MIP Formulations for Production/Distribution and Production/Sequencing

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Contents

The Single-Item Lot-Sizing Set X^{LS-C}

$$\begin{aligned} s_{t-1} + x_t = d_t + s_t \ \ \forall t, \\ x_t \leq C_t y_t \ \ \forall t, \\ x, s \geq 0, y \in \{0,1\}^T. \end{aligned}$$

When $C_t = C$ for all t, we call it “Constant Capacity", denoted X^{LS-CC}.
When $C_t = M > d_{1t}$ for all t, we call it “Uncapacitated", denoted X^{LS-U}.
Notation. Throughout we use $d_{kl} \equiv \sum_{u=k}^{l} d_u$.

The Uncapacitated Lot-Sizing Set *LS-U*

We consider X^{LS-U}

$$
\begin{aligned}
s_{t-1} + x_t = d_t + s_t \ \ \forall t,\\
x_t \leq M y_t \ \ \forall t,\\
x, s \geq 0, y \in \{0,1\}^T.
\end{aligned}
$$

Optimization is easy by Shortest Path/Dynamic Programming. $O(T^2)$ or even $O(T \log T)$

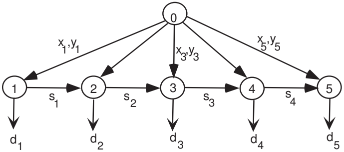
Thus by Grötschel, Lovasz and Schrijver, one can separate in polynomial time.

Thus there is some hope to find a nice description of $\mathrm{conv}(X)$!

Valid Inequalities

A first inequality

$$s_{k-1} \geq d_k(1 - y_k).$$



Generalizing

$$s_{k-1} \geq \sum_{u=k}^{t} d_u(1 - y_k - \cdots - y_u) \ \forall\ 1 \leq k \leq t \leq T.$$

Generalizing further

$$s_{k-1} + \sum_{j \in S} x_j \geq \sum_{u=k}^{t} d_u(1 - \sum_{j \in [k,u] \setminus S} y_j) \ \forall\ 1 \leq k \leq t \leq T,$$

Equivalently with $L = \{1, \ldots, l\}$ and $S \subseteq L$

$$\sum_{j \in L \setminus S} x_j + \sum_{j \in S} d_{jl} y_j \geq d_{1l}.$$

Multi-commodity/facility location reformulation $LS - U$

w_{ut} is fraction of demand in t produced in u

$$
\begin{aligned}
\sum_{u=1}^{t} w_{ut} &= 1 && \forall\, t \\
w_{ut} &\leq y_u && \forall\, 1 \leq u \leq t \leq T
\end{aligned}
$$

$$
x_u = \textstyle\sum_{t=u}^{T} d_t w_{ut} \quad \forall\, u
$$

Solving this an a linear program, one solves the lot-sizing problem. More generally, view objects delivered in period t as the distinct t^{th} commodity.

Discrete Constant Capacity Lot-sizing *DLSI – CC*

$$s_0 + C\sum_{u=1}^{t} y_u \geq d_{1t} \quad 1 \leq t \leq T$$
$$s \in \mathbb{R}^1_+, y \in \{0,1\}$$

$n = 3, d = (6,3,5), C = 10$

$$\begin{array}{ccccccc}
s_0 & +10y_1 & & & \geq & 6 \\
s_0 & +10y_1 & +10y_2 & & \geq & 9 \\
s_0 & +10y_1 & +10y_2 & +10y_3 & \geq & 14 \\
s_0 & \in & \mathbb{R}^1_+, & y & \in & \{0,1\}^3
\end{array}$$

$s_0 \geq 6(1 - y_1), \qquad s_0 \geq 4(2 - y_1 - y_2 - y_3)$

$s_0 \geq 4(2 - y_1 - y_2 - y_3) + (6-4)(1 - y_1) + [(10-6)(1 - y_1 - y_2 - y_3)]$

Constant Capacity Lot-sizing: WW-CC

The Wagner-Whitin relaxation $s_{t-1} + x_t = d_t + s_t,\ x_t \leq Cy_t$

$$s_{k-1} + C\sum_{u=k}^{t} y_u \geq d_{kt} \ 1 \leq k \leq t \leq T$$
$$s \in \mathbb{R}_+^1, y \in \{0,1\}$$

With typical non-speculative costs ($p_t + h_t \geq p_{t+1}\ \forall\ t$), $WW - CC$ solves $LS - CC$.
The convex hull is known either using valid inequalities, or with an extended formulation.

$$\mathrm{conv}(X^{WW-CC}) = \cap_{k=1}^{T} \mathrm{conv}(X_k^{DLSI-CC})$$

Also compact extended formulation with $O(T^2)$ constraints and variables.

