International Doctoral School Algorithmic Decision Theory: MCDA and MOO
Lecture 2: Multiobjective Linear Programming

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Overview

1. Multiobjective Linear Programming
   - Formulation and Example
   - Solving MOLPs by Weighted Sums

2. Biobjective LPs and Parametric Simplex
   - The Parametric Simplex Algorithm
   - Biobjective Linear Programmes: Example

3. Multiobjective Simplex Method
   - A Multiobjective Simplex Algorithm
   - Multiobjective Simplex: Examples
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   - The Parametric Simplex Algorithm
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3. Multiobjective Simplex Method
   - A Multiobjective Simplex Algorithm
   - Multiobjective Simplex: Examples
- Variables \( x \in \mathbb{R}^n \)
- Objective function \( Cx \) where \( C \in \mathbb{R}^{p \times n} \)
- Constraints \( Ax = b \) where \( A \in \mathbb{R}^{m \times n} \) and \( b \in \mathbb{R}^m \)

\[
\min \left\{ Cx : Ax = b, x \geq 0 \right\} \quad (1)
\]

\[
X = \{ x \in \mathbb{R}^n : Ax = b, x \geq 0 \}
\]

is the feasible set in decision space

\[
Y = \{ Cx : x \in X \}
\]

is the feasible set in objective space
Variables $x \in \mathbb{R}^n$

Objective function $Cx$ where $C \in \mathbb{R}^{p \times n}$

Constraints $Ax = b$ where $A \in \mathbb{R}^{m \times n}$ and $b \in \mathbb{R}^m$

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Example

\[
\min \begin{pmatrix}
3x_1 + x_2 \\
-x_1 - 2x_2
\end{pmatrix}
\]

subject to
\[
\begin{align*}
x_2 & \leq 3 \\
3x_1 - x_2 & \leq 6 \\
x & \geq 0
\end{align*}
\]

\[
C = \begin{pmatrix}
3 & 1 \\
-1 & -2
\end{pmatrix} \quad A = \begin{pmatrix}
0 & 1 & 1 & 0 \\
3 & -1 & 0 & 1
\end{pmatrix} \quad b = \begin{pmatrix}
3 \\
6
\end{pmatrix}
\]
Example

\[
\begin{align*}
\text{min} & \quad \left( \begin{array}{c}
3x_1 + x_2 \\
-x_1 - 2x_2
\end{array} \right) \\
\text{subject to} & \quad x_2 \leq 3 \\
& \quad 3x_1 - x_2 \leq 6 \\
& \quad x \geq 0
\end{align*}
\]

\[
C = \begin{pmatrix} 3 & 1 \\ -1 & -2 \end{pmatrix} \quad A = \begin{pmatrix} 0 & 1 & 1 & 0 \\ 3 & -1 & 0 & 1 \end{pmatrix} \quad b = \begin{pmatrix} 3 \\ 6 \end{pmatrix}
\]
Multiobjective Linear Programming
Biobjective LPs and Parametric Simplex
Multiobjective Simplex Method

Formulation and Example
Solving MOLPs by Weighted Sums

Feasible set in decision space

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Feasible set in objective space
Definition

Let $\hat{x} \in X$ be a feasible solution of the MOLP (1) and let $\hat{y} = C\hat{x}$.

- $\hat{x}$ is called **weakly efficient** if there is no $x \in X$ such that $Cx < C\hat{x}$; $\hat{y} = C\hat{x}$ is called **weakly nondominated**.

- $\hat{x}$ is called **efficient** if there is no $x \in X$ such that $Cx \leq C\hat{x}$; $\hat{y} = C\hat{x}$ is called **nondominated**.

- $\hat{x}$ is called **properly efficient** if it is efficient and if there exists a real number $M > 0$ such that for all $i$ and $x$ with $c_i^T x < c_i^T \hat{x}$ there is an index $j$ and $M > 0$ such that $c_j^T x > c_j^T \hat{x}$ and

$$\frac{c_i^T \hat{x} - c_i^T x}{c_j^T x - c_j^T \hat{x}} \leq M.$$
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Formulation and Example
Solving MOLPs by Weighted Sums

Nondominated set

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Let \( \lambda_1, \ldots, \lambda_p \geq 0 \) and consider

\[
LP(\lambda) \quad \min \sum_{k=1}^{p} \lambda_k c_k^T x = \min \lambda^T C x
\]

subject to

\[
Ax = b
\]

\[
x \geq 0
\]

with some vector \( \lambda \geq 0 \) (Why not \( \lambda = 0 \) or \( \lambda \leq 0 \)?)

- \( LP(\lambda) \) is a linear programme that can be solved by the Simplex method
- If \( \lambda > 0 \) then optimal solution of \( LP(\lambda) \) is properly efficient
- If \( \lambda \geq 0 \) then optimal solution of \( LP(\lambda) \) is weakly efficient
- Converse also true, because \( Y \) convex
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$$LP(\lambda) \quad \text{min} \sum_{k=1}^{p} \lambda_k c_k^T x = \min \lambda^T Cx$$

subject to $Ax = b$

$$x \geq 0$$

with some vector $\lambda \geq 0$ (Why not $\lambda = 0$ or $\lambda \leq 0$?)

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Illustration in objective space

\[ \lambda^1 = (2, 1), \lambda^2 = (1, 3), \lambda^3 = (1, 1) \]
Illustration in objective space

\[
\lambda^1 = (2, 1), \quad \lambda^2 = (1, 3), \quad \lambda^3 = (1, 1)
\]
\[ y \in \mathbb{R}^p \text{ satisfying } \lambda^T y = \alpha \text{ define a straight line (hyperplane)} \]

- Since \( y = Cx \) and \( \lambda^T Cx \) is minimised, we push the line towards the origin (left and down).
- When the line only touches \( Y \) nondominated points are found.
- Nondominated points \( Y_N \) are on the boundary of \( Y \).
- \( Y \) is convex polyhedron and has finite number of facets. \( Y_N \) consists of finitely many facets of \( Y \). The normal of the facet can serve as weight vector \( \lambda \).
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- $y \in \mathbb{R}^p$ satisfying $\lambda^T y = \alpha$ define a straight line (hyperplane)
- Since $y = Cx$ and $\lambda^T Cx$ is minimised, we push the line towards the origin (left and down)
- When the line only touches $Y$ nondominated points are found
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\begin{itemize}
  \item $y \in \mathbb{R}^p$ satisfying $\lambda^T y = \alpha$ define a straight line (hyperplane)
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  \item $Y$ is convex polyhedron and has finite number of facets. $Y_N$ consists of finitely many facets of $Y$. The normal of the facet can serve as weight vector $\lambda$
\end{itemize}
Question: Can all efficient solutions be found using weighted sums? If \( \hat{x} \in X \) is efficient, does there exist \( \lambda > 0 \) such that \( \hat{x} \) is optimal solution to

\[
\min \{ \lambda^T Cx : Ax = b, x \geq 0 \}
\]

