Cutting Planes:

A Convex Analysis Perspective

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Mixed Integer Linear Programming

$$\begin{array}{lll} \min & cx & \\ \text{s.t.} & Ax = b & \\ & x_j \in \mathbb{Z} & \text{for } j = 1, \ldots, p \\ & x_j \geq 0 & \text{for } j = 1, \ldots, n. \end{array}$$

Common approach to solving MILP:

- First solve the LP relaxation. Basic optimal solution:

$$x_i \quad = \quad f_i + \sum_{j \in N} r^j x_j \quad \text{for } i \in B.$$

- If $f_i \notin \mathbb{Z}$ for some $i \in B \cap \{1, \ldots, p\}$, add cutting planes:

Gomory 1963 Mixed Integer Cuts, Marchand and Wolsey 2001 MIR inequalities, Balas, Ceria and Cornuéjols 1993 lift-and-project cuts, for instance, are used in commercial codes.

References

This talk

Borozan and Cornuéjols MOR 2009
Basu, Conforti, Cornuéjols and Zambelli SIDMA 2010

Related work (old)

Corner polyhedron

Gomory LAA 1969
Gomory and Johnson MP 1972

Intersection cuts

Balas OR 1971

The work that motivated me

Andersen, Louveaux, Weismantel and Wolsey IPCO 2007
Dey and Richard MOR 2008
Dey and Wolsey Working paper 2009

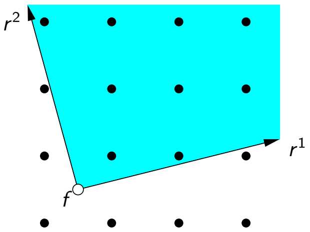
Corner Polyhedron — Gomory 1969

Relax nonnegativity on basic variables x_j.

In our work, we make a further relaxation, as suggested by Andersen, Louveaux, Weismantel and Wolsey:
Relax integrality on nonbasic variables.

$$\begin{aligned} x &= f + \sum_{j=1}^{k} r^j s_j \\ x &\in \mathbb{Z}^q \\ s &\geq 0 \end{aligned}$$

Example



Feasible set $\{ \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} \in \mathbb{Z}^2 :$

$$\begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = f + r^1 s_1 + r^2 s_2$$

where $s_1 \geq 0, s_2 \geq 0 \}$

"Formulas" for Deriving Cutting Planes

$$\begin{aligned} x &= f + \sum_{j=1}^{k} r^j s_j \\ x &\in \mathbb{Z}^q \\ s &\geq 0 \end{aligned}$$

Every inequality cutting off the point $(\bar{x}, \bar{s}) = (f, 0)$ can be expressed in terms of the nonbasic variables s only, in the form $\sum_{j=1}^{k} \alpha_j s_j \geq 1$.

We are interested in "formulas" for deriving such inequalities.

More formally, we are interested in functions $\psi : \mathbb{R}^q \to \mathbb{R}$ such that the inequality

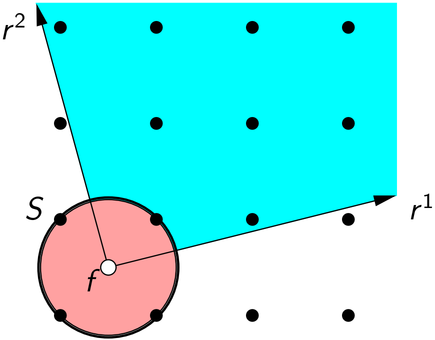
$$\sum_{j=1}^{k} \psi(r^j) s_j \geq 1$$

is valid for every choice of k and vectors $r^1, \ldots, r^k \in \mathbb{R}^q$.

We refer to such functions ψ as *valid functions* (with respect to f).

Intersection Cuts Balas 1971

Assume $f \notin \mathbb{Z}^q$. Want to cut off the basic solution $s = 0, x = f$.



Any convex set S with $f \in int(S)$ and no integer point in $int(S)$.