LS-LIB

LS-LIB is a collection of subroutines/global constraints providing automatically reformulations or cutting plane separation routines for a variety of single item lot-sizing problems. It is implemented in Mosel (Xpress-MP).

Examples: Reformulations and Cut Separation Routines

- LS-U Uncapacitated Lot-Sizing: Facility location reformulation
- WW-CC Wagner-Whitin Relaxation with Constant Capacity
- DLS-CC-SC Discrete Lot-Sizing with Start-up variables

The Challenge

Find ways to use what we have learnt on modeling single item problems to tackle problems with

- more complicated variants
- multiple items
- multiple production levels
- multiple machines

A. Discrete Lot-Sizing with Sales

Set: $s_{t-1} + Cy_t = d_t + w_t + s_t,\; w_t \leq u_t \;\forall\; t$. Equivalently

$$s_0 + \sum_{j=1}^{t} v_j + C \sum_{j=1}^{t} y_j \geq b_{1t}, 0 \leq v_t \leq u_t \;\; t = 1, \ldots, T$$

For all $R \subseteq N$, consider the Wagner-Whitin relaxation

$$(s_0 + \sum_{j \in R} v_j) + C \sum_{j=1}^{t} y_j \geq b_{1t} - \sum_{j \in N \setminus R : j \leq t} u_j \quad t = 1, \ldots, T$$

Intersection of 2^n discrete lot-sizing sets.
For a given objective function min $h_0 s_0 + pv + fy$ with
$0 \leq p_{i_1} \leq \cdots \leq p_{i_n}$, it suffices to take
$R = N, N \setminus \{i_1\}, N \setminus \{i_1, i_2\}, \cdots, \emptyset,$, where $N = \{1, \ldots, T\}$.

B. Discrete Batch Production with Start-Ups

y_t is number of machines producing at capacity in period t.
z_t is increase in number of machines

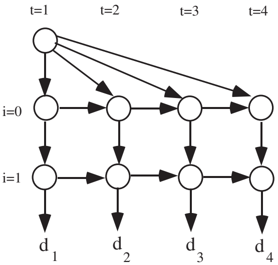
$$s_{t-1} + Cy_t = d_t + s_t \ \forall \ t$$
$$z_t \geq y_t - y_{t-1} \ \forall \ t$$
$$z_t \leq y_t \ \forall \ t$$
$$s \in \mathbb{R}^T_+, y, z \in \mathbb{Z}^T_+$$

WLOG $C = 1$.
Suppose that, given s_{t-1}, the next p units of demand arise in periods $t_1, \cdots, t_p$. Then

$$s_{t-1} \geq \sum_{u=1}^{p} (1 - y_{t+u-1} - z_{t+u} - \cdots - z_{t_u}).$$

($DLSI - CC - SC$) (Van Eijl and Van Hoesel) when $y_t, z_t, , d_t \in \{0, 1\}$.

C. Lot-Sizing in Series with 2 Levels



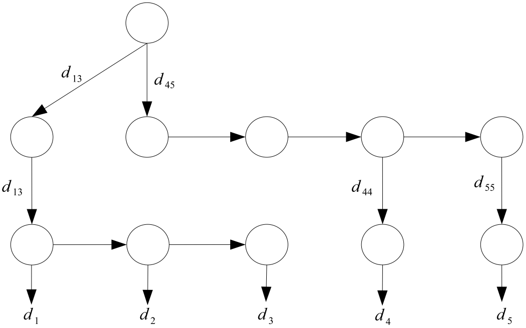
$$
\begin{aligned}
s^0_{t-1} + x^0_t &= x^1_t + s^0_t \ \forall\ t \\
s^1_{t-1} + x^1_t &= d_t + s^1_t \ \forall\ t \\
x^i_t &\leq My^i_t \ \ i = 0, 1,\ \forall\ t \\
x, s \in \mathbb{R}^{2T}_+, &\qquad y \in \{0, 1\}^{2T}
\end{aligned}
$$