Lemma

A feasible solution \( x^0 \in X \) is efficient if and only if the linear programme

\[
\begin{align*}
\max & \quad e^T z \\
\text{subject to} & \quad Ax = b \\
& \quad Cx + lz = Cx^0 \\
& \quad x, z \geq 0,
\end{align*}
\]

where \( e^T = (1, \ldots, 1) \in \mathbb{R}^p \) and \( l \) is the \( p \times p \) identity matrix, has an optimal solution \((\hat{x}, \hat{z})\) with \( \hat{z} = 0 \).
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**Lemma**

A feasible solution $x^0 \in X$ is efficient if and only if the linear programme

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& \quad Cx + Iz = Cx^0 \\
& \quad x, z \geq 0,
\end{align*}$$

where $e^T = (1, \ldots, 1) \in \mathbb{R}^p$ and $I$ is the $p \times p$ identity matrix, has an optimal solution $(\hat{x}, \hat{z})$ with $\hat{z} = 0$. 
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(2)

where \( e^T = (1, \ldots, 1) \in \mathbb{R}^p \) and \( I \) is the \( p \times p \) identity matrix, has an optimal solution \( (\hat{x}, \hat{z}) \) with \( \hat{z} = 0 \).
Proof.

- LP is always feasible with $x = x^0$, $z = 0$ (and value 0)
  - Let $(\hat{x}, \hat{z})$ be optimal solution
  - If $\hat{z} = 0$ then $\hat{z} = Cx^0 - C\hat{x} = 0 \Rightarrow Cx^0 = C\hat{x}$
  - There is no $x \in X$ such that $Cx \leq Cx^0$ because $(x, Cx^0 - Cx)$ would be better solution $\Rightarrow x^0$ efficient
  - If $\hat{x}^0$ efficient there is no $x \in X$ with $Cx \leq Cx^0$
  - $\Rightarrow$ there is no $z$ with $z = Cx^0 - Cx \geq 0$
  - $\Rightarrow \max e^T z \leq 0 \Rightarrow \max e^T z = 0$
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- If \( \hat{z} = 0 \) then \( \hat{z} = Cx^0 - C\hat{x} = 0 \Rightarrow Cx^0 = C\hat{x} \)
- There is no \( x \in X \) such that \( Cx \leq Cx^0 \) because \((x, Cx^0 - Cx)\)
  would be better solution \( \Rightarrow x^0 \) efficient
- If \( \hat{x}^0 \) efficient there is no \( x \in X \) with \( Cx \leq Cx^0 \)
  \( \Rightarrow \) there is no \( z \) with \( z = Cx^0 - Cx \geq 0 \)
  \( \Rightarrow \) max \( e^T z \leq 0 \) \( \Rightarrow \) max \( e^T z = 0 \)
Proof.

- LP is always feasible with \( x = x^0, z = 0 \) (and value 0)
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Proof.

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- If \(\hat{z} = 0\) then \(\hat{z} = Cx^0 - C\hat{x} = 0 \Rightarrow Cx^0 = C\hat{x}\)
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- If \(\hat{x}^0\) efficient there is no \(x \in X\) with \(Cx \leq Cx^0\)
- \(\Rightarrow\) there is no \(z\) with \(z = Cx^0 - Cx \geq 0\)
- \(\Rightarrow \max e^T z \leq 0 \Rightarrow \max e^T z = 0\)
Lemma

A feasible solution \( x^0 \in X \) is efficient if and only if the linear programme

\[
\begin{align*}
\min & \quad u^T b + w^T C x^0 \\
\text{subject to} & \quad u^T A + w^T C \geq 0 \\
& \quad w \geq e \\
& \quad u \in \mathbb{R}^m
\end{align*}
\]

(3)

has an optimal solution \((\hat{u}, \hat{w})\) with \( \hat{u}^T b + \hat{w}^T C x^0 = 0 \).

Proof.

The LP (3) is the dual of the LP (2) \( \square \)
**Lemma**

A feasible solution \( x^0 \in X \) is efficient if and only if the linear programme

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**Proof.**

The LP (3) is the dual of the LP (2). □
Theorem

A feasible solution \( x^0 \in X \) is an efficient solution of the MOLP (1) if and only if there exists a \( \lambda \in \mathbb{R}^p \) such that

\[
\lambda^T C x^0 \leq \lambda^T C x
\]

for all \( x \in X \).

Note: We already know that optimal solutions of weighted sum problems are efficient.
Theorem

A feasible solution $x^0 \in X$ is an efficient solution of the MOLP (1) if and only if there exists a $\lambda \in \mathbb{R}^p_+$ such that

$$\lambda^T C x^0 \leq \lambda^T C x$$

for all $x \in X$.

Note: We already know that optimal solutions of weighted sum problems are efficient.
Proof.

- Let $x^0 \in X_E$
  
  - By Lemma 4 LP (3) has an optimal solution $(\hat{u}, \hat{w})$ such that

    $$\hat{u}^T b = -\hat{w}^T Cx^0 \quad (5)$$

  - $\hat{u}$ is also an optimal solution of the LP

    $$\min \left\{ u^T b : u^T A \geq -\hat{w}^T C \right\}, \quad (6)$$

    which is (3) with $w = \hat{w}$ fixed

  - $\Rightarrow$ There is an optimal solution of the dual of (6)

    $$\max \left\{ -\hat{w}^T Cx : Ax = b, \ x \geq 0 \right\} \quad (7)$$
Proof.

- Let \( x^0 \in X_E \)

- By Lemma 4 LP (3) has an optimal solution (\( \hat{u}, \hat{w} \)) such that

\[ \hat{u}^T b = -\hat{w}^T C x^0 \quad (5) \]

- \( \hat{u} \) is also an optimal solution of the LP

\[ \min \left\{ u^T b : u^T A \geq -\hat{w}^T C \right\} , \quad (6) \]

which is (3) with \( w = \hat{w} \) fixed

- \( \Rightarrow \) There is an optimal solution of the dual of (6)

\[ \max \left\{ -\hat{w}^T C x : Ax = b, \ x \geq 0 \right\} \quad (7) \]
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  \[
  \hat{u}^T b = -\hat{w}^T C x^0
  \]  
  (5)
- \( \hat{u} \) is also an optimal solution of the LP
  \[
  \min \left\{ u^T b : u^T A \geq -\hat{w}^T C \right\},
  \]  
  (6)
  which is (3) with \( w = \hat{w} \) fixed
- \( \Rightarrow \) There is an optimal solution of the dual of (6)
  \[
  \max \left\{ -\hat{w}^T C x : Ax = b, \ x \geq 0 \right\}
  \]  
  (7)
Proof.

