# Strategic Coloring of a Graph*

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**Abstract.** We study a strategic game where every node of a graph is owned by a player who has to choose a color. A player's payoff is 0 if at least one neighbor selected the same color, otherwise it is the number of players who selected the same color. The social cost of a state is defined as the number of distinct colors that the players use. It is ideally equal to the chromatic number of the graph but it can substantially deviate because every player cares about his own payoff, whatever how bad the social cost is. Following a previous work done by Panagopoulou and Spirakis [15] on the Nash equilibria of the coloring game, we give worst case bounds on the social cost of stable states. Our main contribution is an improved (tight) bound for the worst case social cost of a Nash equilibrium, and the study of strong equilibria, their existence and how far they are from social optima.

## 1 Introduction

We study a VERTEX COLORING game is defined as follows: given a simple graph $G = (V, E)$, each vertex is a player who has to choose (deterministically) one color out of $n = |V|$. A player's payoff is 0 if he selects the same color as one of his neighbors, otherwise it is the number of vertices with same color.

Panagopoulou and Spirakis [15] introduced the game and studied its set of *pure strategy Nash equilibria*, denoted by $PNE(G)$. Nash equilibria (NE in short) are sustainable and rational states of the game. Interestingly, $PNE(G)$ is nonempty for every graph $G$ and there exists a polynomial time procedure to compute an element of $PNE(G)$ [15]. However Nash equilibria are known to deviate from a socially optimal state in many situations (e.g. the prisoner's dilemma). The *social cost* associated with a graph $G$ and a strategy profile $\sigma$, denoted by $\mathsf{SC}(G, \sigma)$, is defined as the number of distinct colors selected by the players. Panagopoulou and Spirakis give upper bounds on $\mathsf{SC}(G, \sigma)$ when $\sigma \in PNE(G)$. These bounds depend on several parameters of the graph and often match known bounds on the chromatic number of $G$.

We continue the work done by Panagopoulou and Spirakis [15] and give improved bounds on $\mathsf{SC}(G, \sigma)$ when $\sigma \in PNE(G)$. We also study the set of *pure strong equilibria* of the vertex coloring game, denoted by $PSE(G)$. A strong

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equilibrium (SE in short) [3] is a state where no unilateral deviation by a non empty coalition of players is profitable to *all* its members. This solution concept refines the pure strategy Nash equilibrium, and it is more sustainable. In this paper we mainly show that a strong equilibrium always exists but it is **NP**-hard to compute one. In addition we provide upper bounds on the social cost $\mathsf{SC}(G, \sigma)$ when $\sigma \in PSE(G)$.

### 1.1 Previous work and contribution

The vertex coloring problem is a central optimization problem in graph theory (see for instance [11, 12]) and several games defined upon it exist in the literature. Bodlaender [5] study a 2-player game where, given a graph, an ordering on the set of vertices, and a finite set of colors $C$, the players alternatively assign a color $c \in C$ to the uncolored vertex that comes first in the ordering, and such that two neighbors have distinct colors. Bodlaender considers several variants of the game (e.g. a player looses if he cannot move) and focuses on the existence of a winning strategy.

In [7] Chaudhuri, Chung and Jamall study a coloring game defined by a set of available colors and a graph $G = (V, E)$ where each node represents a player. The game is played in rounds; each round, the players choose a color simultaneously. A player's payoff is 0 if one of his neighbors uses the same color, and 1 otherwise. The main result in [7] is that for a coloring game played on a network on $n$ vertices with maximum degree $\Delta$, if the number of colors available to each vertex is $\Delta + 2$ or more, and if each player plays a simple greedy strategy, then the coloring game converges in $O(\log n)$ steps with high probability. The game adressed by Chaudhuri, Chung and Jamall was initiated by Kearns, Suri and Montfort [10] who performed an experimental study. A possible motivation of the game is a scenario where faculty members wish to schedule classes in a limited number of classrooms, and must avoid conflicts with other faculty members [10].

The coloring game studied by Panagopoulou and Spirakis [15] and the game introduced by Kearns et al [10, 7] mainly differ in the definition of a player's payoff. In [15] a player gets 0 if one of his neighbors selects the same color, otherwise his payoff is the number of players using the same color. The other difference is that $n$ colors are available to each node. This paper is mainly dedicated to this model (an edge coloring game is also investigated). The motivation given in [15] is the analysis of a local search algorithm for the vertex coloring problem with provably good worst case distance of local optima to global optima. Interestingly they choose to illustrate their results via a game-theoretic analysis where local optima correspond to the Nash equilibria of the coloring game. Nevertheless the coloring game has applications in selfish routing in particular networks [8, 1, 9, 4] where every player has to choose a facility (i.e. a wavelength, a time-slot, etc) that is not used by another player with which he is incompatible (a detailed motivation is given in the appendix). Then most results in [15] are seen as bounds on the loss of efficiency in stable states of a strategic game, and it is the topic of many papers since the seminal papers [14] and [16].

Panagopoulou and Spirakis [15] prove that every NE of the vertex coloring game is a feasible, and locally optimum, vertex coloring of $G$. It is noteworthy that a feasible coloring (in particular a social optimum) is not necessarily a NE. However at least one social optimum of the VERTEX COLORING game is a NE. As we will see later, this property does not hold for strong equilibria. It is also shown in [15] that a Nash equilibrium $\sigma$ of the VERTEX COLORING game on a graph $G = (V, E)$ satisfies:

$$\mathsf{SC}(G,\sigma) \leq \min\{\Delta_2(G)+1, \frac{n+\omega(G)}{2}, \frac{1+\sqrt{1+8m}}{2}, n-\alpha(G)+1\} \qquad (1)$$

where $n = |V|$, $\omega(G)$ is the clique number of $G$ (maximum size of a clique), $m = |E|$, $\alpha(G)$ is the stability number of $G$ (maximum size of a stable set), $\mathcal{N}_G(v)$ is the neighborhood of a vertex $v$ (its set of adjacent vertices in $G$), $d_G(v)$ is the degree of $v$ in $G$, $\Delta(G)$ is the maximum degree and $\Delta_2(G) = \max_{v\in V} \max\{d_G(u) : u \in \mathcal{N}_G(v) \text{ and } d_G(u) \leq d_G(v)\}$.

We separate the bounds given in (1) into three groups according to the dominant parameter: $(a)$ $\Delta_2(G)$, $(b)$ $n$ and $(c)$ $m$. Hence, we obtain $(a)$ $\mathsf{SC}(G,\sigma) \leq \Delta_2(G)+1$, $(b)$ $\mathsf{SC}(G,\sigma) \leq \min\{\frac{n+\omega(G)}{2}, n-\alpha(G)+1\}$ and $(c)$ $\mathsf{SC}(G,\sigma) \leq \frac{1+\sqrt{1+8m}}{2}$. It is not difficult to prove that the bounds given in $(a)$ and $(b)$ are tight for every value of $\Delta_2(G) \geq 2$, $\omega(G) \geq 2$ or $\alpha(G) \geq 2$. Since a NE must be a social optimum if the following (independent) cases: $\Delta_2(G) = 1$, $\omega(G) \leq 1$ and $\alpha(G) = 1$, we will always assume $\chi(G) \geq 2$ (the case $\chi(G) = 1$ corresponds to $\omega(G) = 0$). However the bound $(c)$ is not sharp as we will see in Theorem 2.

In this article, we first deal with NE in Section 3. We propose a graph characterization of NE and, based on this characterization, we propose tight bounds depending on $m$ and $\chi(G)$ for the number of colors used in a NE, improving the one of [15]. Then, we show that the situation greatly improves in trees, since in this case the number of colors in a NE is only logarithmic.

In Section 4, we study SE in the same spirit: we propose a graph characterization and show almost tight bounds on the number of colors used in a SE. This allows us to derive that the *strong price of anarchy*, the worst case value of $\mathsf{SC}(G,\sigma)/\chi(G)$ for $\sigma \in PSE(G)$, is logarithmic.

We conclude this article in Section 5 by some additional results dealing with $k$-strong equilibria (strong equilibria for coalition of size a most $k$) for the vertex coloring game, new payoff functions that can alleviate the social cost and an edge coloring game (the same game up to the fact that we want to color the edges of the input graph).

Due to space limitations, some proofs are put in the appendix.

## 2 Notations and definitions

**Graph Theory** We use standard notations in graph theory. A *stable* set is a subset of pairwise non adjacent vertices. A stable set $S$ is *maximal* if, for every vertex $x \in V \setminus S$, $S \cup \{x\}$ is not a stable set. The *stability number* $\alpha(G)$ is the maximum size of a stable set. A *coloring* is a partition of $V$ into stable sets $\mathcal{S} = (S_1, \ldots, S_q)$. The *chromatic number* $\chi(G)$ is the minimum size of a coloring. It is well known (see for instance [6]) that

$$\chi(G) \geq \omega(G) \text{ and } \chi(G)\alpha(G) \geq n \tag{2}$$

**Strategic games** A strategic game $\varGamma$ is a tuple $\langle N, (\varSigma_i)_{i\in N}, (u_i)_{i\in N}\rangle$ where $N$ is the set of players and $\varSigma_i$ is the strategy set of player $i$. Each player $i$ has to choose a strategy in $\varSigma_i$. Then $\times_{i\in N}\varSigma_i$ is the set of all possible pure states (or strategy profiles) of the game. We only study pure strategy states, so we often omit the word ”pure”. $u_i : \times_{i\in N}\varSigma_i \to \mathbb{R}$ is the utility function of player $i$ (the higher the better). $\sigma_i$ denotes the strategy of player $i$ in the strategy profile $\sigma \in \times_{i\in N}\varSigma_i$. For a subset of players $N' \subset N$, $\sigma_{N'}$ (resp., $\sigma_{-N'}$) refers to $\sigma$, restricted to (resp., without) the strategies of $N'$. Hence, given two states $\sigma'$ and $\sigma$, $\sigma'' = (\sigma_{-N'}, \sigma'_{N'})$ denotes the state where $\sigma''_i = \sigma_i$ if $i \in N \setminus N'$ and $\sigma''_i = \sigma'_i$ if $i \in N'$. We often use the following abusive notations: $\sigma_i$ and $\sigma_{-i}$ instead of $\sigma_{\{i\}}$ and $\sigma_{-\{i\}}$ respectively. Finally $\sigma' = (\sigma_{-i}, j)$ denotes the state where $\sigma'_i = j$ and $\sigma'_{i'} = \sigma_{i'}$ for every $i' \neq i$. A state $\sigma$ is a *pure Nash equilibrium* (NE in short) if for any $i \in N$ and any strategy $j \in \varSigma_i$, $u_i(\sigma_{-i}, j) \leq u_i(\sigma)$. Hence no player has an incentive to deviate unilaterally from a NE. A strategy profile $\sigma$ is a *strong equilibrium* if for every non empty subset of players $S$ and every assignment $\sigma'$, at least one player $i \in S$ satisfies $u_i(\sigma_{-S}, \sigma'_S) \leq u_i(\sigma)$. In other words, any joint deviation by a coalition can not be profitable to all its members. A $k$-strong equilibrium is defined similarly for coalitions involving at most $k$ players. In particular, Nash equilibria and strong equilibria are respectively 1-strong and $|N|$-strong equilibria.