Multi-commodity Reformulation

$$\begin{aligned}
\sigma^{0q}_{t-1} + w^{0q}_t &= w^{1q}_t + \sigma^{0q}_t \; \forall \;\; t,q,\; t \le q \\
\sigma^{1q}_{t-1} + w^{1q}_t &= d_q \delta_{tq} + \sigma^{1q}_t \; \forall \; t,q,\; t \le q \\
w^{iq}_t &\le d_q y^i_t \;\; i = 0,1, \;\forall \;\; t,q,\; t \le q \\
w^{iq}_t, \sigma^{iq}_t \in \mathbb{R}^1_+, & \qquad y \in \{0,1\}^{2T} \\
x^i_t &= \sum_{q=t}^{T} w^{iq}_t \\
s^i_t &= \sum_{q=t}^{T} \sigma^{iq}_t
\end{aligned}$$

The multi-commodity reformulation is good in practice, but it is not tight even for just two levels.
Two levels corresponds to the simplest production/transportation or single warehouse/single retailer model.

Uncapacitated Lot-Sizing in Series with 2 Levels: Solution Structure



Solution Structure and DP

If $x_j^1 = d_{jt}$ is produced/shipped at level 1, then $x_i^0 = d_{pq}$ for some periods i, p, q with $i \leq p \leq j \leq t \leq q$.
Let $G(t)$ be the minimum cost of the two-level problem restricted to the periods 1 up to t, and $H(i,t)$ be the minimum cost of satisfying demands from i to t at level one.

$$G(t) = \min_{1 \leq j \leq t} \{G(j-1) + \min_{1 \leq i \leq j} (f_i^0 + p_i^0 d_{jt}) + H(j,t)\},$$

$$H(i,t) = \min_{i \leq j \leq t} \{H(i,j-1) + f_j^1 + p_j^1 d_{jt}\}.$$

Algorithm in $O(T^2 \log T)$.

Lot-Sizing in Series with 2 Levels: Solution Structure

$\upsilon_{ijt} = 1$ if $x_i^0 = d_{jt}$.
$\omega_{pjt} = 1$ $x_j^1 = d_{jt}$ and $x_i^0 = d_{pq}$ with $[j,t]$ a subinterval of $[p,q]$.

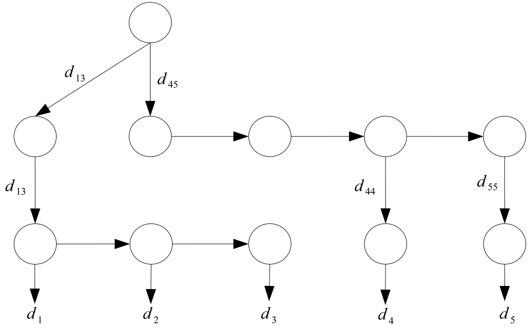


Figure: Solution with $\upsilon_{113} = \upsilon_{245} = 1$ and $\omega_{113} = \omega_{444} = \omega_{455} = 1$

Corresponding Dual

$$\min \sum_{i=1}^{T}\sum_{j=i}^{T}\upsilon_{ijk}(f_i^0+p_i^0 d_{jk})+\sum_{i=1}^{T}\sum_{j=i}^{T}\sum_{k=j}^{T}\omega_{ijk}(f_j^1+p_j^1 d_{jk})$$

$$\sum_{i=1}^{T}\sum_{j=i}^{T}\upsilon_{ijT}=1,$$

$$\sum_{i=1}^{t}\sum_{j=i}^{t}\upsilon_{ijt}-\sum_{i=1}^{t+1}\sum_{j=t+1}^{T}\upsilon_{i,t+1,j}=0 \quad \text{for } 1\le t\le T-1,$$

$$\sum_{i=t}^{l}\omega_{til}-\sum_{i=l+1}^{T}\omega_{t,l+1,i}-\sum_{i=1}^{t}\upsilon_{itl}=0 \quad \text{for } 1\le t\le l\le T,$$