- By weak duality $u^T b \geq -\hat{w}^T C x$ for all feasible solutions $u$ of (6) and for all feasible solutions $x$ of (7).
- We already know that $\hat{u}^T b = -\hat{w}^T C x^0$ from (5).
- $\Rightarrow x^0$ is an optimal solution of (7).
- Note that (7) is equivalent to

$$\min \left\{ \hat{w}^T C x : A x = b, \quad x \geq 0 \right\}$$

with $\hat{w} \geq e > 0$ from the constraints in (3).
Proof.

- By weak duality $u^T b \geq -\hat{w}^T Cx$ for all feasible solutions $u$ of (6) and for all feasible solutions $x$ of (7).
- We already know that $\hat{u}^T b = -\hat{w}^T Cx^0$ from (5).
- ⇒ $x^0$ is an optimal solution of (7).
- Note that (7) is equivalent to

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Overview

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   - Formulation and Example
   - Solving MOLPs by Weighted Sums

2. Biobjective LPs and Parametric Simplex
   - The Parametric Simplex Algorithm
   - Biobjective Linear Programmes: Example

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   - A Multiobjective Simplex Algorithm
   - Multiobjective Simplex: Examples
• Modification of the **Simplex algorithm** for LPs with two objectives

\[
\begin{align*}
\text{min} & \quad ((c^1)^T x, (c^2)^T x) \\
\text{subject to} & \quad A x = b \\
& \quad x \geq 0
\end{align*}
\]  

(8)

• We can find all efficient solutions by solving the parametric LP

\[
\begin{align*}
\text{min} \left\{ \lambda_1 (c^1)^T x + \lambda_2 (c^2)^T x : A x = b, x \geq 0 \right\}
\end{align*}
\]

for all \( \lambda = (\lambda_1, \lambda_2) > 0 \)
Modification of the *Simplex algorithm* for LPs with two objectives

$$\min \quad ((c^1)^T x, (c^2)^T x)$$

subject to

$$\begin{align*}
Ax &= b \\ x &\geq 0
\end{align*}$$

We can find all efficient solutions by solving the parametric LP

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We can divide the objective by $\lambda_1 + \lambda_2$ without changing the optima, i.e. $\lambda_1' = \lambda_1/(\lambda_1 + \lambda_2)$, $\lambda_2' = \lambda_2/(\lambda_1 + \lambda_2)$ and $\lambda_1' + \lambda_2' = 1$ or 

$$\lambda_2' = 1 - \lambda_1'$$

LPs with one parameter $0 \leq \lambda \leq 1$ and parametric objective 

$$c(\lambda) := \lambda c^1 + (1 - \lambda)c^2$$

$$\min \left\{ c(\lambda)^T x : Ax = b, x \geq 0 \right\}$$  \hspace{1cm} (9)
We can divide the objective by $\lambda_1 + \lambda_2$ without changing the optima, i.e. $\lambda_1' = \lambda_1 / (\lambda_1 + \lambda_2)$, $\lambda_2' = \lambda_2 / (\lambda_1 + \lambda_2)$ and $\lambda_1' + \lambda_2' = 1$ or $\lambda_2' = 1 - \lambda_1'$.

LPs with one parameter $0 \leq \lambda \leq 1$ and parametric objective

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Let $\mathcal{B}$ be a feasible basis

- Recall reduced cost $\bar{c}_N = c_N - c_B^T B^{-1} N$
- Reduced cost for the parametric LP

$$\bar{c}(\lambda) = \lambda \bar{c}^1 + (1 - \lambda) \bar{c}^2$$

- Suppose $\hat{\mathcal{B}}$ is an optimal basis of (9) for some $\hat{\lambda}$
- $\bar{c}(\hat{\lambda}) \geq 0$
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Case 1: $\bar{c}^2 \geq 0$

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Case 2: There is at least one $i \in \mathcal{N}$ with $\bar{c}^2_i < 0$

- $\Rightarrow$ there is $\lambda < \hat{\lambda}$ such that $\bar{c}(\lambda)_i = 0$
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- Below this value $\hat{B}$ is not optimal
\[ \mathcal{I} = \{ i \in \mathcal{N} : \bar{c}_i^2 < 0, \bar{c}_i^1 \geq 0 \} \]

\[ \lambda' := \max_{i \in \mathcal{I}} \frac{-\bar{c}_i^2}{\bar{c}_i^1 - \bar{c}_i^2}. \] (11)

\( \hat{B} \) is optimal for all \( \lambda \in [\lambda', \hat{\lambda}] \)

As soon as \( \lambda < \lambda' \) new bases become optimal

Entering variable \( x_s \) has to be chosen where the maximum in (11) is attained for \( i = s \)
\[ \mathcal{I} = \{ i \in \mathcal{N} : \bar{c}_i^2 < 0, \bar{c}_i^1 \geq 0 \} \]

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As soon as \( \lambda < \lambda' \) new bases become optimal

Entering variable \( x_s \) has to be chosen where the maximum in (11) is attained for \( i = s \)
\[ I = \{ i \in N : \bar{c}_i^2 < 0, \bar{c}_i^1 \geq 0 \} \]

\[ \lambda' := \max_{i \in I} \frac{-\bar{c}_i^2}{\bar{c}_i^1 - \bar{c}_i^2}. \]  

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Entering variable \( x_s \) has to be chosen where the maximum in (11) is attained for \( i = s \)
Algorithm (Parametric Simplex for biobjective LPs)

**Input:** Data $A$, $b$, $C$ for a biobjective LP.

**Phase I:** Solve the auxiliary LP for Phase I using the Simplex algorithm. If the optimal value is positive, STOP, $X = \emptyset$. Otherwise let $B$ be an optimal basis.

**Phase II:** Solve the LP (9) for $\lambda = 1$ starting from basis $B$ found in Phase I yielding an optimal basis $\hat{B}$. Compute $\tilde{A}$ and $\tilde{b}$.

**Phase III:** While $I = \{i \in N : \bar{c}^2_i < 0, \bar{c}^1_i \geq 0\} \neq \emptyset$.

$$\lambda := \max_{i \in I} \frac{-\bar{c}^2_i}{\bar{c}^1_i - \bar{c}^2_i}.$$  

$$s \in \text{argmax} \left\{ i \in I : \frac{-\bar{c}^2_i}{\bar{c}^1_i - \bar{c}^2_i} \right\}.$$  

$$r \in \text{argmin} \left\{ j \in B : \frac{\tilde{b}_j}{\tilde{A}_{js}}, \tilde{A}_{js} > 0 \right\}.$$  

Let $B := (B \setminus \{r\}) \cup \{s\}$ and update $\tilde{A}$ and $\tilde{b}$.