The *social cost* of a strategy profile $\sigma$ for the game $\varGamma$ is a real number which characterizes how costly to the whole set of players $\sigma$ is. It is denoted by $\mathsf{SC}(\varGamma, \sigma)$ (we will sometimes omit $\varGamma$ if not necessary). Hence the social cost is minimized for some states called *social optima*. The *price of anarchy* (PoA) [14] for pure Nash equilibria is defined as the worst case value of $\max_{\sigma\in PNE(\varGamma)} \mathsf{SC}(\varGamma, \sigma)/\mathsf{SC}(\varGamma, \sigma^*)$, over all instances of the game, where $PNE(\varGamma)$ is the set of all pure NE of $\varGamma$ and $\sigma^*$ is a social optimum. The PoA captures the lack of coordination between independent players. The *strong price of anarchy* (SPoA) [2] is defined similarly, just replace $PNE(\varGamma)$ by $PSE(\varGamma)$ (the set of all pure strategy strong equilibria of $\varGamma$) in the previous definition.

**The vertex coloring game** The VERTEX COLORING game is a strategic game where $N = V$ and $\varSigma_i = \{1, \cdots, n\}$ for all $i$. The utility of player $i$ in $\sigma$ is $u_i(\sigma) = |\{i' \in V : \sigma_{i'} = \sigma_i\}|$ if the set $\{i' \in \mathcal{N}(i) : \sigma_{i'} = \sigma_i\} = \emptyset$ and $u_i(\sigma) = 0$ otherwise. To any state $\sigma$ corresponds a coloring $\mathcal{S}(\sigma)$ defined as $(S_1(\sigma), \ldots, S_q(\sigma))$, where $S_j(\sigma) = \{i \in V : \sigma_i = j\}$.

Let $PNE(G)$ (resp., $PSE(G)$) be the set of all pure Nash equilibria (resp., pure strong Nash equilibria) of the VERTEX COLORING game for a simple graph $G$. It is known that $PNE(G) \neq \emptyset$ but, up to our knowledge, nothing is known about the existence of a strong equilibrium.

Given a simple graph $G = (V, E)$, a *social optimum* of the VERTEX COLORING game is an optimal coloring. Hence, the optimal social cost is the chromatic number $\chi(G)$.

We always assume that $|S_1(\sigma)| \geq \cdots \geq |S_q(\sigma)|$ and for any $j = 1, \ldots, q$, $f_\sigma(j)$ denotes a player with strategy $j$ in $\sigma$ (if one exists). Then by definition we have:

*Property 1.* For any NE $\sigma$ of a simple graph $G$, the following (in)equalities hold:

$(i)$ For any $j = 1, \ldots, q$, for any $i \in S_j(\sigma)$, $u_i(\sigma) = |S_j(\sigma)|$. We deduce that $u_{f_\sigma(1)}(\sigma) \geq \cdots \geq u_{f_\sigma(q)}(\sigma)$.

$(ii)$ For any $j, j' \in \{1, \ldots, q\}$, for any $i \in S_j(\sigma)$, $i' \in S_{j'}(\sigma)$, $j \leq j'$ implies that $u_i(\sigma) \geq u_{i'}(\sigma)$.

$(iii)$ $n = \sum_{j=1}^{q} u_{f_\sigma(j)}(\sigma)$.

## 3 Nash equilibria

We propose a graph-characterization of the Nash equilibria of the VERTEX COLORING game which will be useful in the following. Given a coloring $\mathcal{S} = (S_1, \ldots, S_q)$ where $|S_1| \geq \cdots \geq |S_q|$, the mapping $g$ (depending on $\mathcal{S}$) from $\{1, \ldots, q\}$ to $\{1, \ldots, q\}$ is defined as $g(j) = \min\{i : |S_i| = |S_j|\}$. For instance, we get $g(1) = 1$ and if the stable sets of $\mathcal{S}$ have distinct sizes, then $g(j) = j$.

**Theorem 1.** *Let $G = (V, E)$ be a simple graph. The state $\sigma$ is a NE of $G$ for the* VERTEX COLORING *game iff for every $i = 1, \ldots, q$ the stable set $S_i(\sigma)$ is maximal in $G_{g(i)}$ where $G_t$ is defined as the subgraph of $G$ induced by $S_t(\sigma) \cup \cdots \cup S_q(\sigma)$.*

*Proof.* Consider a simple graph $G = (V, E)$, instance of the VERTEX COLORING game. Let $\sigma$ be a NE with corresponding coloring $S(\sigma) = (S_1(\sigma), \ldots, S_q(\sigma))$. Let $i \in \{1, \ldots, q\}$ and consider a player $j \in S_k$ for some $k \geq g(i)$, $k \neq i$. Since $|S_i(\sigma)| = |S_{g(i)}(\sigma)| \geq |S_k(\sigma)|$, the fact that player $j$ does not want to deviate to set $S_i(\sigma)$ implies that $j$ is adjacent to some vertex in $S_i(\sigma)$. Then we deduce that $S_i(\sigma)$ is a stable set maximal in $G_{g(i)}$.

Conversely, let $\mathcal{S} = (S_1, \ldots, S_q)$ be a coloring of $G$ with $|S_1| \geq \cdots \geq |S_q|$ and such that $S_i$ is a stable set maximal in $G_{g(i)}$. Consider the state $\sigma$ where player $j \in S_i$ plays strategy $\sigma_j = i$ (thus, $S_i(\sigma) = S_i$) and assume by contradiction that $\sigma$ is not a NE. This means that there is a player $j \in S_i$ who can unilaterally replace his strategy by $k$ such that $u_j(\sigma_{-j}, k) > u_j(\sigma)$. Hence, we deduce that $S_k \cup \{j\}$ is a stable set of $G$ and $|S_k| \geq |S_i|$. We obtain a contradiction, since on the one hand $g(k) \leq k \leq i$ and on the other hand $S_k(\sigma)$ is supposed to be a stable set maximal in $G_{g(k)}$. $\square$

Using Theorem 1, we can improve the bound of the PoA given in [15] according to the parameter $m$ (see inequalities (1) and $(c)$).

**Theorem 2.** *For simple graphs $G$ on $m$ edges with chromatic number $\chi(G) \geq 2$, the social cost of a NE $\sigma$ verifies:*

$$\mathsf{SC}(G,\sigma) \leq \frac{\chi(G)+1}{2} + \sqrt{m - (\chi(G)+1)(\chi(G)-1)/4} \tag{3}$$

*This bound is tight for any $\chi(G) \geq 2$ and arbitrarily large $m$.*

*Proof.* Consider a simple graph $G = (V, E)$ on $m$ edges and with chromatic number $\chi(G) \geq 2$, instance of the VERTEX COLORING game. Let $\sigma$ be a NE with corresponding coloring $\mathcal{S}(\sigma) = (S_1(\sigma), \ldots, S_q(\sigma))$ and social cost $\mathsf{SC}(G,\sigma) = q$. We suppose that $q \geq \chi(G)+1$ since otherwise $PoA = 1$. Assume $r = |S_1(\sigma)| \geq \cdots \geq |S_q(\sigma)|$. For $i = 1, \ldots, r$, let $G^i$ be the subgraph of $G$ induced by the stable sets of $\mathcal{S}(\sigma)$ of size $i$ ($G^i$ can be empty for some $i$) and let $p_i$ be the number of the stable sets of $\mathcal{S}(\sigma)$ of size $i$. Using Theorem 1, the number of edges of $G$ is at least:

$$m \geq \sum_{i=1}^{r} \frac{ip_i(p_i-1)}{2} + \sum_{i=1}^{r} ip_i \left( q - \sum_{j=1}^{i} p_j \right) \tag{4}$$

Actually, since the $p_i$ stable sets of size $i$ of $\mathcal{S}(\sigma)$ are also maximal in $G^i$ (using $G^i \subseteq G_{g(j)}$ where $|S_j(\sigma)| = i$ and Theorem 1), there are at least $ip_i(p_i-1)/2$ edges in $G^i$ (for any $v \in V(G^i)$, $d_{G^i}(v) \geq p_i - 1$). Moreover, each stable set $S_j(\sigma)$ of $\mathcal{S}(\sigma)$ of size $i$ is maximal in $G_{g(j)}$, leading to the conclusion that there are at least $i(q - \sum_{j=1}^{i} p_j)$ edges between the vertices of $S_j(\sigma)$ and the graph $G^{i+1} \cup \cdots \cup G^r$. Now, let $p'_2 = \sum_{i=2}^{r} p_i$; one can see that the left part of inequality (4) is greater than $p'_2(p'_2-1) + \frac{p_1(p_1-1)}{2} + p_1(q-p_1)$. In fact, for every $i \in \{1, \ldots, r-1\}$ one can check that we have: $(i+1)p_{i+1}(p_{i+1}-1)/2 + (i+1)p_{i+1}\left(q - \sum_{j=1}^{i+1} p_j\right) + ip_i(p_i-1)/2 + ip_i\left(q - \sum_{j=1}^{i} p_j\right) \geq i(p_i+p_{i+1})(p_i+p_{i+1}-1)/2 + i(p_i+p_{i+1})\left(q - \sum_{j=1}^{i+1} p_j\right)$. Thus, the inequality follows by induction. Finally, observe that we get: $p_1 + p'_2 = q$ by construction and $p_1 \leq \omega(G) \leq \chi(G) \leq q-1$ since $G^1$ is a clique from Theorem 1 (thus, $p_1 = \omega(G^1) \leq \omega(G)$, $q \geq \chi(G)+1$ by hypothesis and $\chi(G) \geq \omega(G)$ from inequality (2)). Hence, we deduce:

$$\begin{aligned} m &\geq p'_2(p'_2-1) + \tfrac{p_1(p_1-1)}{2} + p_1(q-p_1) && = (p_1+p'_2)^2 + \tfrac{p_1^2}{2} - p_1(q-\tfrac{1}{2}) - q \\ &= q^2 + \tfrac{p_1^2}{2} - p_1(q-\tfrac{1}{2}) - q && \geq q^2 - q + \tfrac{\chi(G)^2}{2} - \chi(G)(q-\tfrac{1}{2}) \\ &= \left(q - (\tfrac{\chi(G)+1}{2})\right)^2 + \tfrac{(\chi(G)+1)(\chi(G)-1)}{4} && \end{aligned}$$

In fact, the mapping $z(x) = q^2 + \frac{x^2}{2} - x(q-\frac{1}{2}) - q$ (see second line in the above inequalities with $x = p_1$) is decreasing for $x \leq q - 1/2$. Since, $p_1 \leq \chi(G) \leq q-1 \leq q-1/2$, we deduce $z(p_1) \geq z(\chi(G))$.