$$\upsilon_{ijk}\in\mathbb{R}_+ \quad \text{for } 1\le i\le j\le k\le T,$$

$$\omega_{ijk}\in\mathbb{R}_+ \quad \text{for } 1\le i\le j\le k\le T.$$

D. Multiple Items. Constant Capacity. Joint Set-up Costs

$$\begin{aligned} s^i_{t-1} + x^i_t &= d^i_t + s^i_t \ \forall\ i,t \\ \sum_i x^i_t &\le QY_t \ \forall\ t \\ s, x \in \mathbb{R}^{IT}_+, Y &\in \mathbb{Z}^T_+. \end{aligned}$$

Consider a surrogate item consisting of the aggregation of all the items in $V \subseteq \{1, \dots, m\}$.

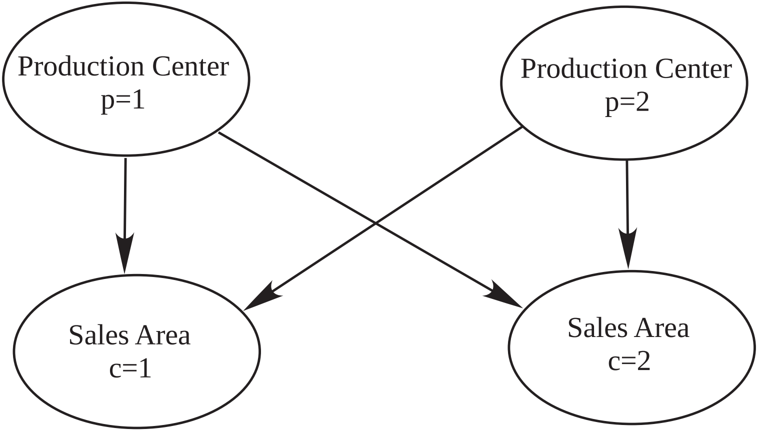
Let $X^V_t = \sum_{i \in V} x^i_t$, $S^V_t = \sum_{i \in V} s^i_t$ and $D^V_t = \sum_{i \in V} d^i_t$ be surrogate variables and demands. We obtain:

$S^V_{k-1} + Q \sum_{u=k}^t Y_u \ge D^V_{kt} \quad 1 \le k \le t \le n.$

Add $\mathrm{conv}(X^{WW-CC})$ for the m surrogate items
$V = \{1\}, \{1,2\}, \cdots, \{1,2,\dots,m\}$.
Optimal integer solution for typical Wagner-Whitin type costs.

E. The multi-item/multi-warehouse/multi-client problem

Two-level problem with multiple items, warehouses and clients. $I \geq 1$ is the number of items, $P \geq 1$ is the number of production sites (warehouses) and $C \geq 1$ is the number of clients.



A basic MIP formulation is

$$
\begin{aligned}
\min \textstyle\sum_{i,p,t} p_t^{0ip} x_t^{0ip} + \sum_{i,p,t} q_t^{0ip} y_t^{0ip} + \sum_{i,p,c,t} p_t^{1ipc} x_t^{1ipc} + \sum_{p,c,t} f_t^{1pc} Y_t^{1pc} \\
s_{t-1}^{0ip} + x_t^{0ip} = \textstyle\sum_{c=1}^{C} x_t^{1ipc} + s_t^{0ip} \quad \text{for } 1 \le i \le I,\ 1 \le p \le P,\ 1 \le t \le T, \\
s_{t-1}^{1ic} + \textstyle\sum_{p=1}^{P} x_t^{1ipc} = d_t^{i,c} + s_t^{1ic} \quad \text{for } 1 \le i \le I,\ 1 \le c \le C, 1 \le t \le T, \\
x_t^{0ip} \le M y_t^{0ip} \quad \text{for } 1 \le i \le I,\ 1 \le p \le P, 1 \le t \le T, \\
\textstyle\sum_{i=1}^{I} x_t^{1ipc} \le Q Y_t^{1pc} \quad \text{for } 1 \le p \le P,\ 1 \le c \le C, 1 \le t \le T, \\
s^0, x^0 \in \mathbb{R}_+^{I \times P \times T},\ s^1 \in \mathbb{R}^{I \times C \times T},\ x^1 \in \mathbb{R}^{I \times P \times C \times T}, \\
y^0 \in \{0,1\}^{I \times P \times T},\ Y^1 \in \{0,1\}^{P \times C \times T}.
\end{aligned}
$$

