consider now the practical issues that arise when the Mean- Variance model is used to construct a portfolio from a large underlying family of assets or stocks.

• the number of assets or stocks $n$ may be in the hundreds or thousands.
Diversification – In general, there is no reason to expect that solutions to the Markowitz model will be well diversified portfolios.

Practitioners often use additional constraints to ensure that the chosen portfolio is well diversified.

1.3.1 Portfolio Optimization with Minimum (and/or Maximum) Transaction Levels

When solving the classical Markowitz model, the optimal portfolio often contains $x_i$ that are too small or too big. In practice, one would like a solution of with the following properties.

With the property that:

$$x_i \leq l_{max}^i \quad (A)$$

where $l_{max}^i$ are given maximum transaction levels.
• this constraint alone can be easily considered as follow

\[ x_i \leq l_{max}^i \quad i = 1, \ldots, n \]

With the property that:

\[ \text{if } x_i > 0 \Rightarrow x_i \geq l_{min}^i \quad (B) \]

where \( l_{min}^i \) are given minimum transaction levels.

• This constraint states that, if an investment is made in a stock, then it must be large enough, for example at least 100 shares

• Because this constraint is not a simple linear constraint (logic constraint), it cannot be handled directly by a solver.

BQP formulation

• The Mean-Variance (MV) portfolio problem with minimum and maximum transaction level requires optimally allocating wealth among a set \( N \) of assets in order to obtain a prescribed level of target return \( R \) while minimizing the risk as measured by the variance of the portfolio.

\[
\begin{align*}
\text{min} & \quad \sum_{i=1}^{n} \sum_{j=1}^{n} Q_{ij} x_i x_j \\
\sum_{i=1}^{n} \mu_i x_i & \geq R \\
\sum_{i=1}^{n} x_i & = 1 \\
x_i & \geq l_{min}^i u_i \quad i = 1, \ldots, n \\
x_i & \leq l_{max}^i u_i \quad i = 1, \ldots, n \\
x_i & \geq 0 \quad i = 1, \ldots, n \\
u_i & \in \{ 0, 1 \} \quad i = 1, \ldots, n
\end{align*}
\]

• where \( \mu_i, l_{min}^i, l_{max}^i \) are respectively the expected unitary return and the minimum and maximum transaction level for asset \( i \),

• This apparently simple model is rather demanding for general-purpose (MIQP) solvers, since the root node gaps of the ordinary continuous relaxation are huge

• In addition classical polyhedral approaches are not effective to improve the lower bounds.

1.3.2 Portfolio Optimization with Transaction Costs

• We can add a portfolio turnover constraint to ensure that the change between the current holdings \( x_0 \) and the desired portfolio \( x \) is bounded by \( h \)

• This constraint is essential when solving large mean-variance models since the covariance matrix (A square matrix which does not have an inverse. A matrix is singular if and only if its determinant is zero.) is almost singular in most practical applications and hence the optimal decision can change significantly with small changes in the problem data
To avoid big changes when reoptimizing the portfolio, turnover constraints can be imposed.

Let $x^0$ denote the current portfolio $(x^0_i \ i = 1, \ldots, n)$.

Let $y_i$ be the amount of asset $i$ bought and $z_i$ the amount sold, and $h$ a bound on the total amount bought and sold:

\begin{align}
  x_i - x^0_i & \leq y_i, \ y_i \geq 0, \quad (27) \\
  x^0_i - x_i & \leq z_i, \ z_i \geq 0, \quad (28) \\
  \sum_{i=1}^{n} y_i + z_i & \leq h \quad (29)
\end{align}

We can introduce transaction costs directly into the model.

Suppose that there is a transaction cost $t_i$ proportional to the amount of asset $i$ bought, and a transaction cost $t'_i$ proportional to the amount of asset $i$ sold.

Then a re-optimized portfolio is obtained by solving

\begin{align}
  \min & \sum_{i=1}^{n} \sum_{j=1}^{n} Q_{ij} x_i x_j \\
  & \sum_{i=1}^{n} \mu_i x_i - t_i y_i - t'_i z_i \geq R \quad (30) \\
  & \sum_{i=1}^{n} x_i = 1 \\
  & x_i - x^0_i \leq y_i \quad i = 1, \ldots, n \quad (33) \\
  & x^0_i - x_i \leq z_i \quad i = 1, \ldots, n \quad (34) \\
  & \sum_{i=1}^{n} y_i + z_i \leq h \quad (35) \\
  & x_i \geq 0 \quad i = 1, \ldots, n \quad (36) \\
  & y_i \geq 0 \quad i = 1, \ldots, n \quad (37) \\
  & z_i \geq 0 \quad i = 1, \ldots, n \quad (38)
\end{align}

1.4 Portfolio Rebalancing Problem

we consider an approach which tries to rebalance the portfolio at minimum cost.