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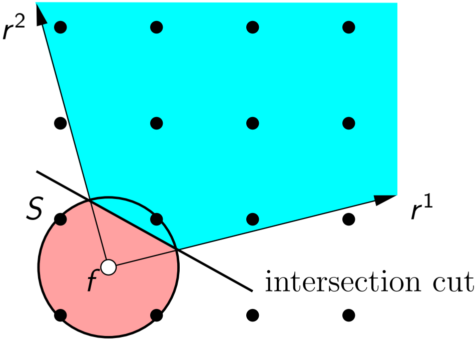


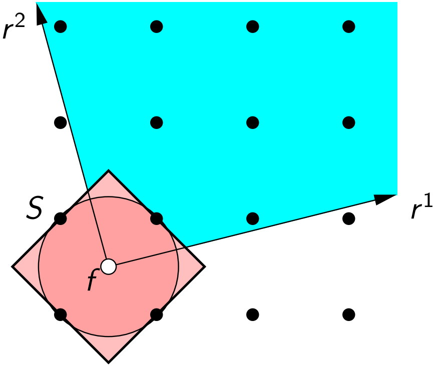
Any convex set S with $f \in int(S)$ and no integer point in $int(S)$.

The gauge of $S - f$, i.e. $\psi(r) = inf\{\lambda \geq 0 : \; r \in \lambda(S - f)\}$ is a valid function.

Intersection cut: $\psi(r^1)s_1 + \psi(r^2)s_2 \geq 1$.

Minimal Valid Functions

Our main interest is in minimal valid functions $\psi(r) : \mathbb{R}^q \to \mathbb{R}$, i.e. there is no valid function $\psi' \leq \psi$ where $\psi'(r) < \psi(r)$ for at least one $r \in \mathbb{R}^q$.



Bigger convex set: [Balas 1971]

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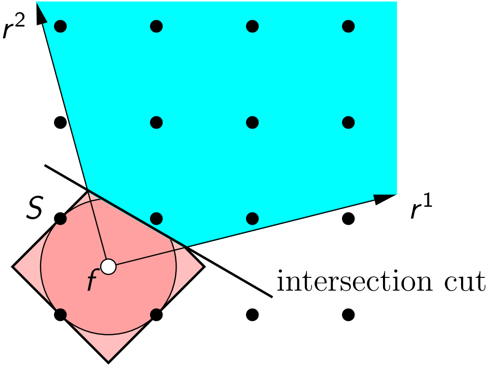


Bigger convex set: [Balas 1971]

Better cut: $\psi(r^1)s_1 + \psi(r^2)s_2 \geq 1$.

Theorem Borozan and Cornuéjols MOR 2009

(extension due to Basu, Conforti, Cornuéjols, Zambelli 2009)

Let $f \in \mathbb{R}^q \setminus \mathbb{Z}^q$.

If $\psi : \mathbb{R}^q \to \mathbb{R}$ is a minimal valid function, then ψ is

- nonnegative
- piecewise linear
- positively homogeneous
- and convex.

Furthermore $B_\psi := \{x \in \mathbb{R}^q : \psi(x - f) \leq 1\}$ is a maximal lattice-free convex set containing f in its interior.

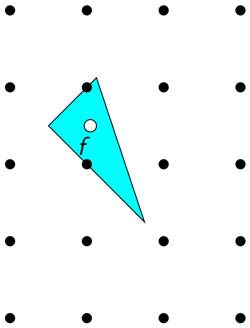
Conversely, for any maximal lattice-free convex set B containing f in its interior, the gauge of $B - f$ is a minimal valid function ψ.

DEFINITION A convex set is lattice-free if it does not have any integral point in its interior. However, it may have integral points on its boundary.