**End while.**

**Output:** Sequence of $\lambda$-values and sequence of optimal BFSs.
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Example

\[
\begin{align*}
\text{min} & \quad \begin{pmatrix}
3x_1 + x_2 \\
-x_1 - 2x_2
\end{pmatrix} \\
\text{subject to} & \quad \begin{align*}
x_2 & \leq 3 \\
3x_1 - x_2 & \leq 6 \\
x & \geq 0
\end{align*}
\end{align*}
\]

\[LP(\lambda)\]

\[
\begin{align*}
\text{min} & \quad (4\lambda - 1)x_1 + (3\lambda - 2)x_2 \\
\text{subject to} & \quad \begin{align*}
x_2 + x_3 & = 3 \\
3x_1 - x_2 + x_4 & = 6 \\
x & \geq 0
\end{align*}
\end{align*}
\]
Use Simplex tableaus showing reduced cost vectors $\bar{c}^1$ and $\bar{c}^2$

- Optimal basis for $\lambda = 1$ is $B = \{3, 4\}$, optimal basic feasible solution $x = (0, 0, 3, 6)$
- Start with Phase 3
• Use Simplex tableaus showing reduced cost vectors $\bar{c}^1$ and $\bar{c}^2$

• Optimal basis for $\lambda = 1$ is $B = \{3, 4\}$, optimal basic feasible solution $x = (0, 0, 3, 6)$

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Use Simplex tableaus showing reduced cost vectors $\bar{c}^1$ and $\bar{c}^2$

Optimal basis for $\lambda = 1$ is $B = \{3, 4\}$, optimal basic feasible solution $x = (0, 0, 3, 6)$

Start with Phase 3
Iteration 1:

| $\bar{c}^1$ | 3 | 1 | 0 | 0 | 0 |
| $\bar{c}^2$ | -1 | -2 | 0 | 0 | 0 |
| $x_3$ | 0 | 1 | 1 | 0 | 3 |
| $x_4$ | 3 | -1 | 0 | 1 | 6 |

$\lambda = 1, \bar{c}(\lambda) = (3, 1, 0, 0), \mathcal{B}^1 = \{3, 4\}, x^1 = (0, 0, 3, 6)$

$I = \{1, 2\}, \lambda' = \max \left\{ \frac{1}{3+1}, \frac{2}{1+2} \right\} = \frac{2}{3}$

$s = 2, r = 3$
Iteration 2

<table>
<thead>
<tr>
<th>$\bar{c}^1$</th>
<th>3</th>
<th>0</th>
<th>-1</th>
<th>0</th>
<th>-3</th>
</tr>
</thead>
<tbody>
<tr>
<td>$\bar{c}^2$</td>
<td>-1</td>
<td>0</td>
<td>2</td>
<td>0</td>
<td>6</td>
</tr>
<tr>
<td>$x^2$</td>
<td>0</td>
<td>1</td>
<td>1</td>
<td>0</td>
<td>3</td>
</tr>
<tr>
<td>$x^4$</td>
<td>3</td>
<td>0</td>
<td>1</td>
<td>1</td>
<td>9</td>
</tr>
</tbody>
</table>

$\lambda = 2/3$, $\bar{c}(\lambda) = (5/3, 0, 0, 0)$, $B^2 = \{2, 4\}$, $x^2 = (0, 3, 0, 9)$

$I = \{1\}$, $\lambda' = \max \left\{ \frac{1}{3+1} \right\} = \frac{1}{4}$

$s = 1$, $r = 4$
Iteration 3

<table>
<thead>
<tr>
<th>$\bar{c}^1$</th>
<th>0</th>
<th>0</th>
<th>-2</th>
<th>-1</th>
<th>-12</th>
</tr>
</thead>
<tbody>
<tr>
<td>$\bar{c}^2$</td>
<td>0</td>
<td>0</td>
<td>7/3</td>
<td>1/3</td>
<td>9</td>
</tr>
<tr>
<td>$x_2$</td>
<td>0</td>
<td>1</td>
<td>1</td>
<td>0</td>
<td>3</td>
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<td>0</td>
<td>1/3</td>
<td>1/3</td>
<td>3</td>
</tr>
</tbody>
</table>

$\lambda = 1/4, \bar{c}(\lambda) = (0, 0, 5/4, 0), B^3 = \{1, 2\}, x^3 = (3, 3, 0, 0), I = \emptyset$
Weight values $\lambda^1 = 1, \lambda^2 = 2/3, \lambda^3 = 1/4, \lambda^4 = 0$

Basic feasible solutions $x^1, x^2, x^3$

In each iteration $\bar{c}(\lambda)$ can be calculated with the previous and current $\bar{c}^1$ and $\bar{c}^2$.

Basis $B^1 = (3, 4)$ and BFS $x^1 = (0, 0, 3, 6)$ are optimal for $\lambda \in [2/3, 1]$.

Basis $B^2 = (2, 4)$ and BFS $x^2 = (0, 3, 0, 9)$ are optimal for $\lambda \in [1/4, 2/3]$, and

Basis $B^3 = (1, 2)$ and BFS $x^3 = (3, 3, 0, 0)$ are optimal for $\lambda \in [0, 1/4]$.

Objective vectors for basic feasible solutions: $Cx^1 = (0, 0)$, $Cx^2 = (3, -6)$, and $Cx^3 = (12, -9)$
Weight values $\lambda^1 = 1, \lambda^2 = 2/3, \lambda^3 = 1/4, \lambda^4 = 0$

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Weight values $\lambda^1 = 1, \lambda^2 = 2/3, \lambda^3 = 1/4, \lambda^4 = 0$

Basic feasible solutions $x^1, x^2, x^3$

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- Basis $B^1 = (3, 4)$ and BFS $x^1 = (0, 0, 3, 6)$ are optimal for $\lambda \in [2/3, 1]$.
- Basis $B^2 = (2, 4)$ and BFS $x^2 = (0, 3, 0, 9)$ are optimal for $\lambda \in [1/4, 2/3]$, and
- Basis $B^3 = (1, 2)$ and BFS $x^3 = (3, 3, 0, 0)$ are optimal for $\lambda \in [0, 1/4]$.