This approach assumes that we have identified important characteristics of the market index we would like to achieve. Such characteristics might be:

- the fraction $f_k$ of companies with market capitalization in various range
- the fraction $f_k$ of companies that pay no dividends
- ...

Let us assume that there are $m$ such characteristics that we would like our index fund to track as well as possible.

Let $a_{ki} = 1$ if company $i$ has characteristic $k$ and 0 if it does not.
• Let $x_i$ denote the optimum weight of asset $i$ in the portfolio.

• Assume that initially, the portfolio has weights $x_i^0$.

• Let $y_i$ denote the fraction of asset $i$ bought and $z_i$ the fraction sold.

The problem of rebalancing the portfolio at minimum cost is the following:

\[
\min \sum_{i=1}^{n} (y_i + z_i) \tag{39}
\]

\[
\sum_{i=1}^{n} a_{ki} x_i = f_i \tag{40}
\]

\[
\sum_{i=1}^{n} x_i = 1 \tag{41}
\]

\[
x_i - x_i^0 \leq y_i \tag{42}
\]

\[
x_i^0 - x_i \leq z_i \tag{43}
\]

\[
x_i \geq 0 \tag{44}
\]

\[
y_i \geq 0 \tag{45}
\]

\[
z_i \geq 0 \tag{46}
\]

1.5 Maximizing the Sharpe Ratio

• It is a method to uniquely define the “optimal” portfolio

• Recall that we denote with $R_{\min}$ and $R_{\max}$ the minimum and maximum target expected returns of efficient portfolios

• Let us define the function $\sigma(R)$ where $R$ is the target expected return:

\[
\sigma(R) : [R_{\min}, R_{\max}] \rightarrow \mathbb{R}
\]

\[
\sigma(R) := (x_R^\top Q x_R)
\]

$\Rightarrow x_R$ denotes the unique optimal solution of the MVO problem for each $R$

• Since we assumed that $Q$ is positive definite, it is possible to show that the function $\sigma(R)$ is strictly convex in its domain

• the efficient frontier is the graph

\[
E = \{(R, \sigma(R)) : R \in [R_{\min}, R_{\max}]\}
\]

• A set of “optimal” portfolios – Which one is the “best”?

• We now consider a riskless asset whose expected return is $r_f \geq 0$. We can assume that $r_f < R_{\min}$, since the portfolio $x_{\min}$ has a positive risk associated with it while the riskless asset does not.

• The Sharpe ratio measures the excess return (or risk premium) per unit of variance (risk) of a portfolio (named after William F. Sharpe)

\[
h(x) = \frac{\mu^\top x - r_f}{x^\top Q x}
\]
The portfolio that maximizes the Sharpe ratio is found by solving the following problem:

$$\max \frac{\mu^\top x - rf}{x^\top Qx}$$

subject to

$$e^\top x = 1$$

$$x \geq 0$$

Where $$Q$$ is a positive definite matrix, i.e., $$x^\top Qx > 0$$ for every non-zero column vector $$x$$ of $$n$$ real numbers. Or equivalently all the eigenvalues of $$Q$$ are positive.

- In this form, this problem is not easy to solve. Although it has a nice polyhedral feasible region, its objective function is somewhat complicated, and worse, is possibly non-concave. It is not a convex optimization problem.

- An effective way of computing the optimal risky portfolio is based instead on the following graphic interpretation of the Sharpe Ratio

**Graphic interpretation of the Sharpe Ratio**

- Return/risk profiles of different combinations of a risky portfolio with the riskless asset can be represented as a straight line – capital allocation line (CAL) –. This line start in $$rf$$ (all budget allocated in the riskless asset) and it passes thought $$x^R Q x^R$$ (all budget allocated in the portfolio)

- The optimal CAL never goes above a point on the efficient frontier (otherwise you can have the same return with a smaller risk). The optimal CAL touches in only one point the efficient frontier.

- The point where the optimal CAL touches the efficient frontier corresponds to the optimal risky portfolio.

![Figure 3: Capital Allocation Line](image)

- Alternatively, one can think of the optimal CAL as the CAL with the biggest slope ($$\frac{df(x)}{dx}$$). Mathematically, this can be expressed as the portfolio $$x$$ that maximizes the quantity

$$h(x) = \frac{\mu^\top x - rf}{x^\top Qx}$$

- The effective strategy to find the portfolio maximizing the Sharpe ratio (optimal risky portfolio) is the following:
First, one traces out the efficient frontier on a two dimensional target return vs. variance graph.

Then, the point on this graph corresponding to the optimal risky portfolio is found as the tangency point of the line going through the point representing the riskless asset and tangent to the efficient frontier.

Once this point is identified, one can recover the composition of this portfolio from the information generated and recorded while constructing the efficient frontier.
2 Value-at-Risk and Conditional Value-at-Risk

- Financial activities involve risk
- Financial and other institutions must manage risk using sophisticated mathematical techniques
- Necessity of quantitative risk measures that adequately reflect the vulnerabilities of a company
- Value-at-Risk and Conditional Value-at-Risk are widely used measure of risk in finance
- They can be used instead of the variance of a portfolio as in the Markowitz model
- This is achieved through stochastic programming
- The random events are modeled by a large but finite set of scenarios, leading to a linear programming equivalent of the original stochastic program.