Maximal Lattice-Free Convex Sets

...are polyhedra (Lovász' theorem 1989)

- Lattice-free convex set contains no integral point in its interior



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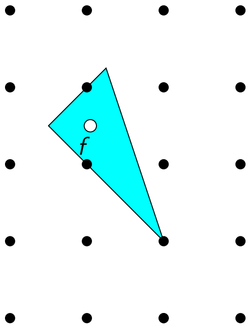

Maximal: each edge contains an integral point in its relative interior.

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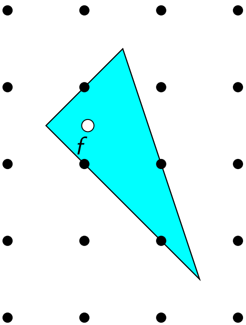
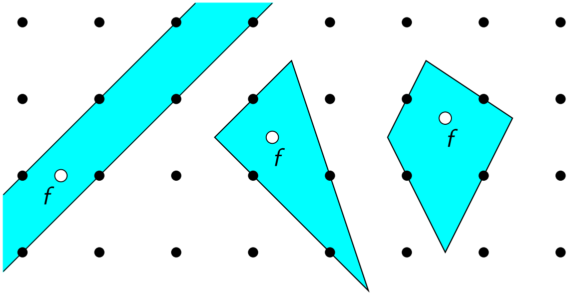


Maximal: each edge contains an integral point in its relative interior.

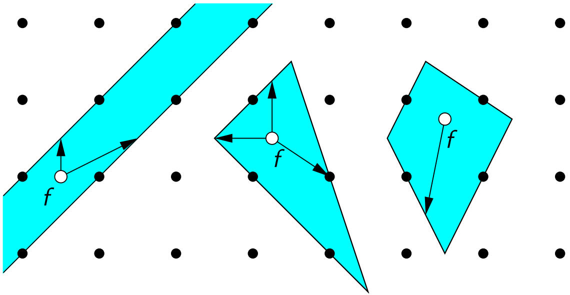
Maximal Lattice-Free Sets in the Plane

Split, triangles and quadrilaterals



Maximal Lattice-Free Sets in the Plane

Split, triangles and quadrilaterals



generate split, triangle and quadrilateral inequalities $\sum \psi(r) s_r \geq 1$, where the function ψ is the gauge of $S - f$.

Equivalently: Let $S = \{x \in \mathbb{R}^2 : \; a_i(x - f) \leq 1, i = 1, \ldots, t\}$. Then $\psi(r) = max_{i=1,\ldots,t} \; a_i r$.

Definitions

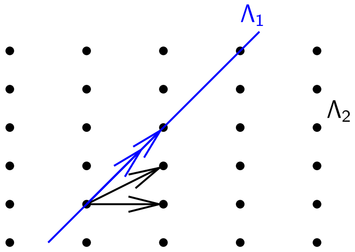
Let $a_1, \ldots, a_m \in \mathbb{R}^n$. The set $\Lambda = \{\lambda_1 a_1 + \ldots \lambda_m a_m$ where $\lambda_1, \ldots, \lambda_m \in \mathbb{Z}\}$ is called a finitely generated additive group of $\mathbb{R}^n$.

If $a_1, \ldots, a_m$ are linearly independent, Λ is called a lattice of the linear space $< a_1, \ldots, a_m >$. The vectors $a_1, \ldots, a_m$ are called a basis of the lattice.

EXAMPLES

$\Lambda_1 = \{\lambda_1 \begin{pmatrix} 2 \\ 2 \end{pmatrix} + \lambda_2 \begin{pmatrix} \sqrt{2} \\ \sqrt{2} \end{pmatrix}$ where $\lambda_1, \lambda_2 \in \mathbb{Z}\}$ is a finitely generated additive group but is not a lattice.

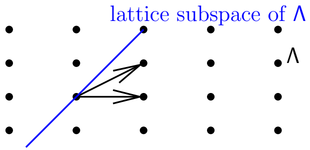
$\Lambda_2 = \{\lambda_1 \begin{pmatrix} 2 \\ 0 \end{pmatrix} + \lambda_2 \begin{pmatrix} 2 \\ 1 \end{pmatrix}$ where $\lambda_1, \lambda_2 \in \mathbb{Z}\}$ is a lattice.



