On s-t paths and trails in edge-colored graphs

Laurent Gourvès[a], Adria Lyra[b,c], Carlos Martinhon[b]
Jérôme Monnot[a], Fábio Protti[b]

[a] *CNRS, FRE 3234, Université de Paris-Dauphine, LAMSADE, F-75775, France*

[b] *Inst. of Computation - Fluminense Federal University, 24210-240, Brazil*

[c] *Fed. Center of Techn. Educ. Celso S. Fonseca, CEFET/RJ, 26041-271, Brazil* [1]

Abstract

In this paper we deal from an algorithmic perspective with different questions regarding monochromatic and properly edge-colored s-t paths/trails on edge-colored graphs. Given a c-edge-colored graph G^c without properly edge-colored closed trails, we present a polynomial time procedure for the determination of properly edge-colored s-t trails visiting all vertices of G^c a prescribed number of times. As an immediate consequence, we polynomially solve the Hamiltonian path (resp., Eulerian trail) problem for this particular class of graphs. In addition, we prove that to check whether G^c contains 2 properly edge-colored s-t paths/trails with length at most $L > 0$ is **NP**-complete in the strong sense. Finally, we prove that, if G^c is a general c-edge-colored graph, to find 2 monochromatic vertex disjoint s-t paths with different colors is **NP**-complete.

Keywords: Edge-colored graphs, properly edge-colored paths/trails, monochromatic paths.

1 Introduction

In the last few years a great number of applications have been modelled as problems in edge-colored graphs. To solve them, we can explore some inter-

[1] Email addresses: `fabio@ic.uff.br`, `mart@dcc.ic.uff.br`, `alyra@ic.uff.br`, `{laurent.gourves,monnot}@lamsade.dauphine.fr`

esting connections between edge-colored graphs and the theory of cycles and paths in directed and undirected graphs, matching theory, and other branches of graph theory [2]. For instance, problems in molecular biology [12], transportation and connectivity problems [7] (where connection costs are associated to pair of colors at adjacent edges), social sciences [5], between others.

Given a graph $G = (V, E)$, a walk ρ from s to t in G (called *s-t walk*) is a sequence $\rho = (v_0, e_0, v_1, e_1, \ldots, e_k, v_{k+1})$ where $v_0 = s$, $v_{k+1} = t$ and $e_i = v_i v_{i+1}$ for $i = 0, \ldots, k$. A trail and a path from s to t in G (called *s-t trail* and *s-t path*) is a walk $\rho = (v_0, e_0, v_1, e_1, \ldots, e_k, v_{k+1})$ from s to t where $e_i \neq e_j$ for $i \neq j$ and $v_i \neq v_j$ for $i \neq j$ respectively. The *length* of the path (trail, walk) is the number of its edges. Now, let $I_c = \{1, 2, \ldots, c\}$ be a set of given colors, with $c \geq 2$. In this work, G^c denotes a simple graph whose edges are colored by colors of I_c and with no parallel edges linking the same pair of vertices. The vertex and edge sets of G^c are denoted by $V(G^c)$ and $E(G^c)$, respectively, where $\mid V(G^c) \mid= n$ and $\mid E(G^c) \mid= m$. For a given color i, $E^i(G^c)$ denotes the set of edges of G^c colored by i.

If H^c is a subgraph of G^c then $N^i_{H^c}(x)$ denotes the set of vertices of H^c, linked to x with an edge colored by i. The degree of x in G^c is $d_{G^c}(x) = \sum_{i \in I_c} |N^i_{H^c}(x)|$. An undirected (or a non-oriented) edge between two vertices x and y is denoted by xy, its color by $c(xy)$.

From now on, we denote *properly edge-colored* by just PEC, for short. A subgraph of G^c containing at least two edges is said to be a PEC *subgraph* if any two adjacent edges differ in color. A PEC *path* or *trail* is a path or trail such that any two successive edges have different colors. However, observe that the edges in this trail need not to form a PEC subgraph since we can have adjacent but not successive edges with the same color. A PEC path or trail in G^c is said to be *closed* if its end-vertices coincide and its first and last edges differ in color. They are also refereed, respectively, as PEC *cycles* and PEC *closed trails*. Finally, a *monochromatic* path in G^c contains all its edges colored with the same color.

1.1 Some Related work

The determination of PEC s-t paths was first solved by Edmonds for two colors (see Lemma 1.1 in [11]) and then extended by Szeider[14] to include any number of colors. In Abouelaoualim *et al.*[1], the authors also deal with PEC trails and present polynomial time procedures for several versions of the s-t path/trail problem: as the shortest PEC s-t path/trail on general c-edge-colored graphs and the longest PEC path (resp., trail) for graphs without PEC cycles (resp., closed trails). A polynomial characterization of c-edge-colored graphs without PEC cycles was first presented by Yeo [16] and generalized in [1] for PEC closed trails. In addition, the authors in [1] prove that deciding whether there exist k pairwise vertex/edge disjoint PEC s-t paths/trails in a c-edge-colored graph G^c is **NP**-complete even for $k = 2$ and $c = \Omega(n^2)$. Moreover, they prove that these problems remain **NP**-complete for c-edge-colored graphs containing no PEC cycles and $c = \Omega(n)$. They conclude their paper with some approximation results for the associated maximization problems together with polynomial results for some particular classes of c-edge-colored graphs.