2.1 Value-at-Risk – VaR

**Definition** – Take a random variable $X$ that represents for example the loss of an investment over a fixed period of time (a negative value for $X$ indicates gains). Given a probability level $\alpha$, $\alpha$-VaR of the random variable $X$ is given by the following relation:

$$\text{VaR}_\alpha(X) = \min\{\gamma : P(X \leq \gamma) \geq \alpha\}$$

**VaR**: represents the predicted maximum loss with a specified probability level (e.g., 95%) over a certain period of time (e.g., one day).

![Figure 4: 0.95-VaR on a portfolio loss distribution plot](image-url)
Example of application

- take a generic portfolio
- you know its current market value, at the beginning of the day while it is not known its market value at the end of the day
- The investment bank that holds this portfolio may declare that its portfolio has a 1-day VaR of 0.1 million with a confidence level of 95%
- It means that the bank expects that, with a probability of 95%, the portfolio’s loss will not be less than 0.1 million during the day
  - This implies that the bank expects that the loss of its portfolio at the end of the day will be more than 0.1 million, with a probability of 5%
  - So the bank expects that 5 times out of 100, the portfolio loss will be greater greater than 0.1 million, while below this threshold will be 95 times out of 100

Features:

- VaR is widely used by people in the financial industry and VaR calculators are common features in most financial software
- VaR can be estimated either parametrically (for example, variance-covariance VaR or delta-gamma VaR) or nonparametrically (for examples, historical simulation VaR or resampled VaR)
- 85% of large banks use historical simulation. The other 15% used Monte Carlo methods

Problems:

- it lacks of sub-additivity
- a sub-additive function \( f(x_1 + x_2) \leq f(x_1) + f(x_2) \) \( \forall x_1, x_2 \)
- a good Risk measures should instead respect the principle that diversification reduces risk
- When VaR is computed by generating scenarios, it turns out to be a non-smooth and non-convex function.
- Therefore, when one tries to optimize VaR computed in this manner, the optimization is difficult (global optimization algorithms)
- Another criticism on VaR is that it pays no attention to the magnitude of losses beyond the VaR value.

Example:

- Consider two independent and identical investment opportunities:
  - 1 of gain with probability 96%
  - 2 of loss with probability 4%
- Then, 0.95-VaR for both investments are -1.
- Now consider the sum of these two investment opportunities. Because of independence, this sum has the following loss distribution:
  - 4 of loss with probability 1.6% (0.04 * 0.04)
  - 1 of loss with probability 7.68% (2 * 0.96 * 0.04)
  - 2 of gain with probability 92.16% (0.96 * 0.96)
- Therefore, the 0.95-VaR of the sum of the two investments is 1, which exceeds −2, the sum of the 0.95-VaR values for individual investments (lacks of sub-additivity).
Example of calculation

- Consider the following losses of a portfolio over a time window of 20 days

<table>
<thead>
<tr>
<th>day</th>
<th>loss</th>
<th>day</th>
<th>loss</th>
<th>day</th>
<th>loss</th>
<th>day</th>
<th>loss</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>10</td>
<td>6</td>
<td>-30</td>
<td>11</td>
<td>20</td>
<td>16</td>
<td>-5</td>
</tr>
<tr>
<td>2</td>
<td>20</td>
<td>7</td>
<td>-40</td>
<td>12</td>
<td>15</td>
<td>17</td>
<td>-10</td>
</tr>
<tr>
<td>3</td>
<td>10</td>
<td>8</td>
<td>10</td>
<td>13</td>
<td>-5</td>
<td>18</td>
<td>10</td>
</tr>
<tr>
<td>4</td>
<td>15</td>
<td>9</td>
<td>15</td>
<td>14</td>
<td>-10</td>
<td>19</td>
<td>15</td>
</tr>
<tr>
<td>5</td>
<td>-30</td>
<td>10</td>
<td>10</td>
<td>15</td>
<td>-10</td>
<td>20</td>
<td>10</td>
</tr>
</tbody>
</table>

- We can now compute and plot the portfolio loss distribution and cumulative loss distribution

<table>
<thead>
<tr>
<th>loss</th>
<th>probability</th>
<th>cumulative</th>
</tr>
</thead>
<tbody>
<tr>
<td>-40</td>
<td>5.00</td>
<td>100.00</td>
</tr>
<tr>
<td>-30</td>
<td>10.00</td>
<td>95.00</td>
</tr>
<tr>
<td>-10</td>
<td>15.00</td>
<td>85.00</td>
</tr>
<tr>
<td>-5</td>
<td>10.00</td>
<td>70.00</td>
</tr>
<tr>
<td>10</td>
<td>30.00</td>
<td>60.00</td>
</tr>
<tr>
<td>15</td>
<td>20.00</td>
<td>30.00</td>
</tr>
<tr>
<td>20</td>
<td>10.00</td>
<td>10.00</td>
</tr>
</tbody>
</table>