Lovász' theorem

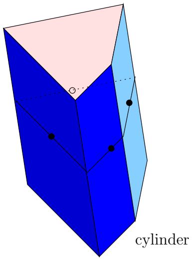
Let Λ be a lattice of $\mathbb{R}^n$.

DEFINITION A linear subspace $L \subseteq \mathbb{R}^n$ is a lattice subspace if there exists a basis of L contained in Λ.



THEOREM Lovász 1989

Every unbounded maximal lattice-free convex set is a cylinder above a polytope in some lattice subspace.



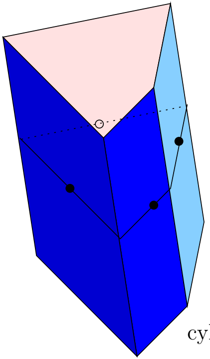
Lovász' Theorem

Actually, the correct statement is the following.

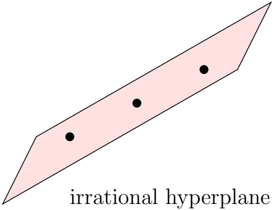
THEOREM

Every unbounded maximal lattice-free convex set S is either

- a cylinder above a polytope in some lattice subspace, or
- an affine hyperplane $v + L$ where $v \in S$ and L is not a lattice-subspace of $\mathbb{R}^n$.



cylinder



irrational hyperplane

Ingredients of the proof of Lovász' theorem

THEOREM
Let Λ be an additive group generated by $a_1, \ldots, a_m \in \mathbb{R}^n$.
Then Λ is a lattice of the linear space $< a_1, \ldots, a_m >$ if and only if there exists $\epsilon > 0$ such that $||y|| \geq \epsilon$ for every $y \in \Lambda \setminus \{0\}$.

(See, for example, Barvinok 2002: A Course in Convexity.)

THEOREM Dirichlet
Given any real numbers $\alpha_1, \ldots, \alpha_n, \epsilon$ with $0 < \epsilon < 1$,
there exist integer $p_1, \ldots, p_n, q$ such that

$$|\alpha_i - \frac{p_i}{q}| < \frac{\epsilon}{q}, \text{ for } i = 1, \ldots, n, \text{ and } 1 \leq q \leq \frac{1}{\epsilon}.$$

Ingredients of the proof of Lovász' theorem

Let L be a linear subspace of $\mathbb{R}^n$.

LEMMA If L is not a lattice-subspace of $\mathbb{R}^n$ then for every $\epsilon > 0$, there exists $y \in \Lambda \setminus L$ at distance less than ϵ from L.

LEMMA Suppose $\dim\, L = n - 1$, and let $v \in \mathbb{R}^n$. Then $H = v + L$ is a maximal Λ-free convex set if and only if L is not a lattice subspace of $\mathbb{R}^n$.

PROOF SKETCH Let S be a maximal Λ-free convex set.
If $\dim\, S < n$ then S is contained in some affine hyperplane $H = v + L$ and by maximality of S we have $S = H$. The above lemma shows that L is not a lattice subspace of $\mathbb{R}^n$.

Therefore we assume $\dim\, S = n$.
Let C be the recession cone of S and L its lineality space. We show first that $C = L$ and then that L is a lattice subspace of $\mathbb{R}^n$.
Finally, if $S \subset \mathbb{R}^n$ is a full-dimensional bounded maximal Λ-free convex set, then S is a polytope.

Proof sketch of the Borozan-Cornuéjols theorem

THEOREM A minimal valid function ψ is nonnegative, piecewise linear, positively homogeneous and convex.

IDEA OF PROOF Nonnegativity, positive homogeneity and convexity are proved directly.