Objective vectors for basic feasible solutions: $Cx^1 = (0, 0), Cx^2 = (3, -6)$, and $Cx^3 = (12, -9)$
- Weight values $\lambda^1 = 1$, $\lambda^2 = 2/3$, $\lambda^3 = 1/4$, $\lambda^4 = 0$
- Basic feasible solutions $x^1, x^2, x^3$
- In each iteration $\bar{c}(\lambda)$ can be calculated with the previous and current $\bar{c}^1$ and $\bar{c}^2$.
- Basis $B^1 = (3, 4)$ and BFS $x^1 = (0, 0, 3, 6)$ are optimal for $\lambda \in [2/3, 1]$.
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Contour lines for weighted sum objectives in decision are parallel to efficient edges.

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\frac{2}{3}(3x_1 + x_2) + \frac{1}{3}(-x_1 - 2x_2) &= \frac{5}{3}x_1 \\
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Feasible set in decision space and efficient set
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Feasible set in decision space and efficient set
Weight vectors \((2/3, 1/3)\) and \((1/4, 3/4)\) are normal to nondominated edges.

Objective space and nondominated set.
Algorithm finds **all nondominated extreme points** in objective space and **one efficient bfs** for each of those

Algorithm **does not find all efficient solutions** just as Simplex algorithm does not find all optimal solutions of an LP

**Example**

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\begin{align*}
\min & \quad (x_1, x_2)^T \\
\text{subject to} & \quad 0 \leq x_i \leq 1 \quad i = 1, 2, 3 \\
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Efficient set: \( \{ x \in \mathbb{R}^3 : x_1 = x_2 = 0, 0 \leq x_3 \leq 1 \} \)
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Overview

1. Multiobjective Linear Programming
   - Formulation and Example
   - Solving MOLPs by Weighted Sums

2. Biobjective LPs and Parametric Simplex
   - The Parametric Simplex Algorithm
   - Biobjective Linear Programmes: Example

3. Multiobjective Simplex Method
   - A Multiobjective Simplex Algorithm
   - Multiobjective Simplex: Examples
\( \min \{ Cx : Ax = b, x \geq 0 \} \)

- Let \( B \) be a basis and \( \bar{C} = C - C_B A_B^{-1} A \) and \( R = \bar{C}_N \)
- How to calculate “critical” \( \lambda \) if \( p > 2 \)?

- At \( B_1 : \bar{C}_N = \begin{pmatrix} 3 & 1 \\ -1 & -2 \end{pmatrix} \), \( \lambda' = 2/3 \), \( \lambda = (2/3, 1/3)^T \) and \( \lambda^T \bar{C}_N = (5/3, 0)^T \)
- At \( B_2 : \bar{C}_N = \begin{pmatrix} 3 & -1 \\ -1 & 2 \end{pmatrix} \), \( \lambda' = 1/4 \), \( \lambda = (1/4, 3/4)^T \) and \( \lambda^T \bar{C}_N = (0, 5/4)^T \)
- Find \( \lambda \in \mathbb{R}^p, \lambda > 0 \) such that \( \lambda^T R \geq 0 \) (optimality) and \( \lambda^T r^j = 0 \) (alternative optimum) for some column \( r^j \) of \( R \)
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\begin{itemize}
  \item \( \min \{ Cx : Ax = b, \ x \geq 0 \} \)
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Lemma

If $\mathcal{X}_E \neq \emptyset$ then $\mathcal{X}$ has an efficient basic feasible solution.

Proof.

- There is some $\lambda > 0$ such that $\min_{x \in \mathcal{X}} \lambda^T C x$ has an optimal solution.
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Definition

1. A feasible basis $\mathcal{B}$ is called **efficient basis** if $\mathcal{B}$ is an optimal basis of $\text{LP}(\lambda)$ for some $\lambda \in \mathbb{R}^p$.

2. Two bases $\mathcal{B}$ and $\hat{\mathcal{B}}$ are called **adjacent** if one can be obtained from the other by a single pivot step.

3. Let $\mathcal{B}$ be an efficient basis. Variable $x_j$, $j \in \mathcal{N}$ is called **efficient nonbasic variable** at $\mathcal{B}$ if there exists a $\lambda \in \mathbb{R}^p$ such that $\lambda^T R \geq 0$ and $\lambda^T r^j = 0$, where $r^j$ is the column of $R$ corresponding to variable $x_j$.

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It is not possible to define efficient nonbasic variables by the existence of a column in $R$ with positive and negative entries.

Example

$$ R = \begin{pmatrix} 3 & -2 \\ -2 & 1 \end{pmatrix} $$

- $\lambda^T r^2 = 0$ requires $\lambda_2 = 2\lambda_1$
- $\lambda^T r^1 \geq 0$ requires $-\lambda_1 \geq 0$, an impossibility for $\lambda > 0$
Lemma

Let $\mathcal{B}$ be an efficient basis and $x_j$ be an efficient nonbasic variable. Then any efficient pivot from $\mathcal{B}$ leads to an adjacent efficient basis $\hat{\mathcal{B}}$.

Proof.

- $x_j$ efficient entering variable at basis $\mathcal{B}$
- $\Rightarrow$ there is $\lambda \in \mathbb{R}^p_+$ with $\lambda^T R \geq 0$ and $\lambda^T r_j = 0$
- $\Rightarrow x_j$ is nonbasic variable with reduced cost 0 in LP($\lambda$)
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How to identify efficient nonbasic variables?

**Theorem**

Let $B$ be an efficient basis and let $x_j$ be a nonbasic variable. Variable $x_j$ is an efficient nonbasic variable if and only if the LP

$$\begin{align*}
\text{max} & \quad e^t v \\
\text{subject to} & \quad Rz - r^j \delta + lv = 0 \\
& \quad z, \delta, v \geq 0
\end{align*}$$

has an optimal value of 0.

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\text{min} \quad & 0^T \lambda = 0 \\
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- (13) is equivalent to

\[
\begin{cases}
\min & 0 \,^T \lambda = 0 \\
\text{subject to} & R \,^T \lambda \geq 0 \\
& - (r^j)^T \lambda \geq 0 \\
& I \lambda \geq e
\end{cases}
\]  

(14)
Proof.

The dual of (14) is

\[
\begin{align*}
\max & \quad e^T v \\
\text{subject to} & \quad Rz - r^j \delta + l v = 0 \\
& \quad z, \delta, v \geq 0.
\end{align*}
\] (15)
Need to show: ALL efficient bases can be reached by efficient pivots

Definition

Two efficient bases $\mathcal{B}$ and $\hat{\mathcal{B}}$ are called connected if one can be obtained from the other by performing only efficient pivots.

Theorem

All efficient bases are connected.
Need to show: ALL efficient bases can be reached by efficient pivots

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Two efficient bases $\mathcal{B}$ and $\hat{\mathcal{B}}$ are called connected if one can be obtained from the other by performing only efficient pivots.