![Figure 5: Portfolio loss distribution](image1)

![Figure 6: Portfolio cumulative loss distribution](image2)

- The 0.95-VaR is the first point for which the cumulative exceeds 100-95 = 20
- The 0.70-VaR is the first point for which the cumulative exceeds 100-70 = 15

2.2 VaR – with continuous and discrete probability distribution

- We consider a portfolio of assets with random returns
- We denote the portfolio choice vector with $x$ and the random events by the vector $y$
- Let $f(x, y)$ denote the loss function when we choose the portfolio $x$ from a set $X$ of feasible portfolios and $y$ is the realization of the random events
- We assume that the random vector $y$ has a probability density function denoted by $p(y)$

For a fixed decision vector $x$, we compute the cumulative distribution function of the loss associated with that vector $x$: 
Continuous probability distribution:

\[ \Psi(x, y) = \int_{f(x,y) < \gamma} p(y) \, dy \]

Discrete probability distributions:

\[ \Psi(x, y) = \sum_{j : f(x,y) < \gamma} p_j \]

Then, for a given confidence level \( \alpha \), the \( \alpha \)-VaR associated with portfolio \( x \) is given as:

\[ \text{VaR}_\alpha(x) = \min\{\gamma \in \mathbb{R} : \Psi(x,y) \geq \alpha\} \]

### 2.3 Conditional Value-at-Risk – CVaR

We define the \( \alpha \)-CVaR associated with portfolio \( x \) as:

\[ \text{CVaR}_\alpha(x) = \frac{1}{1 - \alpha} \int_{f(x,y) \geq \text{VaR}_\alpha(x)} f(x, y)p(y) \, dy \]

For a discrete probability distribution (where event \( y_j \) occurs with probability \( p_j \), for \( j = (1, \ldots, n) \)), the above definition of CVaR becomes:

\[ \text{CVaR}_\alpha(x) = \frac{1}{1 - \alpha} \sum_{j : f(x,y) \geq \text{VaR}_\alpha(x)} p_j f(x, y_j) \]

Features:

- CVaR of a portfolio is always at least as big as its VaR
- Consequently, portfolios with small CVaR also have small VaR
- However, in general minimizing CVaR and VaR are not equivalent

Example:

- Suppose we are given the loss function \( f(x, y) \) for a given decision \( x \) as:
- \( f(x, y) = -y_j \) where \( y_j = 75 - j \) with probability 1% for \( j = 0, \ldots, 99 \)
- We would like to determine the maximum loss incurred with 95% probability
- This is the Value-at-Risk \( \text{VaR}_\alpha(x) \) for \( \alpha = 95\% \)
- We have \( \text{VaR}_{95\%}(x) = 20 \) since the loss is less than 20 with probability 95%
- To compute the Conditional Value-at-Risk, we use the above formula:
- \( \text{CVaR}_{95\%}(x) = \frac{1}{0.05} \times 1\% = 22 \)

Since the definition of CVaR involves the VaR function explicitly, it is difficult to work with and optimize this function. Instead, we consider the following simpler auxiliary function:

\[ F_\alpha(x, \gamma) = \gamma + \frac{1}{1 - \alpha} \int_{f(x,y) \geq \gamma} (f(x, y) - \gamma)p(y) \, dy \]

Alternatively, we can write

\[ F_\alpha(x, \gamma) = \gamma + \frac{1}{1 - \alpha} \int (f(x, y) - \gamma)^+ p(y) \, dy \]

This function, viewed as a function of \( \gamma \), has the following important properties that makes it useful for the computation of VaR and CVaR:
• $F_\alpha(x, \gamma)$ is a convex function of $\gamma$
• VaR$_\alpha$ (x) is a minimizer over $\gamma$ of $F_\alpha(x, \gamma)$
• The minimum value over $\gamma$ of the function $F_\alpha(x, \gamma)$ is CVaR$_\alpha$ (x)

As a consequence of the listed properties, we immediately deduce that, in order to minimize CVaR$_\alpha$ (x) over x, we need to minimize the function $F_\alpha(x, \gamma)$ with respect to x and $\gamma$ simultaneously

$$\min_{x \in X} CVaR_\alpha(x) = \min_{x \in X, \gamma} F_\alpha(x, \gamma)$$

• Consequently, we can optimize CVaR directly, without needing to compute VaR first
• If the loss function $f(x, y)$ is a convex (linear) function of the portfolio variables x, then $F_\alpha(x, \gamma)$ is also a convex (linear) function of x
• In this case, provided the feasible portfolio set X is also convex, the optimization problems above are smooth convex optimization problems that can be solved using well known optimization techniques for such problems.