Hence, we obtain $\mathsf{SC}(G,\sigma) \leq \frac{\chi(G)+1}{2} + \sqrt{m - (\chi(G)+1)(\chi(G)-1)/4}$ and the inequality (6) follows. Tight examples are given in the appendix. □

For instance, for connected bipartite graphs with $m \geq 1$ edges we obtain $\mathsf{SC}(G,\sigma) \leq \frac{3}{2} + \sqrt{m - \frac{3}{4}}$ which is an improvement on the bound given in [15].

Theorem 2 states that the bound of $\frac{3}{2} + \sqrt{m - \frac{3}{4}}$ is tight in bipartite graphs (the lower bound is obtained with a bipartite graph). To conclude this section, we tackle the problem when the graph $G$ is a tree and show that the social cost drops significantly: from $3/2 + \sqrt{m - 3/4}$ in bipartite graphs to $\log(n) + 1$. This bound being tight, we obtain as a conclusion that the PoA in trees is $\frac{\log(n)+1}{2}$.

**Theorem 3.** *In trees, the social cost of a NE is at most* $\log(n) + 1$. *This bound is tight for arbitrarily large* $n$.

## 4 Strong equilibria

First of all, note that when studying strong equilibria, we can restrict ourselves to coalitions where all the players of the coalition choose the same color (in their new strategy) since any coalition $S$ for this game can be decomposed into several coalitions $S_i$ which group the players that switch to the same color. Moreover, the coalition $S$ is improving (i.e., the utility of each member of the coalition increases) iff each coalition $S_i$ is improving. As a consequence, we only need to consider coalitions of size at most $\alpha(G)$.

For SE, we can state a graph-characterization similar to Theorem 1, by replacing "maximal stable set" by "maximum stable set". Actually, we do not need the mapping $g$ anymore.

**Theorem 4.** *Let* $G = (V,E)$ *be a simple graph. The state* $\sigma$ *is a SE of* $G$ *for the* VERTEX COLORING *game iff for every* $i = 1, \ldots, q$, *for every* $j \in S_i(\sigma)$, *we get* $u_j(\sigma) = \alpha(G_i)$ *where* $G_i$ *is the subgraph of* $G$ *induced by* $S_i(\sigma) \cup \cdots \cup S_q(\sigma)$.

In particular, Theorem 4 gives a proof of the existence of SE and a procedure to find it. On the other hand, it also shows that finding a SE within polynomial time is impossible unless **P**=**NP**.

**Corollary 1.** *Finding a SE of the* VERTEX COLORING *game is **NP**-hard.*

Note that we will tackle in Section 5.1 the case of $k$-strong equilibra, *i.e.* strong equilibria restricted to coalitions of size at most $k$. We will show in particular that for $k = 2, 3$, finding such an equilibrium is polynomial, while the problem is left open for $k \geq 4$.

When the chromatic number is one, that is when $G$ contains no edge, the PoA (and then the SPoA) of the VERTEX COLORING game is 1. Thus, we focus on graphs $G$ with $\chi(G) \geq 2$. In [15] it is shown that at least one optimal coloring is a NE. For the strong equilibrium, it is not the case.

**Proposition 1.** *For every* $k \geq 2$, *there are some graphs with chromatic number* $k$ *where no optimal coloring is a SE.*

*Proof.* For any $k \geq 2$, consider the following split graph $G_k = (K_k, S_{2k}; E_k)$ on $3k$ vertices where $K_k = \{x_1, \ldots, x_k\}$ is a clique of size $k$ and $S_{2k} = \{y_1, z_1, \ldots, y_k, z_k\}$ is a stable set of size $2k$. Moreover, each vertex $x_i \in K_k$ is linked to 2 vertices $y_i, z_i \in S_{2k}$. See Figure 1 for an example of graphs $G_2$ and $G_3$.

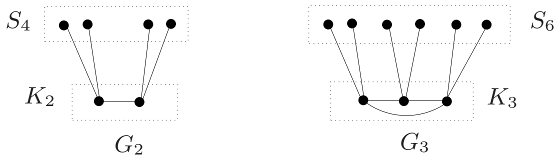

**Fig. 1.** Graphs $G_2$ and $G_3$.

Clearly, $S_{2k}$ is the only maximum stable set of $G_k$. Indeed, a stable set of $G_k$ has at most one vertex of $K_k$ since $K_k$ is a clique and if a stable set has one such a vertex, then it has at most $2k - 2$ vertices of $S_{2k}$. Thus, using Theorem 4, the strategy profile $\sigma$ defined by $\sigma_i = 1$ if $v_i \in S_{2k}$ and $\sigma_{x_j} = 1 + j$ for $j = 1, \ldots, k$, is the only SE using $k + 1$ colors. On the other hand, $\chi(G_k) = k$ since $\chi(G_k) \geq \omega(G_k) = |K_k| = k$ and a coloring using $k$ colors is given for instance by $\mathcal{S}^* = ((S_{2k} \setminus \{y_1, z_1\}) \cup \{x_1\}, \{y_1, z_1\} \cup \{x_2\}, \{x_3\} \ldots, \{x_k\})$. □

Now, we study the SPoA of the VERTEX COLORING game according to parameters $\Delta_2(G)$, $n$ or $m$. From inequality (1), we deduce that $PoA \leq \frac{\Delta_2(G)+1}{\chi(G)}$ for graphs with chromatic number $\chi(G)$ and $PoA \leq \frac{\Delta_2(G)+1}{2}$ for general (non trivial) graphs. Actually, as a corollary of Proposition 2 (see below), we deduce that this bound is tight for $\chi(G) = 2$ and for any value $\Delta_2(G) \geq 2$. More precisely, we prove that according to the parameter $\Delta_2(G)$, the SPoA and the PoA of the VERTEX COLORING game are equal.

**Proposition 2.** *The social cost of a SE of the* VERTEX COLORING *game is at most $\Delta_2(G) + 1$ for simple bipartite graphs $G$ on $n$ vertices. This bound is tight even if we consider the class of trees and arbitrarily large values of $\Delta_2(G)$.*

However, note that Proposition 2 does not imply that in trees every NE are SE. For instance in the $P_{2k+1}$ (the induced path on $n = 2k + 1$ vertices), from Theorem 4 it is easy to prove that there is only one SE corresponding to the optimal coloring (i.e., $SPoA(P_{2k+1}) = 1$ for any $k \geq 1$). On the other hand, in the $P_{6k+1}$ on vertex set $\{1, \ldots, 6k + 1\}$ for every $k \geq 1$, the state $\sigma$ defined by $\sigma_{3i+1} = 1$ for $i = 0, \ldots, 2k$, $\sigma_{3i+2} = 2$ and $\sigma_{3i+3} = 3$ for $i = 0, \ldots, 2k - 1$, is a NE using 3 colors (and then, it is not a SE).

Now, we analyze the SPoA of the VERTEX COLORING game according to the parameter $n$ (the number of vertices). In [15], it is indicated that unless **NP**$\subseteq$**co-RP**, the PoA of the VERTEX COLORING game is at least $n^{1-\varepsilon}$ for simple graphs on $n$ vertices and for every $\varepsilon \in (0;1)$. Here, we prove that the SPoA of the VERTEX COLORING game is much better. In trees, we already know that the SPoA is exactly $(1+\log(n))/2$. We prove in the following that this bound of $O(\log n)$ is in fact also an upper bound for the SPoA of the VERTEX COLORING game in general graphs.

**Theorem 5.** *The social cost of a SE in the* VERTEX COLORING *game is at most* $\chi(G)-1+\log_a\left(\frac{n}{\chi(G)-1}\right)$ *where* $a=\frac{\chi(G)}{\chi(G)-1}$ *for any simple graph* $G$ *on* $n$ *vertices with chromatic number* $\chi(G) \geq 2$.

*Consequently, considering graphs of* $n$ *vertices, the SPoA is at most* $\ln(n)+o(\ln(n))$.

Since in a simple graphs on $n$ vertices there are $m \leq n(n-1)/2$ edges, we deduce from Theorem 5 that the SPoA is at most $2\ln(m)+o(\ln(m))$. From the lower bound in trees, we also get that in (connected) graphs on $m$ edges the SPoA is at least $\log(m)/2+o(\log(m))$.

Using Theorem 5 and the lower bound in trees, we deduce that the SPoA of the VERTEX COLORING game equals $\frac{1}{2}\log n+\frac{1}{2}$ in bipartite graphs on $n$ vertices. It is a notable improvement relatively to the PoA since it is noticed in [15] that the PoA at least $\frac{n}{4}+\frac{1}{2}$ for bipartite graphs.

Dealing with the bound of Theorem 5 depending on both $\chi(G)$ and $n$, we can produce a lower bound which is not tight but closed to being so.

**Proposition 3.** *For any* $\chi(G) \geq 2$*, there are some simple graphs* $G$ *on* $n$ *vertices admitting a SE with social cost at least* $1+(\chi(G)-1)\log_\chi n$.

## 5 Final results and concluding remarks

### 5.1 *k*-strong equilibria

In Sections 3 and 4, we provided a characterization of NE and SE respectively. A natural question is to provide such a characterization for $k$-Strong equilibria, a solution concept which is in between NE and SE. We answer this question by giving a slightly more complex characterization.

Given a coloring $S=(S_1,\cdots,S_q)$ (sorted in non increasing size order), let us define for any $j=1,\cdots,q$ and for any $i \leq j$ the graph $\tilde{G}_{i,j}$ as the subgraph of $G$ induced by $S_i \cup S_{i+1} \cup \ldots \cup S_j$. Let us also remind that $G_j$ is the subgraph of $G$ induced by $S_j(\sigma) \cup \cdots \cup S_q(\sigma)$ and $g(j)$ is the smallest index $i$ such that $|S_i|=|S_j|$.

**Theorem 6.** *Let* $G=(V,E)$ *be a simple graph. The state* $\sigma$ *corresponding to a coloring* $S=(S_1,\cdots,S_q)$ *is a* $k$*-SE of* $G$ *for the* VERTEX COLORING *game iff for every* $j=1,\ldots,q$ *we have the following conditions:*

- *For any* $i<j$ *such that* $|S_i| \leq |S_j|+k-1$*, the size of a maximum stable set containing* $S_j$ *in* $\tilde{G}_{i,j}$ *is at most* $|S_i|$*;*

- *If* $|S_j| < k$ *then* $S_j$ *is a maximum stable set in* $G_j$*;*
- *If* $|S_j| \geq k$ *then* $S_j$ *is a maximal stable set in* $G_{g(j)}$*.*

Note that all the items given in the characterization can be tested in polynomial time provided that $k$ is a fixed constant.