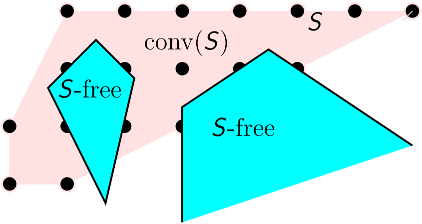
To prove that ψ is piecewise linear, consider $B_\psi := \{x \in \mathbb{R}^q : \psi(x - f) \leq 1\}$.

First we prove that B_ψ is a maximal lattice-free convex set. Thus by Lovász' theorem, B_ψ is a polyhedron.

Second, we prove that f is in its interior.
Thus $B_\psi = \{x \in \mathbb{R}^q : a_i(x - f) \leq 1, i = 1, \ldots, t\}$.
It follows that ψ is piecewise linear (one piece per facet of B_ψ). □

A Stronger Model Dey and Wolsey 2009

$$x = f + \sum_{j=1}^{k} r^j s_j$$
$$x \in S = \mathbb{Z}^q \cap P$$ where P is a rational polyhedron.
$$s \geq 0$$



THEOREM Basu, Conforti, Cornuéjols, Zambelli 2009

A set $B \subset \mathbb{R}^q$ is a maximal S-free convex set if and only if:

(i) B is a polyhedron s.t. $B \cap conv(S)$ has nonempty interior, or

(ii) B is a half-space supporting $conv(S)$, or

(iii) B is an irrational hyperplane of $\mathbb{R}^q$.

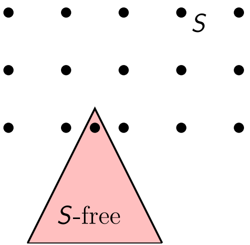
Theorem Basu, Conforti, Cornuéjols, Zambelli 2009

Let $S = \mathbb{Z}^q \cap P$ where P is a rational polyhedron and $f \in conv(S) \setminus \mathbb{Z}^q$.

Let B be a maximal S-free convex set with f in its interior
$B = \{x \in \mathbb{R}^q : \; a_i(x - f) \leq 1, i = 1, \ldots, t\}$
and let $\psi_B(r) = max_{i=1,\ldots,t} \; a_i r$.

Then ψ_B is a minimal valid function.

Conversely, for every valid function ψ, there exists a maximal S-free convex set B with f in its interior such that ψ_B dominates ψ.



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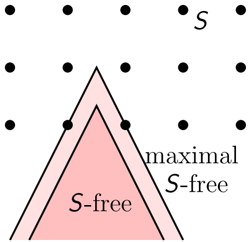

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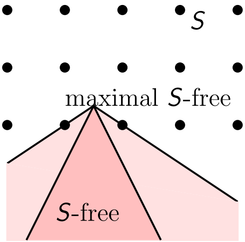

Idea of the proof

LEMMA Every valid function is dominated by a sublinear valid function.

Let ψ be a sublinear valid function and $K = \{r \in \mathbb{R}^n : \psi(r) \leq 1\}$.
Let $\hat{K} = \{y \in K^* : \; \exists x \in K$ such that $xy = 1\}$ and
let $\rho_K(r) = \sup_{y \in \hat{K}} ry$ be the support function of $\hat{K}$.

ρ_K is a sublinear valid function and $K = \{r \in \mathbb{R}^n : \rho_K(r) \leq 1\}$.

LEMMA $\rho_K \leq \psi$. [our proof uses Straszewicz' theorem]

LEMMA There exists a maximal S-free convex set
$B = \{x \in \mathbb{R}^q : \; a_i(x - f) \leq 1, \; i = 1, \ldots, k\}$ such that
$a_i \in cl(conv(\hat{K}))$ for $i = 1, \ldots, k$.

Then $\psi(r) \geq \rho_K(r) = \sup_{y \in \hat{K}} yr = \sup_{y \in cl(conv(\hat{K}))} yr$
$\geq \max_{i=1,\ldots,k} \; a_i r = \psi_B(r)$. □