Uncapacitated when $Q = M$ (large trucks can transport all the items)

Some Computational Results: Single Production Site, Uncapacitated

Group	Dimensions
G4	$I = 5,\ C = 10,\ T = 15$
G5	$I = 5,\ C = 20,\ T = 18$
G6	$I = 20,\ C = 10,\ T = 15$

Table: Instances

	Standard			Echelon stock				MC		New
	LP	XLP	Gap	LP	XLP	Gap	Sec	LP	Sec	Sec
G4	68	83	21	99.3	99.8	0	41	100	2	13
G5	72	86	20	99.3	99.6	3.5	300	100	22	129
G6	58	80	22	99.4	99.7	3.7	300	100[a]	75	247

Table: Results for instances with multiple items and multiple clients

And Then?

- Conclusion 1: Best to use Multi-commodity Formulation - add indices i, c, p, q to each variable!
- Conclusion 2: Use the multi-item family set-up model D to deal with vehicle capacities

What about larger problems?

- Use an approximate multi-commodity formulation to handle larger problems
- Improve separation of valid inequalities in the original space
- Other extensions: sales, etc.

Computation: Capacitated Vehicles

Capacitated Vehicles, Single Production Site
5 items, 12 clients, 12 periods

	LP	XLP	BLB	BIP	gap	secs	nodes
Xpress	6662	7277	7809	8749	10.7	300	6542
mc	8461	8464	8492	8563	0.8	300	1301
mc+WW-CC	8490	8539	8559	8559	0	29	323

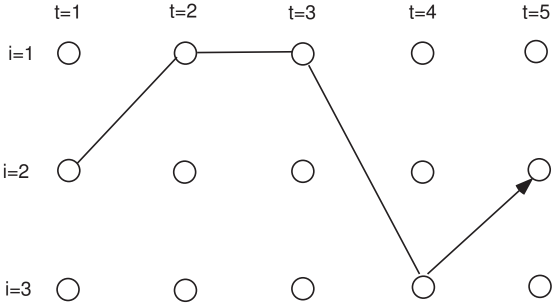
Table: Capacities at transportation level

F. The multi-item/**multi-machine**/changeover cost problem

I is the number of items, K is the number of machines. One item is produced per period, and (later) discrete/all or nothing production.

$$\min \sum_{i,k} p_t^{ik} x_t^{ik} + \sum_i h_t^i s_t^i + \sum_{i,j,k,t} q^{ijk} \chi_t^{ijk}$$
$$s_t^i + \sum_k C y_t^{ik} = d_t^i + s_t^i \ \forall\ i,t$$
$$\sum_i y_t^{ik} = 1 \ \forall\ k,t$$
$$\chi_t^{ijk} \geq y_{t-1}^{ik} + y_t^{jk} - 1 \ \forall\ i,j,k,t$$
$$s \in \mathbb{R}_+^1, x \in \{0,1\}, \chi \in \{0,1\}$$

Improved Formulation of Changeovers for Machine k



$$\sum_i y_1^i = 1$$

$$\sum_i \chi_t^{ij} = y_t^j \;\forall\, j,t$$

$$\sum_j \chi_t^{ij} = y_{t-1}^i \;\forall\, j, t>1$$

$$0 \leq y_t^i, \chi_t^{ij} \;\forall\, i,j,t.$$

Linking Changeovers and Lot-Sizing

Define the start-up and switch-off variables in terms of the changeover variables

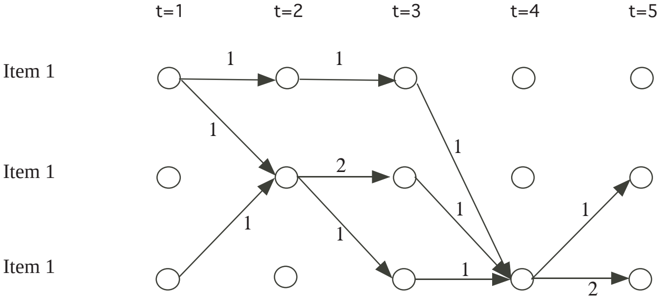
$$z_t^j = \sum_{i:i\neq j} \chi_t^{ij} \text{ and } w_{t-1}^i = \sum_{j:j\neq i} \chi_t^{ij}.$$

Now can use single item models with stocks, set-up and start-up variables z_t^j.
Use relaxations $WW-U-SC$ and $WW-CC$ for which the convex hull is known and relatively compact, and $DLSI-CC-SC$ assuming $d_t^i \in \{0,1\}$.