Some interesting questions regarding monochromatic paths can be found in the literature [13,10]. For instance, the authors in [9], show that it is NP-hard to find the minimum number of vertex disjoint monochromatic paths which cover the vertices of the graph G^c. They also show that there is no constant factor approximation algorithm for this problem unless P = NP.

2 Paths and trails in edge-colored graphs without PEC closed trails

In this section, we are interested in c-edge-colored (undirected) graphs without PEC closed trails and with an arbitrary number of colors c.

2.1 Finding two vertex/edge disjoint PEC s-t paths with bounded length in graphs with no PEC closed trail

It is proved in [1] that deciding whether a c-edge-colored graph on n vertices (even with $\Omega(n^2)$ colors) contains two vertex/edge disjoint PEC s-t paths is **NP**-complete. However, the complexity of this problem for graphs without PEC closed trails is an open problem raised in [1]. Here, we propose and solve a weak version of this problem. Given a graph G^c ($c \geq 2$) without PEC closed trails and a constant $L > 0$, we prove that finding two vertex/edge disjoint PEC s-t paths, each having at most L edges is **NP**-complete in the strong sense.

Dealing with uncolored graphs, it is proved in [15] that finding two vertex/edge disjoint s-t paths, each having at most L edges is **NP**-complete. In weighted versions of these problems, each edge (or arc) e has a non negative length $\ell(e)$ and the total length of a path ρ is defined as $\ell(\rho) = \sum_{e \in \rho} \ell(e)$. In [8], it is shown that the weighted 2 edge disjoint directed s-t paths problem is (weakly) **NP**-hard, even for acyclic digraphs. Using Häggkvist's transformation (i.e., we change arcs $\vec{xy}$ with cost $w(\vec{xy})$ by edges xz, zy with colors 1 and 2, resp., and assign edge costs $w(xz) = w(zy) = \frac{w(xy)}{2}$), we deduce that the weighted 2-edge disjoint PEC s-t path problem is (weakly) **NP**-hard, even for c-edge-colored graphs without PEC closed trails. Unfortunately, the length assigned to the edges in the proof of **NP**-hardness are not polynomially bounded. So with the following result, we establish a **NP**-completeness result in the strong sense.

Theorem 2.1 *Let G^c be a 2-edge-colored graph without* PEC *closed trails and a constant $L > 0$. The problem of finding 2 vertex/edge disjoint* PEC *s-t paths, each having at most L edges in G^c is* ***NP****-complete, even for graphs with maximum vertex degree equal to 4.*

2.2 The determination of PEC *s-t trails visiting vertices a prescribed number of times*

In the work of Das and Rao [6], they characterize those 2-edge-colored complete graphs K_n^c which contain a PEC closed trail visiting each vertex x of K_n^c exactly $f(x) > 0$ times. Generalizing this theorem Bang-Jensen and Gutin [3] solved the problem of determining the length of a longest closed PEC trail visiting each vertex x in 2-edge-colored complete multigraphs at most $f(x) > 0$ times.

When $G^c = (V, E)$ is a c-edge-colored graph containing no PEC closed trails, we propose a more general version of these problems and we show how to polynomially find, provided that one exists, a PEC s-t trail visiting all vertices of G^c a prescribed number of times (defined by an interval associated to each vertex). Formally, given two mappings f_{min} and f_{max} over $W = V \setminus \{s, t\}$ to $\mathbb{N}$ such that $0 \leq f_{min}(x) \leq f_{max}(x) \leq \lfloor d_{G^c}(x)/2 \rfloor$, we show how to find, if any, a PEC trail between vertices s and t, and visiting all vertices of W exactly $f(x)$ times with $f(x) \in \{f_{min}(x), \ldots, f_{max}(x)\}$. Using the concepts of trail-path graph [1] and Edmonds-Szeider graph [14], we can prove the following result:

Theorem 2.2 *Let $G^c = (V, E)$ be a c-edge-colored graph without* PEC *closed trails and $s, t \in V$. Then we can find within polynomial time, if one exists, a* PEC *s-t trail visiting all vertices $x \in W$ exactly $f(x)$ times with $f_{min}(x) \leq f(x) \leq f_{max}(x)$.*

Using Theorem 2.2, we can easily find a PEC hamiltonian s-t path (set $f_{min}(x) = f_{max}(x) = 1, \forall x \in W$), a PEC s-t trail visiting all vertices of G^c (set $f_{min}(x) = 1$ and $f_{max}(x) = \lfloor d_{G^c}(x)/2 \rfloor, \forall x \in W$), or an Eulerian PEC s-t trail if G^c has all vertices of W with even degrees and $s, t \in V$ with odd degrees. In this case, we add 2 new vertices s', t' and edges $s's$ and tt' both colored with one unused color and set $f_{min}(x) = f_{max}(x) = d_{G^c}(x)/2, \forall x \in V$. Now, we find an Eurelian PEC s'-t' trail in this new graph, if any. This last result is not so interesting since we recall that a polynomial algorithm is already known for finding PEC Eulerian s-t trail (if one exists) in general c-edge-colored graphs, [4].