Scenarios
• Often it is not possible or desirable to compute/determine the joint density function $p(y)$ of the random events in our formulation
• Instead, we may have a number of scenarios (with equal probability), say $y_s$ for $s = 1, \ldots, S$, which may represent some historical values of the random events or some values obtained via computer simulation.

In this case, we obtain the following approximation to the function:

$$\tilde{F}_\alpha(x, \gamma) = \gamma + \frac{1}{(1-\alpha)S} \sum_{s=1}^{S} (f(x, y_s) - \gamma)$$

Now, the problem:

$$\min_{x \in X} CVaR_\alpha(x) = \min_{x \in X, \gamma} \frac{1}{(1-\alpha)S} \sum_{s=1}^{S} (f(x, y_s) - \gamma)$$

To solve this optimization problem, we introduce artificial variables $z_s$ to replace $(f(x, y_s) - \gamma)$, then obtaining:

$$\min_{x, z, \gamma} \gamma + \frac{1}{(1-\alpha)S} \sum_{s=1}^{S} z_s$$

$$z_s \geq 0 \quad s = 1, \ldots, S$$

$$z_s \geq f(x, y_s) - \gamma \quad s = 1, \ldots, S$$

$$x \in X$$

In the case that $f(x, y)$ is linear in x, all the constraints are linear constraints and therefore the problem is a linear programming problem that can be solved using the simplex method or alternative LP algorithms.
3 Asset Pricing and Arbitrage

3.1 Pricing and Hedging of Options

We first start with a description of some of the well-known financial options.

**European call option:** is a contract with the following conditions

- At a prescribed time in the future, known as the *expiration date*, the holder of the option has the right, but not the obligation to purchase a prescribed asset, known as the *underlying*, for a prescribed amount, known as the *strike price* or *exercise price*.

**European put option:** is similar, except that it confers the right to sell the underlying asset.

**American option:** is like a European option, but it can be exercised anytime before the expiration date.

In order to define the price of the option we consider no-arbitrage opportunities:

**Definition** An *arbitrage* is a trading strategy:

- that has a positive initial cash flow and has no risk of a loss later (type A)
- that requires no initial cash input, has no risk of a loss, and a positive probability of making profits in the future (type B).

**How can we determine the price of the options?**

We start with an example and then we formalize the problem.

- To find the *fair value of an option*, we need to solve a *pricing problem*.
- Option pricing problems are often solved using the following strategy:
  - We try to determine a portfolio of assets with known prices which, if updated properly through time, will produce the same payoff (value) as the option.
  - Here a portfolio is the portfolio resulting from a decision of buying or selling stocks and/or selling or borrowing money.
  - Since the portfolio and the option will have the same eventual payoffs, we conclude that they must have the same value today (otherwise, there is arbitrage) and we can therefore obtain the price of the option.
  - A portfolio of other assets that produces the same payoff as a given financial instrument is called a *replicating portfolio* (or a *hedge*).
  - Finding the right portfolio, of course, is not always easy and leads to a replication (or hedging) problem.

Let us consider a simple example to illustrate these ideas:

- We know that one share of stock $\alpha$ is currently valued at €40
- Assume that the value of $\alpha$ will either double or halve with equal probabilities (random values):
- Today, we purchase a European call option to buy one share of stock $\alpha$ for €50 a month from today. What is the fair price of this option?
Assumptions

- we can borrow or lend money with no interest between today and next month
- we can buy or sell any amount of the $\alpha$ stock without any commissions
- assume that $\alpha$ will not pay any dividends within the next month.

To solve the option pricing problem, we consider the following hedging problem:

- Can we form a portfolio of the underlying stock (bought or sold) and cash (borrowed or lent) today, such that the payoff of the portfolio at the expiration date of the option will match the payoff of the option?
- Note that the option payoff will be $e^{30}$ if the price of the stock goes up and $e^{0}$ if it goes down.
- Assume this portfolio has $\Delta$ shares of $\alpha$ and $B$ cash.

This portfolio would be worth $40\Delta + B$ today. Next month, payoffs for this portfolio will be:

\[
\begin{align*}
P_0 &= 40\Delta + B \\
80\Delta + B &= P_1(u) \\
20\Delta + B &= P_1(d)
\end{align*}
\]

Let us choose $\Delta$ and $B$ such that:

\[
\begin{align*}
80\Delta + B &= 30 \\
20\Delta + B &= 0
\end{align*}
\]

so that the portfolio replicates the payoff of the option at the expiration date.

- This gives $\Delta = \frac{1}{2}$ and $B = -10$, which is the hedge we were looking for.
- This portfolio is worth $P_0 = 40\Delta + B = e^{10}$ today, therefore, the fair price of the option must also be $e^{10}$.

3.2 Replication Problem

In this Section we generalize the example of the previous paragraph.