**Theorem**

All efficient bases are connected.
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Two efficient bases $\mathcal{B}$ and $\mathcal{B}'$ are called *connected* if one can be obtained from the other by performing only efficient pivots.

**Theorem**

*All efficient bases are connected.*
Proof.

- $B$ and $\hat{B}$ two efficient bases
- $\lambda, \hat{\lambda} \in \mathbb{R}^p$ such that $B$ and $\hat{B}$ are optimal bases for LP($\lambda$) and LP($\hat{\lambda}$)
- Parametric LP ($\Phi \in [0, 1]$) with objective function
  \[
  c(\Phi) = \Phi \hat{\lambda}^T C + (1 - \Phi)\lambda^T C
  \]  
  (16)
- Assume $\hat{B}$ is first basis (for $\Phi = 1$)
- After several pivots get an optimal basis $\tilde{B}$ for LP($\lambda$)
- Since $\lambda^* = \Phi \hat{\lambda} + (1 - \Phi)\lambda \in \mathbb{R}^p$ for all $\Phi \in [0, 1]$ all bases are optimal for LP($\lambda^*$) for some $\lambda^* \in \mathbb{R}^p$, i.e. efficient
- If $\tilde{B} = B$, done
- Otherwise obtain $B$ from $\tilde{B}$ by efficient pivots: they are alternative optima for LP($\lambda$)
Proof.

- \( B \) and \( \hat{B} \) two efficient bases
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\]

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3 cases
- $\mathcal{X} = \emptyset$, infeasibility
- $\mathcal{X} \neq \emptyset$ but $\mathcal{X}_E = \emptyset$, no efficient solutions
- $\mathcal{X} \neq \emptyset, \mathcal{X}_E \neq \emptyset$

result in three phase multiobjective Simplex algorithm

- Phase I: Solve $\min \{e^T z : Ax + lz = b, x \geq 0, z \geq 0\}$
  If optimal value is nonzero, $\mathcal{X} = \emptyset$
  Otherwise find bfs of $Ax = b, x \geq 0$ from optimal solution

- Phase II: Find efficient bfs by solving appropriate LP($\lambda$)
  Note: LP($\lambda$) can be unbounded even if $\mathcal{X}_E \neq \emptyset$
  Solve $\min \{u^T b + w^T Cx^0 : u^T A + w^T C \geq 0, w \geq e\}$
  If unbounded then $\mathcal{X}_E = \emptyset$
  Otherwise find optimal $\hat{w}$ and solve
  $\min \{\hat{w} Cx : Ax = b, x \geq 0\}$
  Optimal bfs $x^1$ exists and is efficient bfs for MOLP

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result in three phase multiobjective Simplex algorithm

- **Phase I:** Solve $\min \{ e^T z : Ax + Iz = b, x \geq 0, z \geq 0 \}$
  
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  Otherwise find bfs of $Ax = b, x \geq 0$ from optimal solution

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  Note: $LP(\lambda)$ can be unbounded even if $\mathcal{X}_E \neq \emptyset$
  
  Solve $\min \{ u^T b + w^T Cx^0 : u^T A + w^T C \geq 0, w \geq e \}$
  
  If unbounded then $\mathcal{X}_E = \emptyset$
  
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Algorithm (Multicriteria Simplex Algorithm.)

Input: Data $A$, $b$, $C$ of an MOLP.

Initialization: Set $L_1 := \emptyset$, $L_2 := \emptyset$.

Phase I: Solve the LP $\min \{ e^T z : Ax + Iz = b, x, z \geq 0 \}$.  
If the optimal value of this LP is nonzero, STOP, $X = \emptyset$.  
Otherwise let $x^0$ be a basic feasible solution of the MOLP.

Phase II: Solve the LP 
$\min \{ u^T b + w^T Cx^0 : u^T A + w^T C \geq 0, w \geq e \}$. 
If the problem is infeasible, STOP, $X_E = \emptyset$.  
Otherwise let $(\hat{u}, \hat{w})$ be an optimal solution.  
Find an optimal basis $B$ of the LP $\min \{ \hat{w}^T Cx : Ax = b, x \geq 0 \}$.  
$L_1 := \{ B \}$, $L_2 := \emptyset$. 
Algorithm

Phase III:

While $L_1 \neq \emptyset$

Choose $B$ in $L_1$, set $L_1 := L_1 \setminus \{B\}$, $L_2 := L_2 \cup \{B\}$.

Compute $\tilde{A}$, $\tilde{b}$, and $R$ according to $B$.

$\mathcal{EN} := \mathcal{N}$.

For all $j \in \mathcal{N}$.

Solve the LP $\max \{e^T v : Ry - r^i \delta + lv = 0; y, \delta, v \geq 0\}$.

If this LP is unbounded $\mathcal{EN} := \mathcal{EN} \setminus \{j\}$.

End for.

For all $j \in \mathcal{EN}$.

For all $i \in B$.

If $B' = (B \setminus \{i\}) \cup \{j\}$ is feasible and $B' \not\in L_1 \cup L_2$ then $L_1 := L_1 \cup B'$.

End for.

End for.

End while.

Output: $L_2$. 
Example

- There can be exponentially many efficient bfs
  - \[
  \begin{align*}
  \min \quad & x_i & i = 1, \ldots, n \\
  \min \quad & -x_i & i = 1, \ldots, n \\
  \text{subject to} \quad & x_i \leq 1 & i = 1, \ldots, n \\
  & -x_i \leq 1 & i = 1, \ldots, n.
  \end{align*}
  \]

- \( n \) variables, \( m = 2n \) constraints, \( p = 2n \) objective functions
- all \( 2^n \) extreme points of the feasible set are efficient
Example

- There can be exponentially many efficient bfs

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\end{align*}
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& -x_i \geq 1 \quad i = 1, \ldots, n.
\end{align*}
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Overview

1. Multiobjective Linear Programming
   - Formulation and Example
   - Solving MOLPs by Weighted Sums

2. Biobjective LPs and Parametric Simplex
   - The Parametric Simplex Algorithm
   - Biobjective Linear Programmes: Example

3. Multiobjective Simplex Method
   - A Multiobjective Simplex Algorithm
   - Multiobjective Simplex: Examples
Example

\[
\begin{align*}
\text{min} & \quad \begin{pmatrix} 3x_1 + x_2 \\ -x_1 - 2x_2 \end{pmatrix} \\
\text{subject to} & \quad x_2 \leq 3 \\
& \quad 3x_1 - x_2 \leq 6 \\
& \quad x \geq 0
\end{align*}
\]