Now, we prove that starting from a feasible coloring, computing a 3-strong equilibrium can be done in $O(n^3)$ steps, each step corresponding to an improvement for a coalition of at most 3 players. So, computing a $k$-strong equilibrium for $k = 1, 2, 3$ can be done in polynomial time. For $k = 1$, the result was already known from [15] since the authors proved that a NE (i.e., a 1-strong equilibrium) can be found in $O(n\alpha(G))$ steps. Actually, we believe that the result holds for any constant $k \geq 1$, but we are not able to prove this.

**Proposition 4.** *A* 3*-strong equilibrium of the* VERTEX COLORING *game can be computed in polynomial time.*

Proposition 4 is proved via a potential function argument, i.e. one can assign a real positive value to every state that is $O(n^3)$. Interestingly enough, it can be shown that a similar approach would not work for coalitions of size at most $k$, where $k \geq 4$. Indeed, in this case the weight associated to an independent set of size $i$ has to be exponential in $i$.

### 5.2 Alleviating the social cost with a new utility function

In the model of Kearns et al [10, 7] a player's payoff is 0 if one of his neighbors uses the same color and 1 otherwise. Then a player is satisfied if he is in an independent set, whatever how large the set is. With social cost considerations in mind (and supposing that an unbounded number of colors is available), this payoff function would be very expensive ($n$ actually). In the model of Panagopoulou and Spirakis [15], the players are incentivized to be in a large independent set because their payoff grows with the size of their set. As we have seen this payoff function ensures better bounds on the social cost (compared to previous the model).

An interesting question would be to provide a different utility function in order to improve the global efficiency of the system[1]. Trying to overcome the limits of the adopted utility function, we propose the following one. Instead of considering the size of a stable set, we consider the number of edges incident to a stable set. Formally, given a simple graph $G = (V, E)$ and a strategy profile $\sigma$, the utility of player $i$ in $\sigma$ now becomes $u_i(\sigma) = \sum_{v_j:\sigma_j=\sigma_i} d(v_j)$ if $\{v_j : \sigma_j = \sigma_i\}$ is a stable set and 0 otherwise.

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[1] We exclude the following solution which requires to solve an **NP**-hard problem: compute an optimal coloring, give 1 to the players who follow this optimum and give 0 to the others.

It is easy to see that the characterization of a SE is the same for this new utility function: instead of considering maximum stable set, we just have to consider maximum weight stable set, where the weight of a stable set is the sum of the degree of the vertices it contains. More precisely, a state $\sigma$ corresponding to a coloring $S(\sigma) = (S_1(\sigma), \cdots, S_q(\sigma))$ (the sets being in non decreasing weight order) is a SE iff for any $i$ $S_i(\sigma)$ is a maximum weight stable set in $G_i$.

Using this utility function, we get a simple but nice result for bipartite graphs.

**Proposition 5.** *Using the above utility function, any SE is an optimum coloring in bipartite graphs.*

This is a nice improvement compared to the bound of $\theta(\log(n))$ for the initial utility function. Unfortunately, this does not generalize as soon as $\chi(G) \geq 3$.

**Proposition 6.** *Using the above utility function, the SPoA is at least* $(\log(n/3)+1)/3$ *in 3-colorable graphs.*

In our opinion, finding a utility function that alleviates the social cost is an interesting question that deserves further research.

### 5.3 An edge coloring game

The edge coloring problem on a simple graph $G = (V, E)$ can be viewed as the vertex coloring problem on $L(G)$ where $L(G)$ is the line graph of $G$ (each edge $e_i \in E$ becomes a vertex $x_i$ of $L(G)$ and there is an edge $[x_i, x_j]$ in $L(G)$ if $e_i$ and $e_j$ are adjacent in $G$). Here, for simplicity, we refer to the edge model. Thus, an edge coloring $\mathcal{M} = (M_1, \ldots, M_q)$ of a simple graph $G = (V, E)$ is a partition of $E$ into matchings $M_i$. The minimum number of matchings partitioning the edges of $G$ is called *chromatic index* of $G$ and is denoted by $\chi_i(G)$. It is well known that the chromatic index of any simple graph $G$ of maximum degree $\Delta(G)$ verifies:

$$\Delta(G) \leq \chi_i(G) \leq \Delta(G) + 1 \tag{5}$$

Hence, the EDGE COLORING game is the VERTEX COLORING game on line graphs. In particular, Theorems 1 and 4 are valid when we replace vertex coloring $\mathcal{S}(\sigma) = (S_1(\sigma), \ldots, S_q(\sigma))$ by edge coloring $\mathcal{M}(\sigma) = (M_1(\sigma), \ldots, M_q(\sigma))$. Also, $G_i$ becomes the partial subgraph of $G$ induced by $M_i(\sigma) \cup \cdots \cup M_q(\sigma)$. However, now computing a SE of the EDGE COLORING game is polynomial (using the characterization of Theorem 4) since it consists of finding inductively a maximum matching of the current graph which is polynomial [6]. Finally, we always assume that $\Delta(G) \geq 2$ since otherwise $\mathsf{SC}(G, \sigma) = \chi_i(G)$ for every NE $\sigma$ of the EDGE COLORING game for graphs $G$ with $\Delta(G) = 1$.

**Theorem 7.** *The PoA and the SPoA of the* EDGE COLORING *game are* $2 - \frac{1}{\Delta(G)}$ *for simple bipartite graphs $G$. Moreover, these results hold even if we consider the class of trees with arbitrarily large values of $\Delta(G)$.*

On the other hand, if we restrict ourselves to regular bipartite graphs, any SE is a social optimum for the EDGE COLORING game.

**Proposition 7.** *A SE for the* EDGE COLORING *game is a social optimum for simple regular bipartite graphs $G$.*

**Theorem 8.** *The SPoA of the* EDGE COLORING *game is at most* $1 - \frac{1}{\Delta(G)+1} + \frac{1}{\Delta(G)} \log_a \left( \frac{m}{\Delta(G)-1} \right)$ *where* $a = \frac{\Delta(G)+1}{\Delta(G)}$ *for simple graphs $G$ on $m$ edges and of maximum degree $\Delta(G) \geq 2$.*

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# 6 Appendix

## 6.1 Motivation of the coloring game

We are given a set of $n$ users of a network such that every user $i$ wants to connect one source node $s_i$ to a destination node $t_i$. To avoid packet losses or simply guarantee the consistency of data, it is assumed that two users can open the same *facility* (e.g. a time slot or a wavelength) to establish a connection if their respective source-destination paths are disjoint. It is also assumed that every user has a unique way to connect his source to his destination. These restrictions occur for example in SS/TDMA network switches [8, 1] and optical tree networks [9]. Several optimization problems related to the above routing problem were addressed. The goal is to devise a central algorithm which groups connections in order to minimize the number of open facilities or to maximize the number of connections for a limited number of available facilities. This paper deals with the case where the connections are not monitored by a central algorithm. Instead, each user chooses which facility to open in order to establish his own connection. Then we consider a strategic game where all users (the players) have the same strategy space, a set $\Sigma = \{1, \dots, n\}$, representing the facilities. Actually $i$ plays $j$ means facility $j$ is open to send a packet along the path $s_i - t_i$.

A possible configuration is when one facility per user is opened. It corresponds to a very poor utilization of the resource if several connections can be done simultaneously. In the worst case, $n$ facilities are opened while only one suffices. Thus we consider the case where the agents are incentivized to use a minimal number of facilities as follows: the facilities are opened serially by their non increasing number of users. Hence it is in every user's interest to select the most "populated" compatible facility.

The situation is better represented as a strategic vertex coloring game on a graph of incompatibility $G = (V, E)$. Each node of $V$ is controlled by a player with strategy set $\Sigma = \{1, \dots, n\}$ (also called the set of colors) and there is an edge $[i, i']$ in $E$ iff the paths $s_i - t_i$ and $s_{i'} - t_{i'}$ overlap. A facility is represented by a color, and the payoff of a user is the number of users that use the same facility as himself.

In [4] Bampas *et al* study a similar game where several paths connecting $s_i$ to $t_i$ may exist. Hence the strategy of a player is composed of a path and a facility.

## 6.2 Tightness of Theorem 2

**Theorem 2.** *For simple graphs $G$ on $m$ edges with chromatic number $\chi(G) \geq 2$, the social cost of a NE $\sigma$ verifies:*

$$\mathsf{SC}(G, \sigma) \leq \frac{\chi(G)+1}{2} + \sqrt{m - (\chi(G)+1)(\chi(G)-1)/4} \tag{6}$$

*This bound is tight for any $\chi(G) \geq 2$ and arbitrarily large $m$.*

*Proof.* Let us prove that inequality (6) is tight for some graphs. Let $\gamma \geq 2$ and for any $k \geq 1$, consider the graph $H_k^\gamma$ on $n = 2k + \gamma$ vertices and $m = k^2 + k(\gamma - 1) + \frac{\gamma(\gamma-1)}{2}$ edges described as follows:

- $V(H_k^\gamma) = \{x_i, y_i : i = 1, \ldots, k\} \cup \{v_1, \ldots, v_\gamma\}$,
- $[x_i, y_j] \in E(H_k^\gamma)$ if $i \neq j$ and $i, j = 1, \ldots, k$,
- $[v_1, y_j] \in E(H_k^\gamma)$ for $j = 1, \ldots, k$, and $[v_i, x_j] \in E(H_k^\gamma)$ for $i = 2, \ldots, \gamma$ and $j = 1, \ldots, k$,
- $[v_i, v_j] \in E(H_k^\gamma)$ for $1 \leq i < j \leq \gamma$.

For instance, we can observe that $H_k^2$ is isomorphic to $K_{k+1,k+1} - kK_2$ where $K_{k+1,k+1}$ is the complete bipartite graph where each part of the bipartition has $k+1$ vertices. An example for $k = 3$ and $\gamma = 2$ is given in Figure 2.

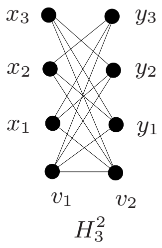

**Fig. 2.** The bipartite graph $H_3^2$ on 8 vertices and 13 edges.

It is easy to prove that the optimal social cost is $\gamma$ (i.e., $\chi(H_k^\gamma) = \gamma$) and that $\sigma$ is a NE with social cost $\mathsf{SC}(G, \sigma) = k + \gamma$ where $\sigma_{x_i} = \sigma_{y_i} = i$ for $i = 1, \ldots, k$ and $\sigma_{v_j} = k + j$ for $j = 1, \ldots, \gamma$. Since, $m = k^2 + k(\gamma - 1) + \frac{\gamma(\gamma-1)}{2} = \left(k + (\frac{\gamma-1}{2})\right)^2 + \frac{(\gamma+1)(\gamma-1)}{4}$ and $\mathsf{SC}(G, \sigma) = k + \gamma$, we deduce that $m = \left(\mathsf{SC}(G, \sigma) - (\frac{\chi+1}{2})\right)^2 + \frac{(\chi+1)(\chi-1)}{4}$ and the tightness follows. □

### 6.3 Proof of Theorem 3

**Theorem 3.** *In trees, the social cost of a NE is at most* $\log(n) + 1$. *This bound is tight for arbitrarily large* $n$.

