Optimization in Finance

project for groups of maximum 3 students – due date: January 11th – sent your project to: fabio.furini@dauphine.fr

Construction of an Index Fund

Index fund – Definition  Given a target population of $n$ stocks, select $q$ stocks and their weights in the index fund (a portfolio), to represent the target population as closely as possible.

- consider the following ILP model aiming at aggregating a broad market index of $n$ securities into a representative index fund:

\[
\begin{align*}
\text{max} & \quad \sum_{i=1}^{n} \sum_{j=1}^{n} \alpha_{ij} x_{ij} \\
\sum_{j=1}^{n} y_{j} &= q \quad (2) \\
\sum_{j=1}^{n} x_{ij} &= 1 \quad i = 1, \ldots, n \quad (3) \\
x_{ij} &\leq y_{j} \quad i = 1, \ldots, n, j = 1, \ldots, n \quad (4) \\
x_{ij} &\in \{0, 1\} \quad i = 1, \ldots, n, j = 1, \ldots, n \quad (5) \\
y_{j} &\in \{0, 1\} \quad j = 1, \ldots, n \quad (6)
\end{align*}
\]

- where the decision variables represent:

\[
\begin{align*}
y_{j} &= \begin{cases} 
1 & \text{if stock } j \text{ is in the index fund} \\
0 & \text{otherwise}
\end{cases} \\
x_{ij} &= \begin{cases} 
1 & \text{if } j \text{ is the most similar stock to stock } i \text{ in the index fund} \\
0 & \text{otherwise}
\end{cases}
\end{align*}
\]

- $q$ represents the number of securities in the index fund and

\[
\alpha_{ij} = \text{similarity between stock } i \text{ and stock } j \quad (7)
\]
Case Study – Investigate the performance of the model

- Consider the securities and the historical data provided by email. Compute the covariance matrix (positive semidefinite)

\[ Q_{ij} = \frac{1}{T} \sum_{t=1}^{T} (r_{ti}^t - r_{ai}^t)(r_{tj}^t - r_{aj}^t) \]

correlation matrix

\[ \rho_{ij} = \frac{Q_{ij}}{\sigma_i \sigma_j} \]
of these securities.

- Set up the ILP model and solve it with CPLEX for different values of \( q \) using the covariance and the correlation matrix as a proxy of similarity between the stocks. Analyse which are the securities selected according to different values of \( q \) and similarity measure. Determine a portfolio using the selected stocks and justify your choice.

- Compare the portfolio performance constructed in period \( t \) (based on historical data up to period \( t \)) with the market performances by observing the portfolio performances in period \( t + 1 \) (repeat the analysis for different periods).

- Discuss the advantages and the disadvantages of the index funds constructed in this manner and propose possible solutions for the principal problems.

- Determine a good index fund and a portfolio analyzing the trade off between \( q \) (index funds with small \( q \) are in general preferable) and the performances.

Business game:

- Construct an index fund considering one of the national markets (S&P 500 index, NASDAQ index) using the previous ILP model. Present and motivate the adopted strategy.

- The dimension and the computational complexity of the model increase with the total number of stocks and the value of \( q \). Analyse the variation in the computational time according to the total number of stocks and the value of \( q \). Determine the maximum dimension of instances which can be solved to proven optimality (within an acceptable computing time of 10 minutes).

- Discuss the results as you were in a financial consulting lab trying to convince the management to adopt (or not) the proposed portfolios.