\text{LP(}\lambda\text{)}

\[
\begin{align*}
\text{min} & \quad (4\lambda - 1)x_1 + (3\lambda - 2)x_2 \\
\text{subject to} & \quad x_2 + x_3 = 3 \\
& \quad 3x_1 - x_2 + x_4 = 6 \\
& \quad x \geq 0.
\end{align*}
\]
Phase I: MOLP is feasible
\[ x^0 = (0, 0) \]

Phase II: Optimal weight
\[ \hat{w} = (1, 1) \]

Phase II: First efficient solution
\[ x^2 = (0, 3) \]

Phase III: Efficient entering variables \( s^1, x^2 \)

Phase III: Efficient solutions
\[ x^1 = (0, 0), x^3 = (3, 3) \]

Phase III: No more efficient entering variables
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- Phase III: Efficient solutions
  \[ x^1 = (0, 0), x^3 = (3, 3) \]
- Phase III: No more efficient entering variables

![Graphical representation of Phase I, II, and III of a Multiobjective Simplex Algorithm](image)
Example

\[
\begin{align*}
\text{min} & \quad -x_1 - 2x_2 \\
\text{min} & \quad -x_1 + 2x_3 \\
\text{min} & \quad x_1 - x_3 \\
\text{subject to} & \quad x_1 + x_2 \leq 1 \\
& \quad x_2 \leq 2 \\
& \quad x_1 - x_2 + x_3 \leq 4.
\end{align*}
\]

Slack variables \(x_4, x_5, x_6\) introduced to write the constraints in equality form \(Ax = b\)
Phase I: $B = \{4, 5, 6\}$ is a basis with bfs $x^0 = (0, 0, 0, 1, 2, 4)$

Phase II:

$$\begin{align*}
\min & \quad u_1 + 2u_2 + 4u_3 \\
\text{subject to} & \quad u^T \begin{pmatrix} 1 & 1 & 0 & 1 & 0 & 0 \\ 0 & 1 & 0 & 0 & 1 & 0 \\ 1 & -1 & 1 & 0 & 0 & 1 \end{pmatrix} + w^T \begin{pmatrix} -1 & -2 & 0 & 0 & 0 & 0 \\ -1 & 0 & 2 & 0 & 0 & 0 \\ 1 & 0 & -1 & 0 & 0 & 0 \end{pmatrix} \geq 0 \\
& \quad w \geq e
\end{align*}$$

$\hat{w} = (1, 1, 1)$

$$\begin{align*}
\min & \quad -x_1 - 2x_2 + x_3 : Ax = b, x \geq 0 \\
B^1 = \{2, 5, 6\}, \quad x^1 = (0, 1, 0, 0, 1, 3) \text{ is efficient bfs,} \\
L_1 = \{\{2, 5, 6\}\}
\end{align*}$$
Phase I: $\mathcal{B} = \{4, 5, 6\}$ is a basis with bfs $x^0 = (0, 0, 0, 1, 2, 4)$

Phase II:

$$
\begin{array}{l}
\min \quad u_1 + 2u_2 + 4u_3 \\
\text{subject to} \quad u^T \begin{pmatrix}
1 & 1 & 0 & 1 & 0 & 0 \\
0 & 1 & 0 & 0 & 1 & 0 \\
1 & -1 & 1 & 0 & 0 & 1 \\
\end{pmatrix} + w^T \begin{pmatrix}
-1 & -2 & 0 & 0 & 0 & 0 \\
-1 & 0 & 2 & 0 & 0 & 0 \\
1 & 0 & -1 & 0 & 0 & 0 \\
\end{pmatrix} \succeq 0 \\
w \succeq e
\end{array}
$$

$\hat{w} = (1, 1, 1)$

$$
\min \{ -x_1 - 2x_2 + x_3 : Ax = b, x \succeq 0 \}
$$

$\mathcal{B}^1 = \{2, 5, 6\}, \ x^1 = (0, 1, 0, 0, 1, 3) \text{ is efficient bfs,}$

$\mathcal{L}_1 = \{\{2, 5, 6\}\}$
Phase III
Iteration 1:
\( B^1 = \{2, 5, 6\}, \; L_1 = \emptyset, \; L_2 = \{\{2, 5, 6\}\} \)

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\( \mathcal{E}\mathcal{N} := \{1, 3, 4\} \)
### Check $x_1$

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LP has optimal solution, $x_1$ is efficient

### Check $x_3$

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LP has optimal solution, $x_3$ is efficient
- **Check** $x_1$

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LP has optimal solution, $x_1$ is efficient

- **Check** $x_3$

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LP has optimal solution, $x_3$ is efficient
Check $x_4$

\[
\begin{array}{cccccc|c}
1 & 1 & 2 & -2 & 0 & 0 & 0 & 0 \\
1 & 0 & 2 & -2 & 1 & 0 & 0 & 0 \\
-1 & 2 & 0 & 0 & 0 & 1 & 0 & 0 \\
1 & -1 & 0 & 0 & 0 & 0 & 1 & 0 \\
\end{array}
\]

LP is unbounded, $x_4$ is not efficient

$\mathcal{EN} = \{1, 3\}$

Feasible pivot $x_1$ enters and $x_2$ leaves: basis $B^2 = \{1, 5, 6\}$

Feasible pivot $x_3$ enters and $x_6$ leaves: basis $B^3 = \{2, 3, 5\}$

$\mathcal{L}_1 := \{\{1, 5, 6\}, \{2, 3, 5\}\}$
Check $x_4$

\[
\begin{array}{ccccccc}
1 & 1 & 2 & -2 & 0 & 0 & 0 \\
1 & 0 & 2 & -2 & 1 & 0 & 0 \\
-1 & 2 & 0 & 0 & 0 & 1 & 0 \\
1 & -1 & 0 & 0 & 0 & 0 & 1 \\
\end{array}
\]

LP is unbounded, $x_4$ is not efficient

$\mathcal{E}_N = \{1, 3\}$

Feasible pivot $x_1$ enters and $x_2$ leaves: basis $\mathcal{B}_2 = \{1, 5, 6\}$

Feasible pivot $x_3$ enters and $x_6$ leaves: basis $\mathcal{B}_3 = \{2, 3, 5\}$