*Proof.* We first show the upper bound. Let $\sigma$ be a NE whose corresponding coloring is $\mathcal{S}(\sigma) = (S_1(\sigma), \ldots, S_q(\sigma))$. We assume that $|S_1(\sigma)| \geq |S_2(\sigma)| \geq \ldots \geq |S_q(\sigma)| \geq 1$. We show by recurrence that for any $i = 1, \cdots, q-1$, $|S_i(\sigma)| \geq 2^{q-1-i}$. Note that if it is true, we get $n = \sum_{i=1}^{q} |S_i(\sigma)| \geq 1 + \sum_{i=1}^{q-1} 2^{q-1-i} = 2^{q-1}$, and then $q \leq 1 + \log(n)$. Since $\chi(G) = 2$ in trees, we get the upper bound.

The inequality $|S_i(\sigma)| \geq 2^{q-1-i}$ is obviously true for $i = q - 1$. Suppose that it is true for $i = j + 1, \cdots, q - 1$. Then since $\sigma$ is a NE, any vertex in $S_l(\sigma)$ is adjacent to at least one vertex in each $S_k(\sigma)$ for $k < l$. Then the forest induced by the vertices in $S_j(\sigma)\cup\cdots\cup S_q(\sigma)$ contains at least $\sum_{i=j+1}^{q}(i-j)|S_i(\sigma)|$ edges. Since in the forest the number of edges is at most the number of vertices minus 1, *i.e.*,$\sum_{i=j}^{q}|S_i(\sigma)|-1$, we get:

$$\begin{aligned}
&\sum_{i=j}^{q}|S_i(\sigma)|-1\geq\sum_{i=j+1}^{q}(i-j)|S_i(\sigma)| \\
&\Leftrightarrow|S_j(\sigma)|\geq 1+\sum_{i=j+1}^{q}(i-j-1)|S_i(\sigma)| \\
&\Rightarrow|S_j(\sigma)|\geq 1+(q-j-1)+\sum_{i=j+1}^{q-1}(i-j-1)2^{q-1-i} \qquad (7)
\end{aligned}$$

where (7) uses our recurrence and the fact that $|S_q|\geq 1$.

Let us now simplify the expression $N=\sum_{i=j+1}^{q-1}(i-j-1)2^{q-1-i}$.

$$N=\sum_{i=0}^{q-j-2}(q-j-2-i)2^i=(q-j-2)(2^{q-j-1}-1)-\sum_{i=0}^{q-j-2}i2^i$$

Now, we use the fact (which can be easily verified) that $\sum_{i=0}^{k}i2^i=2+(k-1)2^{k+1}$ holds for any $k$. This gives:

$$N=(q-j-2)(2^{q-j-1}-1)-2-(q-j-3)2^{q-j-1}=2^{q-j-1}-(q-j)$$

Then Inequality (7) gives $|S_j(\sigma)|\geq 2^{q-j-1}$.

To show the tightness, we consider the following trees $T_k$ ($k\geq 0$) on $2^{k+1}$ vertices built inductively as follows: $T_0$ has obviously two vertices $\{u,v\}$ and one edge $[u,v]$. Denote $S_0=\{u\}$ and $S_1=\{v\}$. $T_{k+1}$ is built from $T_k$ by adding an independent set $S_{k+2}$ of size $2^{k+1}=|V(T_k)|$ and a perfect matching between $S_{k+2}$ and the vertices of $T_k$. Then $T_{k+1}$ is a tree the leaves of which are $S_{k+2}$.

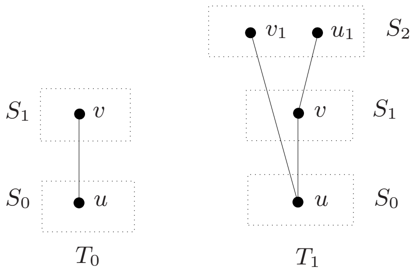

**Fig. 3.** Two trees $T_0$ and $T_1$.

Now, consider the tree $T_k$, and the coloring $(S_{k+1},S_k,\cdots,S_1,S_0)$. Obviously, by construction, for any $i\geq 0$ $S_{i+1}$ is a maximal independent set in $T_i$, hence

Theorem 1 shows that the state corresponding to this coloring is a NE. It uses $k+2$ colors where $n=2^{k+1}$, hence it uses $\log(n)+1$ colors. □

### 6.4 Proof of Theorem 4

**Theorem 4.** *Let $G=(V,E)$ be a simple graph. The state $\sigma$ is a SE of $G$ for the* VERTEX COLORING *game iff for every $i=1,\dots,q$, for every $j \in S_i(\sigma)$, we get $u_j(\sigma)=\alpha(G_i)$ where $G_i$ is the subgraph of $G$ induced by $S_i(\sigma)\cup\cdots\cup S_q(\sigma)$.*

*Proof.* Consider a simple graph $G=(V,E)$, instance of the VERTEX COLORING game. Let $\sigma$ be a SE. By contradiction, assume that there exists $i \in \{1,\dots,q\}$ and $j \in S_i(\sigma)$ such that $u_j(\sigma) \neq \alpha(G_i)$. $\sigma$ is also a NE and then $S(\sigma) = (S_1(\sigma),\dots,S_q(\sigma))$ is a coloring. Thus, $u_j(\sigma) < \alpha(G_i)$ since $S_i(\sigma)$ is a stable set of $G_i$ and $u_j(\sigma)=|S_i(\sigma)|$ by (*i*) of Property 1. Let $S^*$ be a stable set of maximum size of $G_i$ and let $\sigma'$ be the state where $\sigma'_j=\sigma_j$ if $j \notin S^*$ and $\sigma'_j=q+1$ if $j\in S^*$. Using (*i*) and (*ii*) of Property 1, we get for every player $\ell \in S^*$, $u_\ell(\sigma')=|S^*|=\alpha(G_i)>u_j(\sigma)\geq u_\ell(\sigma)$ since if $\ell\in S_{i'}(\sigma)$, then $i'\geq i$. Hence, players in $S^*$ may form a coalition and benefits, which is impossible since $\sigma$ is a SE.

Conversely, assume that for every $i=1,\dots,q$, for every $j\in S_i(\sigma)$, we get $u_j(\sigma)=\alpha(G_i)$ and by contradiction suppose that $\sigma$ is not a SE. Thus, there is a coalition $S\subseteq V$ which from state $\sigma$ reach state $\sigma'$. Let $i_0=\min\{i : S_i(\sigma)\cap S\neq\emptyset\}$ and consider a player $\ell\in S_{i_0}(\sigma)$. By construction, $0<u_\ell(\sigma')\leq u_\ell(\sigma')=|S_j(\sigma')|$ with $\sigma'_\ell=j$. Hence, $S_j(\sigma')$ is a stable set of $G$ and of $G_{i_0}$ by construction of $i_0$. We deduce $u_\ell(\sigma')\leq\alpha(G_{i_0})=u_\ell(\sigma)$, contradiction. □

### 6.5 Proof of Corollary 1

**Corollary 1.** *Finding a SE of the* VERTEX COLORING *game is* ***NP****-hard.*

*Proof.* Let $G=(V,E)$ be a simple graph. By contradiction, suppose that we can compute a SE $\sigma$ within polynomial time. We output $S=\text{argmax}\{|S_i(\sigma)| : S_i(\sigma)\in\mathcal{S}(\sigma)\}$ (obviously, it can be done within polynomial time). By Theorem 4, $S$ is a maximum stable set of $G$; hence, we could solve the independent (i.e., stable) set problem within polynomial time, while this problem is known to be **NP**-hard, [13]. □

### 6.6 Proof of Propostion 2

**Proposition 2.** *The social cost of a SE of the* VERTEX COLORING *game is at most $\Delta_2(G)+1$ for simple bipartite graphs $G$ on $n$ vertices. This bound is tight even if we consider the class of trees and arbitrarily large values of $\Delta_2(G)$.*

*Proof.* To see this, we just revisit the proof of Theorem 3 and consider the same trees $T_k$ on $2^{k+1}$ vertices and the NE $\sigma$ corresponding to the coloring $\{S_{k+1},S_k,\cdots,S_1,S_0\}$. Actually, this state is a SE. Indeed, since there is a perfect matching in $T_k$ between the vertices in $S_{k+1}$ and the other vertices, obviously $\alpha(T_k) \leq |S_{k+1}|$, meaning that $S_{k+1}$ is a maximum stable set in $T_k$. Using an obvious recurrence, the characterization of Theorem 4 shows than $\sigma$ is a SE of $T_k$. It uses $k+2$ colors. Finally, let us prove that $\Delta_2(T_k) = k+1$. It is easy to observe that the (only) vertex in $S_0$ and the (only) vertex in $S_1$ have degree $k+1$ is $T_k$ and are the only vertices of maximum degree in $T_k$. So $\Delta(T_k) = k+1$ and, since $\Delta_2(T_k) = \Delta(T_k)$ (there is an edge between the two vertices in $S_0$ and $S_1$), $\sigma$ uses $\Delta_2(T_k)+1$ colors. □

### 6.7 Proof of Theorem 5

**Theorem 5.** *The social cost of a SE in the* VERTEX COLORING *game is at most* $\chi(G)-1+\log_a\left(\frac{n}{\chi(G)-1}\right)$ *where* $a = \frac{\chi(G)}{\chi(G)-1}$ *for any simple graph* $G$ *on* $n$ *vertices with chromatic number* $\chi(G) \geq 2$.

*Consequently, considering graphs of* $n$ *vertices, the SPoA is at most* $\ln(n) + o(\ln(n))$.