- Let $S_0$ be the current price of the underlying security
- Assume that there are two possible outcomes at the end of the period:

\[
\begin{align*}
S_1^u &= S_0 \cdot u \\
S_1^d &= S_0 \cdot d
\end{align*}
\]

(Assume $u > d$.)
• We also assume that there is a fixed interest paid on cash borrowed or lent at rate \( r \) for the given period.

\[
R = 1 + r
\]

• We consider a derivative security which has payoffs of \( C^u_1 \) and \( C^d_1 \) in the up and down states respectively.

\[
\begin{align*}
S_0 & \quad S_1^u = S_0 \cdot u \\
& \quad S_1^d = S_0 \cdot d \\
C_0 & \quad ? \\
C^u_1 & \quad C^d_1
\end{align*}
\]

• The replication problem considers a portfolio of \( \Delta \) shares of the underlying and \( B \) cash.

• For what values of \( \Delta \) and \( B \) does this portfolio have the same payoffs at the expiration date as the derivative security?

• We need to solve the following system of 2 equations and 2 variables (\( \Delta \) and \( B \)):

\[
\begin{align*}
\Delta \cdot S_0 \cdot u + B \cdot R &= C^u_1 \\
\Delta \cdot S_0 \cdot d + B \cdot R &= C^d_1
\end{align*}
\]

• This portfolio is worth \( S_0 \cdot \Delta + B \) today, which corresponds to the fair price of the option as well:

\[
C_0 = \frac{C^u_1 - C^d_1}{u - d} + \frac{u \cdot C^d_1 - d \cdot C^u_1}{R(u - d)}
\]

\[
= \frac{1}{R} \left[ \frac{R - d}{u - d} C^u_1 + \frac{u - R}{u - d} C^d_1 \right]
\]

• In the same way we can compute the hedge

\[
\begin{align*}
\Delta &= \frac{C^u_1 \cdot R - C^d_1 \cdot R}{S_0 \cdot u \cdot R - S_0 \cdot d \cdot R} \\
B &= \frac{S_0 \cdot u \cdot C^d_1 - S_0 \cdot d \cdot C^u_1}{S_0 \cdot u \cdot R - S_0 \cdot d \cdot R}
\end{align*}
\]

**Risk-Neutral Probabilities**

Let us define:

\[
p_u = \frac{R - d}{u - d}
\]

and

\[
p_d = \frac{u - R}{u - d}
\]

• Note that we must have \( d < R < u \) to avoid arbitrage opportunities

• An immediate consequence of this observation is that both \( p_u > 0 \) and \( p_d > 0 \)

• Noting also that \( p_u + p_d = 1 \) one can interpret \( p_u \) and \( p_d \) as probabilities
• these are the so-called risk-neutral probabilities (RNPs) of up and down states, respectively
• Note that they are completely independent from the actual probabilities of these states.
• The price of any derivative security can now be calculated as the present value of the expected value of its future payoffs where the expected value is computed using the risk-neutral probabilities.

Example

In our example above $u = 2$, $d = \frac{1}{2}$ and $r = 0$ so that $R = 1$. Therefore:

\[ p_u = \frac{1}{3} \text{ and } p_d = \frac{2}{3} \]

As a result, we have

\[ S_0 = 40 = \frac{1}{R} (p_u \cdot S_u^1 + p_d \cdot S_d^1) = \frac{1}{3} \cdot 80 + \frac{2}{3} \cdot 20 \]
\[ C_0 = 10 = \frac{1}{R} (p_u \cdot C_u^1 + p_d \cdot C_d^1) = \frac{1}{3} \cdot 30 + \frac{2}{3} \cdot 0 \]

• Using risk neutral probabilities we can also price other derivative securities
• For example, consider a European put option a stock struck at 60€ (this is another way to say “with a strike price of 60€”) and with the same expiration date as the call of the example.

\[ P_0 = \frac{1}{R} (p_u \cdot P_u^1 + p_d \cdot P_d^1) = \frac{1}{3} \cdot 0 + \frac{2}{3} \cdot 20 = 80 \]

\[ P_0 = \begin{cases} P_u^1 = \max\{0, 60 - 80\} = 0 \\ P_d^1 = \max\{0, 60 - 20\} = 40 \end{cases} \]

• We can easily compute:

\[ P_0 = \frac{1}{R} (p_u \cdot P_u^1 + p_d \cdot P_d^1) = \frac{1}{3} \cdot 0 + \frac{2}{3} \cdot 20 = \frac{80}{3} \]

Generalization from a binomial setting to a more general setting of $m$ possible states

Let:

\[ \Omega = \{\omega_1, \omega_2, \ldots, \omega_m\} \]

be the (finite) set of possible future “states.” For example, these could be prices for a security at a future date.

For securities $S^i$ with $i = 1, \ldots, n$:

• let $S^i_1(\omega_j)$ denote the price of this security $i$ at time 1
• let $S^i_0$ denote the current (time 0) price of security $S^i$
• We use $i = 0$ for the “riskless” security that pays the interest rate $r \geq 0$ between time 0 and time 1
• It is convenient to assume that $S^i_0 = 1$ and that $S^i_1(\omega_j) = R = 1 + r, \forall j$. 