$L_1 := \{\{1, 5, 6\}, \{2, 3, 5\}\}$
Iteration 2:
\( B^2 = \{1, 5, 6\} \) with BFS \( x^2 = (1, 0, 0, 0, 2, 3) \)
\( L_1 = \{\{2, 3, 5\}\}, \ L_2 = \{\{2, 5, 6\}, \{2, 3, 5\}\} \)

| \( \bar{c}^1 \) | 0 | -1 | 0 | 1 | 0 | 0 | 1 |
| \( \bar{c}^2 \) | 0 | 1 | 2 | 1 | 0 | 0 | 1 |
| \( \bar{c}^3 \) | 0 | -1 | -1 | -1 | 0 | 0 | -1 |
| \( x_2 \) | 1 | 1 | 0 | 1 | 0 | 0 | 1 |
| \( x_5 \) | 0 | 1 | 0 | 0 | 1 | 0 | 2 |
| \( x_6 \) | 0 | -2 | 1 | -1 | 0 | 1 | 3 |

\( \mathcal{EN} = \{2, 3, 4\} \)
Check $x_2$: Leads back to $B^1 = (2, 5, 6)$

Check $x_3$:

\[
\begin{array}{cccccccc}
-1 & 1 & 1 & -1 & 0 & 0 & 0 & 0 \\
-1 & 0 & 1 & 0 & 1 & 0 & 0 & 0 \\
1 & 2 & 1 & -2 & 0 & 1 & 0 & 0 \\
-1 & -1 & -1 & 1 & 0 & 0 & 1 & 0 \\
\end{array}
\]

$x_3$ not efficient

Check $x_4$:

\[
\begin{array}{cccccccc}
-1 & 1 & 1 & -1 & 0 & 0 & 0 & 0 \\
-1 & 0 & 1 & -1 & 1 & 0 & 0 & 0 \\
1 & 2 & 1 & -1 & 0 & 1 & 0 & 0 \\
-1 & -1 & -1 & 1 & 0 & 0 & 1 & 0 \\
\end{array}
\]

$x_4$ not efficient

$EN = \emptyset$
Check $x_2$: Leads back to $B^1 = (2, 5, 6)$

Check $x_3$:

\[
\begin{array}{cccccccc}
-1 & 1 & 1 & -1 & 0 & 0 & 0 & 0 \\
-1 & 0 & 1 & 0 & 1 & 0 & 0 & 0 \\
1 & 2 & 1 & -2 & 0 & 1 & 0 & 0 \\
-1 & -1 & -1 & 1 & 0 & 0 & 0 & 1 \\
\end{array}
\]

$x_3$ not efficient

Check $x_4$:

\[
\begin{array}{cccccccc}
-1 & 1 & 1 & -1 & 0 & 0 & 0 & 0 \\
-1 & 0 & 1 & -1 & 1 & 0 & 0 & 0 \\
1 & 2 & 1 & -1 & 0 & 1 & 0 & 0 \\
-1 & -1 & -1 & 1 & 0 & 0 & 0 & 1 \\
\end{array}
\]

$x_4$ not efficient

$\mathcal{E}N = \emptyset$
- Check $x_2$: Leads back to $B^1 = (2, 5, 6)$

- Check $x_3$:

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$x_3$ not efficient

- Check $x_4$

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$x_4$ not efficient

\[ \mathcal{E} \mathcal{N} = \emptyset \]
Check $x_2$: Leads back to $B^1 = (2, 5, 6)$

Check $x_3$:

$$
\begin{array}{cccccccc}
-1 & 1 & 1 & -1 & 0 & 0 & 0 & 0 \\
-1 & 0 & 1 & 0 & 1 & 0 & 0 & 0 \\
\hline
1 & 2 & 1 & -2 & 0 & 1 & 0 & 0 \\
-1 & -1 & -1 & 1 & 0 & 0 & 0 & 1
\end{array}
$$

$x_3$ not efficient

Check $x_4$:

$$
\begin{array}{cccccccc}
-1 & 1 & 1 & -1 & 0 & 0 & 0 & 0 \\
-1 & 0 & 1 & -1 & 1 & 0 & 0 & 0 \\
\hline
1 & 2 & 1 & -1 & 0 & 1 & 0 & 0 \\
-1 & -1 & -1 & 1 & 0 & 0 & 0 & 1
\end{array}
$$

$x_4$ not efficient

$\mathcal{E}N = \emptyset$
Iteration 3

\[ B^3 = \{2, 3, 5\} \text{ with bfs } x^3 = (0, 1, 5, 0, 1, 0) \]

\[ L_1 = \emptyset, L_2 = \{\{2, 5, 6\}, \{1, 5, 6\}, \{2, 3, 5\}\} \]

\[
\begin{array}{ccccccc|c}
\bar{c}^1 & 1 & 0 & 0 & 2 & 0 & 0 & 2 \\
\bar{c}^2 & -5 & 0 & 0 & -2 & 0 & -2 & -10 \\
\bar{c}^3 & 3 & 0 & 0 & 1 & 0 & 1 & 5 \\
\hline
x_2 & 1 & 1 & 0 & 1 & 0 & 0 & 1 \\
x_5 & -1 & 0 & 0 & -1 & 1 & 0 & 1 \\
x_3 & 2 & 0 & 1 & 1 & 0 & 1 & 5 \\
\end{array}
\]

\[ \mathcal{EN} = \{1, 4, 6\} \]
### Check $x_1$

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$x_4$ is not efficient

### Check $x_4$

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$x_4$ is not efficient

### Check $x_6$: Leads back to $B^1$
Check $x_1$

\[
\begin{array}{ccccccc|c}
-1 & 1 & -1 & 1 & 0 & 0 & 0 & 0 \\
1 & 2 & 0 & -1 & 1 & 0 & 0 & 0 \\
-5 & -2 & -2 & 5 & 0 & 1 & 0 & 0 \\
3 & 1 & 1 & -3 & 0 & 0 & 1 & 0 \\
\end{array}
\]

$x_4$ is not efficient

Check $x_4$

\[
\begin{array}{ccccccc|c}
-1 & 1 & -1 & -1 & 0 & 0 & 0 & 0 \\
1 & 2 & 0 & -2 & 1 & 0 & 0 & 0 \\
-5 & -2 & -2 & 2 & 0 & 1 & 0 & 0 \\
3 & 1 & 1 & -1 & 0 & 0 & 1 & 0 \\
\end{array}
\]

$x_4$ is not efficient

Check $x_6$: Leads back to $B^1$
Check $x_1$

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$x_4$ is not efficient

Check $x_6$: Leads back to $B^1$
Iteration 4: $\mathcal{L}_1 = \emptyset$, STOP
Output: List of efficient bases
$\mathcal{B}^1 = \{2, 5, 6\}, \mathcal{B}^2 = \{1, 5, 6\}, \mathcal{B}^3 = \{2, 3, 5\}$

\[
x^1 = \{2, 5, 6\}
\]
\[
x^2 = \{1, 5, 6\}
\]
\[
x^3 = \{2, 3, 5\}
\]