*Proof.* Let $G = (V, E)$ be a simple graph on $n$ vertices with $\chi(G) \geq 2$, instance of the VERTEX COLORING game and let $\sigma$ be a worst SE with corresponding coloring $\mathcal{S}(\sigma) = (S_1(\sigma), \ldots, S_{p+\chi(G)-1}(\sigma))$ with value $\mathsf{SC}(G,\sigma) = p + \chi(G) - 1$. One can assume $p \geq 2$ (since otherwise $p = 1$ and we deduce $\mathsf{SC}(G,\sigma) = \chi(G)$). Moreover, remark that $\sum_{j=p+1}^{p+\chi(G)-1} u_{f_\sigma(j)}(\sigma) \geq \chi(G) - 1$ since $u_{f_\sigma(j)}(\sigma) \geq 1$ for a NE. Thus, we obtain:

$$\mathsf{SC}(G,\sigma) = p + \chi(G) - 1 \text{ and } -\sum_{j=p+1}^{p+\chi(G)-1} u_{f_\sigma(j)}(\sigma) \leq -(\chi(G) - 1) \qquad (8)$$

Now, let us focus on the $p$ players $\{f_\sigma(1), \ldots, f_\sigma(p)\}$. Using item $(i)$ of Property 1, inequality (2) and Theorem 5, we get for every $j = 1, \ldots, p$, $u_{f_\sigma(j)}(\sigma) = \alpha(G_j) \geq \frac{n - \sum_{i=1}^{j-1} u_{f_\sigma(i)}(\sigma)}{\chi(G_j)} \geq \frac{n - \sum_{i=1}^{j-1} u_{f_\sigma(i)}(\sigma)}{\chi(G)}$ since on the one hand $G_j$ has $n - \sum_{i=1}^{j-1} u_{f_\sigma(i)}(\sigma)$ vertices and on the other hand $\chi(G_j) \leq \chi(G)$. Thus, we obtain for every $j = 1, \ldots, p$,

$$u_{f_\sigma(j)}(\sigma) + \frac{1}{\chi(G)} \sum_{i=1}^{j-1} u_{f_\sigma(i)}(\sigma) \geq \frac{n}{\chi(G)} \qquad (9)$$

By multiplying inequality (9) by $\left(\frac{\chi(G)-1}{\chi(G)}\right)^{p-j}$ and by summing up for $j = 1, \ldots, p$, the left part of this inequality becomes:

$$\sum_{j=1}^{p} \left(\frac{\chi(G)-1}{\chi(G)}\right)^{p-j} u_{f_\sigma(j)}(\sigma) + \sum_{j=1}^{p} \left(\frac{\chi(G)-1}{\chi(G)}\right)^{p-j} \frac{1}{\chi(G)} \sum_{i=1}^{j-1} u_{f_\sigma(i)}(\sigma) \qquad (10)$$

while the right part of this inequality is:

$$\frac{n}{\chi(G)} \sum_{j=1}^{p} \left( \frac{\chi(G)-1}{\chi(G)} \right)^{p-j} = n \left( 1 - \left( \frac{\chi(G)-1}{\chi(G)} \right)^{p} \right) \tag{11}$$

Now, let us study quantity (10). We get:

$$\begin{aligned}
&\textstyle\sum_{j=1}^{p} \left(\frac{\chi(G)-1}{\chi(G)}\right)^{p-j} u_{f_\sigma(j)}(\sigma) + \sum_{j=1}^{p} \left(\frac{\chi(G)-1}{\chi(G)}\right)^{p-j} \frac{1}{\chi(G)} \sum_{i=1}^{j-1} u_{f_\sigma(i)}(\sigma) \\
&= \textstyle\sum_{i=1}^{p} \left(\frac{\chi(G)-1}{\chi(G)}\right)^{p-i} u_{f_\sigma(i)}(\sigma) + \frac{1}{\chi(G)} \sum_{i=1}^{p-1} u_{f_\sigma(i)}(\sigma) \sum_{j=i+1}^{p} \left(\frac{\chi(G)-1}{\chi(G)}\right)^{p-j} \\
&= \textstyle\sum_{i=1}^{p} \left(\frac{\chi(G)-1}{\chi(G)}\right)^{p-i} u_{f_\sigma(i)}(\sigma) + \frac{1}{\chi(G)} \sum_{i=1}^{p-1} u_{f_\sigma(i)}(\sigma) \sum_{j=0}^{p-i-1} \left(\frac{\chi(G)-1}{\chi(G)}\right)^{j} \\
&= \textstyle\sum_{i=1}^{p} \left(\frac{\chi(G)-1}{\chi(G)}\right)^{p-i} u_{f_\sigma(i)}(\sigma) + \frac{1}{\chi(G)} \sum_{i=1}^{p-1} u_{f_\sigma(i)}(\sigma) \times \chi(G) \left(1 - \left(\frac{\chi(G)-1}{\chi(G)}\right)^{p-i}\right) \\
&= \textstyle\sum_{i=1}^{p} \left(\frac{\chi(G)-1}{\chi(G)}\right)^{p-i} u_{f_\sigma(i)}(\sigma) + \sum_{i=1}^{p-1} u_{f_\sigma(i)}(\sigma) - \sum_{i=1}^{p-1} \left(\frac{\chi(G)-1}{\chi(G)}\right)^{p-i} u_{f_\sigma(i)}(\sigma) \\
&= \textstyle\sum_{i=1}^{p} u_{f_\sigma(i)}(\sigma)
\end{aligned}$$

Using equalities (8) and item $(i)$ of Property 1, we get $n - (\chi(G)-1) \geq n - \sum_{j=p+1}^{p+\chi(G)-1} u_{f_\sigma(j)}(\sigma) = \sum_{i=1}^{p} u_{f_\sigma(i)}(\sigma)$. Hence, from this last equality, quantities (10) and (11) and inequality (9) , we obtain:

$$n - (\chi(G)-1) \geq n \left( 1 - \left( \frac{\chi(G)-1}{\chi(G)} \right)^{p} \right)$$

which is equivalent to

$$p \leq \log_a \left( \frac{n}{\chi(G)-1} \right) \tag{12}$$

where $a = \frac{\chi(G)}{\chi(G)-1}$. Thus, using inequality (12) and equality (8), we deduce:

$$\mathsf{SC}(G,\sigma) \leq \log_a \left( \frac{n}{\chi(G)-1} \right) + \chi(G) - 1 \tag{13}$$

For the bound on the SPoA given as a function of $n$ in the statement of the theorem, let us consider the function $f(x) = x \ln(x/(x-1))$ defined on $[2,n]$. Using the fact that $\ln(x/(x-1)) < 1/(x-1)$, we get that $f'(x) = \ln(x/(x-1)) - 1/(x-1)$ is negative and then $f(x) \geq f(n) = n \ln(n/(n-1)) = 1 + o(1)$. Since the SPoA is at most $1 + \ln(n)/f(\chi(G))$, we get the bound $\ln(n) + o(\ln(n))$. □

### 6.8 Proof of Proposition 3

**Proposition 3** *For any $\chi(G) \geq 2$, there are some simple graphs $G$ on $n$ vertices admitting a SE with social cost at least $1 + (\chi(G)-1)\log_\chi n$.*

*Proof.* Let $\chi \geq 2$ and consider the graphs $G_p$ for $p \geq 1$ build inductively as follows:

- The vertex set of $G_1$ is $X_1^1 \cup \cdots \cup X_\gamma^1 \cup \{x_1, \ldots, x_\gamma\}$ where each block $X_i^1$ for $i = 1, \ldots, \gamma$ is constituted by a collection $X_{i,j}^1$ of size $\gamma - 1$; each group $X_{i,j}^1$ has a size 1. Thus, we obtain $X_i^1 = \cup_{j=1}^{\gamma-1} X_{i,j}^1$ where $|X_{i,j}^1| = 1$. Finally, $[x, y] \in E(G_1)$ if $x \in X_{i,j}^1$ and $y \in X_{i',j'}^1$ and $i \neq i'$, $j \neq j'$; $[x_i, y] \in E(G_1)$ if $y \in X_j^1 \cup \{x_j\}$ and $i \neq j$. Figure 4 illustrates the construction of $G_1$ for $\gamma = 3$.

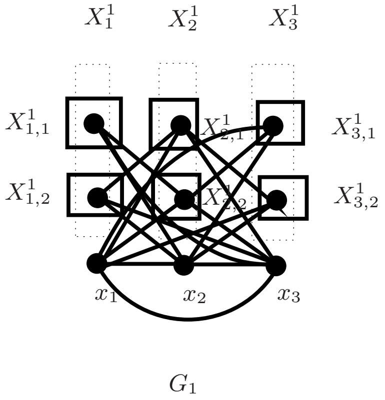

**Fig. 4.** The graph $G_1$ for $\gamma = 3$.

- Given $G_p$ with $p \geq 1$, $G_{p+1}$ contains $G_p$ and we add a set of vertices $X_1^{p+1} \cup \cdots \cup X_\gamma^{p+1}$ where each block $X_i^{p+1}$ for $i = 1, \ldots, \gamma$ is constituted by a collection $X_{i,j}^{p+1}$ of size $\gamma - 1$; each group $X_{i,j}^{p+1}$ has a size $\gamma^p$. Thus, we obtain $X_i^{p+1} = \cup_{j=1}^{\gamma-1} X_{i,j}^{p+1}$ where $|X_{i,j}^{p+1}| = \gamma^p$. Finally, $[x, y] \in E(G_{p+1}) \setminus E(G_p)$ if $x \in X_{i,j}^q$ and $y \in X_{i',j'}^{q'}$ with $q = p + 1$ or $q' = p + 1$ and $i \neq i'$, $j \neq j'$; $[x_i, y] \in E(G_{p+1}) \setminus E(G_p)$ if $y \in X_j^{p+1}$ and $i \neq j$.

It is easy to prove that for any $p \geq 1$, $|V(G_p)| = \gamma(1 + \sum_{\ell=1}^{p} \sum_{j=1}^{\gamma-1} |X_{i,j}^\ell|) = \gamma\left(1 + (\gamma - 1) \sum_{\ell=1}^{p} \gamma^{\ell-1}\right) = \gamma^{p+1}$. Thus,

$$p = \log_\gamma(|V(G_p)|) - 1 \tag{14}$$

It is easy to see that, by construction, $\mathcal{S}^{p+1} = (S_1^{p+1}, \ldots, S_{\gamma-1}^{p+1})$ where $S_j^{p+1} = \cup_{i=1}^{\gamma} X_{i,j}^{p+1}$ for $j = 1, \ldots, \gamma - 1$ is a coloring of $G^{p+1} \setminus Gp$ for $p \geq 1$ and $\mathcal{S}^1 = (S_1^1, \ldots, S_{2\gamma-1}^1)$ where $S_j^1 = \cup_{i=1}^{\gamma} X_{i,j}^1$ for $j = 1, \ldots, \gamma - 1$ and $S_{\gamma-1+j}^1 = \{x_j\}$ for $j = 1, \ldots, \gamma$ is a coloring of $G_1$. Thus, $\mathcal{S}_{p+1} = (\mathcal{S}^{p+1}, \ldots, \mathcal{S}^1)$ is a coloring of $G_{p+1}$. Let us prove that $\mathcal{S}_{p+1}$ corresponds to a SE of $G_{p+1}$ by induction on $p$. For $p = 0$, $G_1$ has $\gamma^2$ vertices and it is easy to observe that $\alpha(G_1) = \gamma$. Actually, on the one hand, $X_i^1$ are stable sets of size $\gamma$ for $i = 1, \ldots, \gamma$, and on the other hand, any stale set $S$ such that $S \not\subseteq X^1_i$ and $S \cap X^1_i \neq \emptyset$ for some $i$ satisfies $|S \cap X^1_i| \leq 1$ for every $i = 1, \ldots, \gamma$. Thus, since $|S^1_1| = \cdots = |S^1_{\gamma-1}| = \gamma$, $S^1_1, \ldots, S^1_{\gamma-1}$ are pairwise disjoint maximum stable sets of $G_1$; moreover, by construction $G_1[S^1_\gamma \cup \ldots \cup S^1_{2\gamma-1}]$ is a clique $K_\gamma$. Hence, using Theorem 4 we conclude that $\mathcal{S}^1$ corresponds to a SE of $G_1$. Now, let us assume that $\mathcal{S}_p$ corresponds to a SE of $G_p$ for $p \geq 1$ and let us prove that $\mathcal{S}_{p+1}$ corresponds to a SE of $G_{p+1}$ for $p \geq 1$. Using similar arguments as previously, we can show that $\alpha(G_{p+1} = \gamma^{p+1}$ (for instance, any set $X^{p+1}_i$ for $i = 1, \ldots, \gamma$ is a maximum stable set). Thus, since $\mathcal{S}^{p+1}$ is a coloring of $G^{p+1} \setminus Gp$ which only uses stable sets of size $\gamma^{p+1}$, we derive from Theorem 4 that $\mathcal{S}^{p+1}$ corresponds to a SE of $G^{p+1} \setminus Gp$ and by using the inductive hypothesis on $G_p$, the expected result follows.