25
**Definition** A risk-neutral probability measure is a vector of positive numbers \((p_1, p_2, \ldots, p_m)\) such that:

\[
\sum_{j=1}^{m} p_j = 1
\]

and for every security \(S^i\), \(i = 0, \ldots, n,\):

\[
S^i_0 = \frac{1}{R} \left( \sum_{j=1}^{m} p_j S^i_j(\omega_j) \right)
\]

### 3.3 The Fundamental Theorem of Asset Pricing

In this section we state the first fundamental theorem of asset pricing and prove it for finite \(\Omega\) using the theory of linear programming.

**Theorem 3.1** (Fundamental Theorem of Asset Pricing). A risk-neutral probability measure exists if and only if there is no arbitrage.

**Proof.**  
- We assume that the state space \(\Omega\) is finite.
  - We assume without loss of generality that every state has a positive probability of occurring (since states that have no probability of occurring can be removed)
  - Given the current prices \(S^i_0\) and the future prices \(S^i_j(\omega_j)\) in each state \(\omega_j\), for securities \(0\) to \(n\), consider the following linear program with variables \(x_i\), for \(i = 0, \ldots, n\):

\[
\begin{align*}
\min & \sum_{i=0}^{n} S^i_0 x_i \\
\text{s.t.} & \sum_{i=0}^{n} S^i_j(\omega_j) x_i \geq 0 & j = 1, \ldots, m
\end{align*}
\]

- Note that type-A arbitrage corresponds to a feasible solution to this LP with a negative objective value
- Since \(x_i = 0\) \((i = 0, \ldots, n)\) is a feasible solution, the optimal objective value is always non-positive
- Furthermore, since all the constraints are homogeneous \((0\ \text{RHS})\), if there exists a feasible solution such that

\[
\sum_{i=0}^{n} S^i_0 x_i < 0
\]

(this corresponds to type-A arbitrage), the problem is unbounded.
- In other words, there is no type-A arbitrage if and only if the optimal objective value of this LP is 0

- Suppose that there is no type-A arbitrage.
- Then, there is no type-B arbitrage if and only if all constraints are tight for all optimal solutions of the previous LP since every state has a positive probability of occurring.
• Consider the dual of the LP:

\[
\max \sum_{j=1}^{m} 0 \cdot p_j \quad (57)
\]

\[
\sum_{j=1}^{m} S_i^j(\omega_j)p_j = S_i^0 \quad i = 0, \ldots, n 
\]  

\[
p_j \geq 0 \quad j = 1, \ldots, m \quad (59)
\]

• Since the dual has a constant (0) objective function, any dual feasible solution is also dual optimal.

• When there is no type-A arbitrage the LP has an optimal solution, and the Strong Duality Theorem indicates that the dual must have a feasible solution.

• If there is no type-B arbitrage also, the complementary slackness theorem indicates that there exists a feasible (and therefore optimal) dual solution \( p^* \) such that \( p > 0 \).

• From the dual constraint corresponding to \( i = 0 \), we have that

\[
\sum_{i=1}^{n} p_j^* = \frac{1}{R}
\]

• Multiplying \( p^* \) by \( R \) one obtains a risk-neutral probability distribution.

• Therefore, the "no arbitrage" assumption implies the existence of RNPs.

• The converse direction is proved in an identical manner.
3.4 Boolean expressions using binary variables

- A Boolean expression, also called propositional logic formula, is built from
  1) boolean-valued functions (also called proposition)
    * functions $f : X \rightarrow B$
    * $X$ is an arbitrary set
    * $B$ is a boolean domain $B \in \{\text{TRUE}, \text{FALSE}\}$
      - example: a function that returns TRUE if a number is even and FALSE otherwise, $X \equiv \mathbb{Z}_+$ and $B \in \{\text{TRUE}, \text{FALSE}\}$
    - a boolean-valued functions can be represented by a binary variable:
      $$x \in \{0, 1\}, \text{ if TRUE } \rightarrow x = 1, \text{ else (if FALSE) } \rightarrow x = 0$$
  2) boolean operators:
    * AND – conjunction – denoted also by $\land$
    * OR – disjunction – denoted also by $\lor$
      - the negation of a boolean-valued functions represented by a binary variable is $\text{NOT } f$, it corresponds to the binary variable $(1 - x)$
  3) parentheses

- A Boolean expression is said to be satisfiable if it can be made TRUE by assigning appropriate logical values (i.e., TRUE, FALSE) to its binary variables

- Example of Boolean expression:
  $$(x_a \text{ OR } x_b) \text{ AND } (x_c \text{ OR } x_d) \text{ AND } (\text{ NOT } x_e)$$