Corollary 2.3 *Let $G^c = (V, E)$ be a c-edge-colored graph without* PEC *closed trails. Then, we can find within polynomial time, a shortest (resp., a longest)* PEC *s-t trail visiting vertices x of G^c at least $f_{min}(x)$ times (resp., at most $f_{max}(x)$ times).*

Now, we extend Theorem 2.2 by forcing the visit of a subset E' of edges.

Theorem 2.4 *Let $G^c = (V, E)$ be a c-edge-colored graph without* PEC *closed trails and let $E' \subseteq E$. Then we can find within polynomial time, a* PEC *s-t trail visiting all edges of E'.*

Note that Theorem 2.4 also allows to find a PEC Eulerian s-t trail in c-edge-colored graph without PEC closed trails.

3 Monochromatic s-t paths in edge-colored graphs

Here, we deal with monochromatic s-t paths in c-edge-colored graphs. We show that finding k vertex disjoint monochromatic s-t paths with different colors is **NP**-complete even for $k = 2$. As an immediate consequence, we show that the same problem over c-edge-colored digraphs is also **NP**-complete. On the other hand, notice that finding one monochromatic s-t path in G^c or 2 monochromatic edge disjoint s-t paths in G^c can be easily done in polynomial time (by using for each $i \in I_c$, a polynomial algorithm for finding one s-t path in $(V, E^i(G^c))$). Formally, we have the following result:

Theorem 3.1 *Let G^c be a c-edge-colored graph with $s, t \in V(G^c)$ with $c \geq 2$ and maximum vertex degree equal to 4. The problem of finding two vertex disjoint monochromatic s-t paths with different colors in G^c is **NP**-complete.*

As a future direction, an interesting question is to study the complexity of this problem for planar c-edge-colored graphs.

References

[1] A. Abouelaoualim, K. C. Das, L. Faria, Y. Manoussakis, C. A. Martinhon and R. Saad. *Paths and trails in edge-colored graphs*, Theor. Comp. Science, 409(3), 497-510, 2008.

[2] J. Bang-Jensen and G. Gutin. *Alternating cycles and paths in edge-coloured multigraphs: a survey*, Discrete Mathematics, 165/166, 39-60, 1997.

[3] J. Bang-Jensen and G. Gutin. *Alternating cycles and trails in 2-edge-coloured complete multigraphs*, Discrete Mathematics, 188, 61-72, 1998.

[4] A. Benkouar, Y. Manoussakis, V. Th. Paschos and R. Saad. *On the Complexity of Some Hamiltonian and Eulerian Problems in Edge-Colored Complete Graphs*, Proc. ISA, LNCS 1991, 190-198, 1991.

[5] W. S. Chou, Y. Manoussakis, O. Megalakaki, M. Spyratos and Z. Tuza. *Paths through fixed vertices in edge-colored graphs*, Mathématiques et Sciences Humaines, 127, 49-58, 1994.

[6] P. Das and S. B. Rao. *Alternating eulerian trails with prescribed degrees in two edge-colored complete graphs*, Discrete Mathematics, 43(1), 9-20, 1983.

[7] I. Gamvros. *Satellite network design, optimization and management*, PhD thesis, University of Maryland, 2006.

[8] A. Itai, Y. Perl, Y. Shiloach. *The complexity of finding maximum disjoint paths with length constraints*, Networks, v.12, n.3, 277-286, 1982.

[9] Z. Jin, X. Li, *Monochromatic and Heterochromatic Subgraphs in Edge-Colored Graphs - A Survey*, Graphs and Combinatorics, 24, 237–263, 2008.

[10] M. Kano, X. Li, *The complexity for partitiong graphs by monochromatic trees, cycles and paths*, Int. Journal of Comp. Math., 1-6, 2004.

[11] Y. Manoussakis, *Properly edge-colored paths in edge-colored complete graphs*, Discrete Applied Mathematics 56, 297-309, 1995.

[12] P. A. Pevzner, *DNA physical mapping and alternating eurelian cycles in colored graphs*, Algorithmica, 13, v.1–2, pp. 77–105, 1995.

[13] B. Sands, N. Sauer, and R. E.Woodrow, *On monochromatic paths in edge-coloured digraphs*, J. Combinatorial Theory (B), 33, 271–275,1982.

[14] S. Szeider. *Finding paths in graphs avoiding forbidden transitions*, Discrete Appl. Math., 126, n.2-3, 261-273, 2003.

[15] S. Tragoudas and Y. L. Varol. *Computing Disjoint Path with Length Constraints*, In Proc. WG '96, LNCS 1197, 375-389, 1996.

[16] A. Yeo. *A note on Alternating Cycles in Edge-coloured Graphs*, Journal of Combinatorial Theory, Series B- 69, (2):222-225, 1997.