Symmetry-breaking?

K Identical Machines and Discrete Lot-Sizing

Let $Y_t^i = \sum_{k=1}^K y_t^{ik}, Z_t^i = \sum_{k=1}^K z_t^{ik}$.
New flow model with integer flows between 0 and K, and demands that are integer between 0 and K. This leads to a much more compact model.



Obs 1: Same LP value for integer and disaggregated models.
Obs 2: Integer flows decompose into a flow on each machine.

Subproblem is discrete batching with start-ups

$$
\begin{aligned}
s^i_{t-1} + Y^i_t &= d^i_t + s^i_t \\
Y_t \geq Z_t &\geq Y_t - Y_{t-1} \\
S \in \mathbb{R}^T_+, Y_t, Z_t &\in \{0, 1, \ldots, K\}
\end{aligned}
$$

Batch start-up Inequalities presented above.
Extended formulation due to Eppen and Martin - very large
Vanderbeck and Wolsey - cutting planes -special purpose
separation heuristics

Computation: identical machines, start-ups

DLS-Identical parallel Machines - Integer Start-Ups
5 machines, 10 items, 60 periods

	LP	XLP	BLB	BIP	gap	secs	nodes
XPress	5060	8723	8979	10150	11.5	300	27200
VI	9722	9847	98916	9891	0	25	87
EF	9883	9889	9891	9891	0	214	1

Table: Start-up Costs

How to use large extended formulations?

- Up to a certain size, the large extended formulation solves the most rapidly, because it provides a very tight LP bound and requires very few nodes in the tree.
- At a certain stage, the LP bound is still very good, but resolving in the branch-and-bound tree takes too long, and very few nodes are enumerated. Adding cuts in the original space produces a weaker bound, but good feasible solutions can be found in the tree.
- At a certain stage, even the LP becomes too big.

Some possible solutions

- Use (approximate) extended formulation for dual bound, and tightened weaker reformulation for primal bounds.
- Use LP solution of extended formulation for variable fixing to find good feasible solutions with the weaker reformulation.
- Use extended formulation for separation while running the weaker formulation.
- Use LP solution of extended formulation to separate different points.(yoyo - Fischetti)
- Use bound of extended formulation in running the weaker formulation. How?

I=10,K=2,T=150 Formulations and Heuristics

	LP	XLP	BLB	BIP	Gap%	Sec	nodes	m	n
Xpress	8968	11325	11621	20611	43.6	600	19500	4	4
VI(30)	15246	15560	15642	16526	5.3	600	5400	9	4
EF	15889	15889				600	0	139	267
EFA(10)	15881	15890				600	0	98	178

Combining Formulations

	LB	UB	Gap%	secs
EF-VI	15899	17011	6.6	300
Heur-EF-VI	15899	16478	3.6	300

Separation with Extended Formulation

LB	LP(10)	XLP	BIP(300)
14246	15514	15591	16477

Heuristic $I = 10, K = 10, T = 50$

Extended Formulation Approx with $\delta = 20$ for initial LP and $Tk = 20$ for Batch Inequalities

I	K	T	instance	LB	BIP	gap	secs1	secs2
10	10	50	pb21	15622	15763	0.9%	105	3
10	10	50	pb22	16127	16314	1.1%	89	1
10	10	50	pb23	14072	14260	1.3%	105	7
10	10	50	pb24	13427	13588	1.2%	85	2
10	10	50	pb25	14005	14075	0.5%	97	6

To conclude

- MIRs and Wagner-Whitin (Mixing) sets are everywhere in lot-sizing.
- Extended formulations are great fun and very useful.
- Approximate extended formulations can help.
- How to combine what is good for the primal with what is good for the dual?

Thank you for your attention.