- Conjunction:
  $$x_a \text{ AND } x_b \Leftrightarrow \begin{cases} x_a \geq 1 \\ x_b \geq 1 \end{cases}$$

- Disjunction:
  $$x_a \text{ OR } x_b \Leftrightarrow x_a + x_b \geq 1$$

- Remarque : the boolean operator AND come first than OR, $x_a \text{ OR } x_b \text{ AND } x_c$ is equivalent to $x_a \text{ OR } (x_b \text{ AND } x_c)$

Conjunctive Normal Form (CNF)

- A Boolean expression is said to in Conjunctive Normal Form if it is expressed as conjunctions of disjunctions

- examples:
  - $x_a \text{ AND } x_b$
  - $(x_a \text{ OR } x_b) \text{ AND } x_c$

- Equivalences to get CNFs:
  - $\text{ NOT } (\text{ NOT } x_a) = x_a$
  - $\text{ NOT } (x_a \text{ OR } x_b) = (\text{ NOT } x_a) \text{ AND } (\text{ NOT } x_b)$
  - $\text{ NOT } (x_a \text{ AND } x_b) = (\text{ NOT } x_a) \text{ OR } (\text{ NOT } x_b)$
  - $x_a \text{ AND } (x_b \text{ OR } x_c) = (x_a \text{ AND } x_b) \text{ OR } (x_a \text{ AND } x_c)$
  - $x_a \text{ OR } (x_b \text{ AND } x_c) = (x_a \text{ OR } x_b) \text{ AND } (x_a \text{ OR } x_c)$
Transforming CNF into linear inequalities

- for each disjunction we have an inequality constraint \( \geq 1 \) (the most simple CNF is composed by only one disjunction)
- the left hand side of the inequality is written replacing:
  - OR with +
  - each \((\text{NOT } x_a)\) with \((1 - x_a)\)

Logical Implications between binary variables

- the logical implication \( x_a \Rightarrow x_b \) is expressed by the following boolean expression \((\text{NOT } x_a) \text{ OR } x_b\)
- this can be written as a linear constraints using the binary variables as follow:
  \[(1 - x_a) + x_b \geq 1 \quad (x_b \geq x_a)\]
- Attention that \( x_a \Rightarrow x_b \) implies also \( \text{NOT } x_b \Rightarrow \text{NOT } x_a = (\text{NOT } (\text{NOT } x_b)) \text{ OR } \text{NOT } x_a = (\text{NOT } x_a) \text{ OR } x_b\)

examples

(1) \((x_a \text{ AND } x_b) \Rightarrow x_c\)

\[
\text{NOT } (x_a \text{ AND } x_b) \text{ OR } x_c \\
(\text{NOT } x_a) \text{ OR } (\text{NOT } x_b) \text{ OR } x_c \\
(1 - x_a) + (1 - x_b) + x_c \geq 1 \\
(x_a + x_b - x_c \leq 1)
\]

(2) \((x_a \text{ OR } x_b) \Rightarrow x_c\)

\[
\text{NOT } (x_a \text{ OR } x_b) \text{ OR } x_c \\
((\text{NOT } x_a) \text{ AND } (\text{NOT } x_b)) \text{ OR } x_c \\
((\text{NOT } x_a) \text{ OR } (x_c)) \text{ AND } ((\text{NOT } x_b) \text{ OR } (x_c)) \\
\begin{cases} 
1 - x_a + x_c \geq 1 \\
1 - x_b + x_c \geq 1
\end{cases}
\]

(3) \(x_a \Rightarrow (x_b \text{ AND } x_c)\)

\[
\text{NOT } (x_a) \text{ OR } (x_b \text{ AND } x_c) \\
((\text{NOT } x_a) \text{ AND } (\text{NOT } x_b)) \text{ OR } x_c \\
((\text{NOT } x_a) \text{ OR } (x_b)) \text{ AND } ((\text{NOT } x_a) \text{ OR } (x_c)) \\
\begin{cases} 
1 - x_a + x_b \geq 1 \\
1 - x_a + x_c \geq 1
\end{cases}
\]

(4) \(x_a \Rightarrow (x_b \text{ OR } x_c)\)

\[
\text{NOT } (x_a) \text{ OR } (x_b \text{ OR } x_c) \\
1 - x_a + x_b + x_c \geq 1 \\
(x_b + x_c \geq x_a)
\]
(5) $x_a \text{ AND ( NOT } x_b) \text{ AND } x_c \Rightarrow \text{ NOT } x_d$

\begin{align*}
\text{NOT (} x_a \text{ AND ( NOT } x_b) \text{ AND } x_c \text{)} \text{ OR ( NOT } x_d) \\
\text{ ( NOT } x_a) \text{ OR } x_b \text{ OR ( NOT } x_c) \text{ OR ( NOT } x_d) \\
1 - x_a + x_b + 1 - x_c + 1 - x_d \geq 1 \\
(x_a - x_b + x_c + x_d \leq 2)
\end{align*}