Thus, the coloring of $G_p$ with $p \geq 1$, $\mathcal{S}_p = (\mathcal{S}^p, \ldots, \mathcal{S}^1)$ uses $(\gamma - 1)(p - 1) + 2\gamma - 1 = (\gamma - 1)p + \gamma$ colors. Let $\sigma$ be the worst SE of $G_p$ with value $\mathsf{SC}(\sigma)$. We deduce $\mathsf{SC}(\sigma) \geq (\gamma - 1)p + \gamma$; using equality (14), we obtain:

$$\mathsf{SC}(\sigma) \geq (\gamma - 1)(\log_\gamma |V(G_p)|) + 1 \tag{15}$$

Now, let us prove that , $\mathcal{S}^* = (S^*_1, \ldots, S^*_\gamma)$ where $S^*_i = X^p_i$ for $i = 1, \ldots, \gamma - 1$ is an optimal coloring of $G_p$ for any $p \geq 1$. The coloring $\mathcal{S}^*$ uses $\gamma$ colors and we have $\omega(G_p) \geq \gamma$ since the subgraph induced by $\{x_1, \ldots, x_\gamma\}$ is a clique of $G_p$; hence, using inequality (2), we get:

$$\chi(G_p) = \gamma \tag{16}$$

Finally, using inequality (15) and equality (16), the expected result follows.

$\square$

### 6.9 Proof of Theorem 6

**Theorem 6.** *Let $G = (V, E)$ be a simple graph. The state $\sigma$ corresponding to a coloring $S = (S_1, \cdots, S_q)$ is a $k$-SE of $G$ for the* VERTEX COLORING *game iff for every $j = 1, \ldots, q$ we have the following conditions:*

- *For any $i < j$ such that $|S_i| \leq |S_j| + k - 1$, the size of a maximum stable set containing $S_j$ in $\tilde{G}_{i,j}$ is at most $|S_i|$;*
- *If $|S_j| < k$ then $S_j$ is a maximum stable set in $G_j$;*
- *If $|S_j| \geq k$ then $S_j$ is a maximal stable set in $G_{g(j)}$.*

*Proof.* Consider a simple graph $G = (V, E)$, instance of the VERTEX COLORING game. Let $\sigma$ be a $k$-SE, defining a coloring $S(\sigma) = (S_1(\sigma), \cdots, S_q(\sigma))$ (sorted in non increasing size order). Since $\sigma$ is a NE, item 3 immediately follows from the characterization of NE. Also, if there is a stable set $S^*$ of size greater than $|S_j|$ in $G_j$, $|S_j| + 1$ players belonging to $S^*$ can form a coalition of size $|S_j| + 1 \leq k$ and play the same color (say a new one). Their utility will be $|S_j| + 1$, greater than the utility they had previously (at most $|S_j|$ since the vertices are in $G_j$). The first item can be proven similarly. Let $S^*$ a stable set of $\tilde{G}_{i,j}$, containing $S_j$

and of size $|S_i|+1$, for some $i<j$ such that $|S_i| \leq |S_j|+k-1$. Then a coalition of at most $k$ players can change their mind and choose $S_j$, obtaining a utility $|S_i|+1$ greater than the one they had before.

Now, we prove that these conditions are sufficient. Assume that we have a state inducing a coloring satisfying the three items, and let a coalition of size $t \leq k$ in the game improving the utility of each of its players. Let $i^*$ be the smallest index $i$ such that $S_i$ was containing a player from the coalition, before they changed their mind. As previously mentioned, we can assume that these players choose the same new strategy, *i.e.,* they are now in the same stable set. Then there are two cases. First, assume that this stable set is a new one (they choose a new color). Then the players of the coalition has a utility $t$, greater than the one they had before. Hence $t>|S_{i^*}|$, which is absurd from the second item.

So, the players of the coalition choose an existing color $S_j$. They receive utility $|S_j|+t$, greater than $|S_{i^*}|$ by hypothesis. Since, by the third item, $S_j$ was maximal in $G_{g(j)}$, we now that each player of the coalition was in a stable set of size greater than $|S_j|$. But $|S_{i^*}|<|S_j|+t \leq |S_j|+k$, and there exists in $\tilde{G}_{i^*,j}$ a stable set containing $S_j$ of size at least $|S_{i^*}|+1$, contradiction with the first item. □

### 6.10 Proof of Proposition 4

**Proposition 4.** *A* 3*-strong equilibrium of the* VERTEX COLORING *game can be computed in polynomial time.*

*Proof.* Let $S_j(\sigma)$ be the set of all nodes playing color $j$ in the state $\sigma$. Let $\pi$ be a permutation of the colors such that $|S_{\pi(1)}(\sigma)| \geq |S_{\pi(2)}(\sigma)| \geq \ldots \geq |S_{\pi(n)}(\sigma)|$. We define $\nabla^\sigma$ as a vector of length $n$, whose $j$-th coordinate $\nabla^\sigma_j$ is equal to $|S_{\pi(j)}(\sigma)|$ (the size of the $j$th largest set). A color may be unused so the coordinate corresponding to this color is equal to 0.

A 3-strong equilibrium can be computed as follows: start from the strategy profile associated with a proper coloring (e.g. one color per node) and, until it is not possible, modify the strategy of at most 3 players such that each of them benefits. The modification of the strategy of two players is done only if the modification of the strategy of one player is not possible. Similarly the modification of the strategy of three players is done only if the modification of the strategy of one or two players is not possible. Moreover one can restrict ourselves to particular modifications, i.e. those which consist of selecting a subset of players and assign them *the same* color (see the justification in the beginning of Section 4)

Let us denote by $\sigma^0$ the initial state while $\sigma^r$ is the state after $r$ improving steps. $\sigma^0$ is a proper coloring and every subsequent state is also a proper coloring then $u_i(\sigma^r) \geq 1$ for every player $i$ and state $\sigma^r$. We consider the potential function $\Phi(\sigma)=\sum_{i=1}^n u_i(\sigma)^2=\sum_{j=1}^n \left(\nabla^\sigma_j\right)^3$. We are going to prove that $\Phi(\sigma^r)<\Phi(\sigma^{r+1})$ for every $r$.

- Suppose that the strategy of only one player, say player 1, is modified. Then $\nabla^{\sigma^r}$ and $\nabla^{\sigma^{r+1}}$ only differ on two coordinates corresponding to the set that player 1 has left and the one that he has joined. The size of these sets, denoted by $b$ and $a$ respectively, has decreased and increased by 1 unit respectively. We deduce that $\Phi(\sigma^{r+1}) - \Phi(\sigma^r) = (a+1)^3 + (b-1)^3 - a^3 - b^3 = 3a^2 + 3a - 3b^2 + 3b$. Since $a + 1 > b$, because player 1 has increased his utility, we get that $\Phi(\sigma^{r+1}) - \Phi(\sigma^r) > 0$.
- Suppose that the strategy of only two players, say players 1 and 2, has been modified. Players 1 and 2 play in $\sigma^{r+1}$ a color, say $j$. Let us suppose that $a$ players selected color $j$ in $\sigma^r$ (if $a = 0$, then this color $j$ is not used in $\sigma^r$), then $a+2$ players select this color in $\sigma^{r+1}$. Suppose that $b$ players, including player 1, use a color $j'$ in $\sigma^r$. If $b \leq a$ then player 1 could alone play $j$ instead of $j'$ and benefit. If $b \geq a + 2$ then player 1 does not benefit in changing his strategy. We deduce that $b = a + 1$. Suppose that $c$ players, including player 2, use a color $j''$ in $\sigma^r$. With similar arguments we have $c = a + 1$. If players 1 and 2 play a different color in $\sigma^r$ then $\Phi(\sigma^{r+1}) - \Phi(\sigma^r) = (a + 2)^3 + 2 * a^3 - 2 * (a + 1)^3 - a^3 = 6a + 6 > 0$. If players 1 and 2 play the same color in $\sigma^r$ then $\Phi(\sigma^{r+1}) - \Phi(\sigma^r) = (a+2)^3 + (a-1)^3 - (a+1)^3 - a^3 = 12a + 6 > 0$.
- Suppose that the strategy of three players, say players 1, 2 and 3, has been modified. Players 1, 2 and 3 play color $j$ in $\sigma^{r+1}$ (if $a = 0$, then this color is a new one). Let us suppose that $a$ players selected color $j$ in $\sigma^r$, then $a + 3$ players select this color in $\sigma^{r+1}$. For at least two players in $\{1, 2, 3\}$, let say 1 and 2, the utility in $\sigma^r$ is at least $a+2$ since otherwise, a profitable deviation by only two players exists. For these players the utility in $\sigma^r$ is at most $a+2$ since otherwise the deviation is not profitable. For the third player (player 3) the utility in $\sigma^r$ is either $a + 2$ or $a + 1$ but not less, since otherwise there exists a profitable deviation by this single player. So for the first case we have $\Phi(\sigma^{r+1}) - \Phi(\sigma^r) = (a + 3)^3 + 3 * (a + 1)^3 - 3 * (a + 2)^3 - a^3 = 6 > 0$, assuming that players 1, 2 and 3 play distinct colors in $\sigma^r$ (if they don't then it is not difficult to see that $\Phi(\sigma^{r+1}) - \Phi(\sigma^r) > 0$). For the second case $\Phi(\sigma^{r+1}) - \Phi(\sigma^r) = (a+3)^3 + (a+1)^3 - 2*(a+2)^3 = 6a+12 > 0$, assuming that players 1 and 2 play distinct colors in $\sigma^r$ (if they don't then it is not difficult to see that $\Phi(\sigma^{r+1}) - \Phi(\sigma^r) > 0$).

Since $n \leq \Phi(\sigma) \leq n^3$ for every state $\sigma$ and each profitable deviation induces a strict increase (by at least 1) of the potential function, we deduce that the algorithm terminates after $O(n^3)$ steps (and the result is a 3-strong equilibrium).

To conclude, finding a profitable deviation by at most 3 players is done in polynomial time: for every subset of at most three players ($\binom{n}{3}$ choices) and every color ($n$ choices), check whether replacing the current color of every member of the subset by the selected color is profitable (done in $O(n^2)$ steps). □

### 6.11 Proof of Proposition 5

**Proposition 5.** *Using the above utility function, any SE is an optimum coloring in bipartite graphs.*

*Proof.* To see this, let $(V_1, V_2)$ be a bipartition of the graph. Then every edge is adjacent to a vertex in $V_1$ (and to one in $V_2$), hence the weight of $V_1$ and of $V_2$ is $m$, so they both are maximum weight stable sets. Conversely, any maximum weight stable set $\tilde{V}$ is such that $(\tilde{V}, V \setminus \tilde{V})$ is a bipartition, since $\tilde{V}$ has to be adjacent to every edge (otherwise its weight would be at most $m-1$). So, using the characterization, any SE has only 2 colors (or 1 if the graph has no edge). □

### 6.12 Proof of Proposition 6

**Proposition 6.** *Using the above utility function, the SPoA is at least* $(\log(n/3)+1)/3$ *in 3-colorable graphs.*

*Proof.* We consider a graph $G_t$ with $n = 3 \times 2^t$ vertices. There are three stable sets $S^1, S^2, S^3$ of size $2^t$. We divide $S^i$ into $t+1$ groups of vertices $V_0^i, V_1^i, \cdots, V_t^i$ where $|V_q^i| = 2^{q-1}$ for $q \neq 0$ (and $|V_0^i| = 1$). We have an edge between a vertex in $V_q^i$ and a vertex in $V_l^j$ iff $q \neq l$ (and $i \neq j$).

Now, let us find the maximum weight stable set. By symmetry, each stable set $S^i$ is adjacent to $2m/3$ edges. But there are $2^{q-1+p-1}$ edges between a group $V_q^i$ and a group $V_p^j$ $(j \neq i)$, hence a group $V_q^i$ is adjacent to $\sum_{p \neq q} 2^{q+p-2}$ edges. If we look at stable sets that contain vertices from different $S^i$, the heaviest one is then $S_1 = V_t^1 \cup V_t^2 \cup V_t^3$: each $V_t^i$ is adjacent to $2 \times 4^{t-1}$ edges, and then $S_1$ is adjacent to $6 \times 4^{t-1}$ edges. The total number of edges in the graph is:

$$m = 6 \sum_{i=1}^{t} 4^{i-1} = 6(4^t - 1)/3 = 2(4^t - 1)$$

Then, $S_1$ is adjacent to more than 2/3 of the edges and is consequently the only heaviest stable set of the graph.

Any SE has $S_1$ as first color. But removing $S_1$ gives the graph $G_{t-1}$. Hence, by a recursive argument, the only SE of $G_t$ has $t+1$ colors. Since $\chi(G) = 3$, the SPoA is $(t+1)/3 = (\log(n/3)+1)/3$. □

### 6.13 Proof of Theorem 7

**Theorem 7.** *The PoA and the SPoA of the* EDGE COLORING *game are* $2 - \frac{1}{\Delta(G)}$ *for simple bipartite graphs* $G$*. Moreover, these results hold even if we consider the class of trees with arbitrarily large values of* $\Delta(G)$*.*

*Proof.* Let $G$ be a simple bipartite graph. Since we have $\Delta(L(G)) \leq 2\Delta(G) - 2$ and $\Delta_2(L(G)) \leq \Delta(L(G))$, we deduce from the result of [15] that for any NE $\sigma$ of the EDGE COLORING game, we get $\mathsf{SC}(G, \sigma) \leq \Delta_2(L(G)) + 1 \leq 2\Delta(G) - 1$. On the other hand, Vizing's theorem (see for instance, [6]) states that the value of a social optimum (i.e., the chromatic index) is $\Delta(G)$ for a bipartite graph $G$. Thus, we get $PoA(G) \leq 2 - \frac{1}{\Delta(G)}$. We show the tightness for the SPoA since we always have $SPoA \leq PoA$.

For the tightness, consider a tree $T_{p+1}$ for any $p \geq 1$ built as follows: $r$ is the root of $T_{p+1}$ and it has $p+1$ neighbors $x_1, \ldots, x_{p+1}$. Each vertex $x_i$ has $p$ other neighbors (distinct from $r$) which are $y^i_1, \ldots, y^i_p$. Thus, $T_{p+1} = \{[r, x_i] : i = 1, \ldots, p+1\} \cup \{[x_i, y^i_j] : i = 1, \ldots, p+1 \text{ and } j = 1, \ldots, p\}$. Figure 5 gives an illustration for $p = 3$.

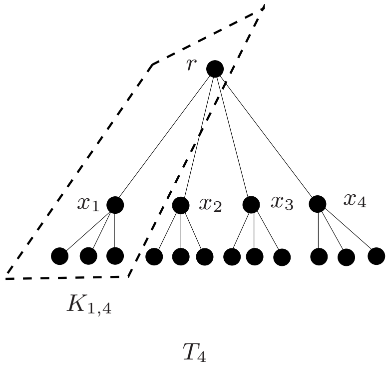

**Fig. 5.** The tree $T_4$.

We can easily prove that $M_1, \ldots, M_p$, where $M_j = \{[x_i, y^i_j] : i = 1, \ldots, p+1\}$, are $p$ maximum matchings of $T_{p+1}$. Actually, $T_{p+1}$ can be viewed as the union of $p+1$ stars $K_{1,p+1}$ sharing a common vertex $r$ where the center of the $i$-th copy of the star $K_{1,p+1}$ is $x_i$ (see Figure 5 for $p = 3$ where a $K_{1,4}$ is indicated in a dotted box). Thus, since any matching of $T_{p+1}$ can share at most one edge with each copy of $K_{1,p+1}$ we deduce that any matching of $T_{p+1}$ has at most $p+1$ edges. In conclusion, using Theorem 4, the state $\sigma$ where $\sigma_{[x_i, y^i_j]} = j$ for $j = 1, \ldots, p$ and $i = 1, \ldots, p+1$, and $\sigma_{[r, x_i]} = p + i$ for $i = 1, \ldots, p+1$ is a SE of $T_{p+1}$ using $\mathsf{SC}(\sigma) = 2p+1$ colors; since $\Delta(T_{p+1}) = p+1$, we deduce $\mathsf{SC}(\sigma) = 2\Delta(T_{p+1}) - 1$. Finally, since a social optimum uses $\Delta(T_{p+1})$ colors (for instance, $\mathcal{M}^* = (M^*_1, \ldots, M^*_p)$ where $M^*_i = \{[r, x_i]\} \cup \{[x_j, y^j_i] : j = 1, \ldots, p+1 \text{ and } j \neq i\}$ is an optimal edge coloring of $T_{p+1}$), we get $SPoA(T_{p+1}) \geq 2 - 1/\Delta(T_{p+1})$. □

### 6.14 Proof of Proposition 7

**Proposition 7.** *A SE for the* EDGE COLORING *game is a social optimum for simple regular bipartite graphs $G$.*

*Proof.* The proof is simple. It is well known that a simple $k$-regular bipartite graph $G = (V, E)$ has a perfect matching $M_1$ [6]. Now, the partial subgraph $G_1 = (V, E \setminus M_1)$ is a $(k-1)$-regular bipartite graph and then it has a perfect matching $M_2$. So, by induction the edge set $E$ of $G$ has a partition into $k = \Delta(G)$ perfect matchings $\mathcal{M} = (M_1, \ldots, M_k)$. Thus, using the edge version of

Theorem 4, we deduce that $\mathcal{M}$ is an edge coloring of $G$ corresponding to a SE $\sigma$. Finally, using inequality (5), we get $\mathsf{SC}(G,\sigma) = \Delta(G) \leq \chi_i(G)$ and then $\mathsf{SC}(G,\sigma) = \chi_i(G)$. Actually, we have proved that every SE $\sigma$ of $G$ satisfies $\mathsf{SC}(G,\sigma) = \chi_i(G)$. □

### 6.15 Proof of Theorem 8

**Theorem 8.** *The SPoA of the* EDGE COLORING *game is at most* $1 - \frac{1}{\Delta(G)+1} + \frac{1}{\Delta(G)} \log_a \left(\frac{m}{\Delta(G)-1}\right)$ *where* $a = \frac{\Delta(G)+1}{\Delta(G)}$ *for simple graphs* $G$ *on* $m$ *edges and of maximum degree* $\Delta(G) \geq 2$.

*Proof.* Let $G = (V,E)$ be a simple graph with $|E| = m$ and of maximum degree $\Delta(G) \geq 2$. Let $\sigma$ be the worst NE of the EDGE COLORING game on $G$. Using the equivalence between the EDGE COLORING game on $G$ and the VERTEX COLORING game on $L(G)$, the line graph of $G$, we deduce from Theorem 5 the following inequality: $\mathsf{SC}(\sigma) \leq \log_a \left(\frac{|V(L(G))|}{\chi(L(G))-1}\right) + \chi(L(G)) - 1$ where $a = \frac{\chi(L(G))}{\chi(L(G))-1}$. The function $\log_{z(x)} = 1/\ln z(x)$ where $z(x) = \frac{x}{x-1}$ is decreasing on $z(x) > 1$ and then is increasing on $x > 1$ (since $z(x)$ is decreasing). Thus, using inequality (5) and $\chi(L(G)) = \chi_i(G)$, we deduce $\log_{z(\chi_i(G))} \leq \log_{z(\Delta(G))+1}$ since $2 \leq \Delta(G) \leq \chi_i(G) \leq \Delta(G)+1$.
Hence using $|V(L(G))| = |E(G)| = m$, we get $SPoA(G) \leq \frac{1}{\chi_i(G)} \log_a \left(\frac{m}{\chi_i(G)-1}\right) + 1 - \frac{1}{\chi_i(G)}$ where $a = \frac{\Delta(G)+1}{\Delta(G)}$. Finally, using inequality (5) and $\Delta(G) \geq 2$, we obtain:

$$SPoA(G) \leq 1 - \frac{1}{\Delta(G)+1} + \frac{1}{\Delta(G)} \log_a \left(\frac{m}{\Delta(G)-1}\right)$$

where $a = \frac{\Delta(G)+1}{\Delta(G)}$ and the result follows. □