Multi-Row Cutting Planes

Corner Polyhedron, Intersection Cuts,

Maximal Lattice-Free Convex Sets

Gérard Cornuéjols

Tepper School of Business
Carnegie Mellon University, Pittsburgh

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Mixed Integer Linear Programming

$$\begin{array}{lll} \min & cx & \\ \text{s.t.} & Ax = b & \\ & x_j \in \mathbb{Z} & \text{for } j = 1, \ldots, p \\ & x_j \geq 0 & \text{for } j = 1, \ldots, n. \end{array}$$

Common approach to solving MILP:

- First solve the LP relaxation. Basic optimal tableau:

$$x_i \quad = \quad \bar{b}_i - \sum_{j \in N} \bar{a}_{ij} x_j \quad \text{for } i \in B.$$

- If $\bar{b}_i \notin \mathbb{Z}$ for some $i \in B \cap \{1, \ldots, p\}$, add cutting planes:

For example Gomory 1963 Mixed Integer Cuts, Marchand and Wolsey 2001 MIR inequalities, Balas, Ceria and Cornuéjols 1993 lift-and-project cuts, are used in commercial codes.

Corner Polyhedron [Gomory 1969]

Initial formulation:

$$\begin{array}{rcll} x_i & = & \bar{b}_i - \sum_{j \in N} \bar{a}_{ij} x_j & \text{for } i \in B. \\ x_j & \in & \mathbb{Z} & \text{for } j = 1, \ldots, p \\ x_j & \geq & 0 & \text{for } j = 1, \ldots, n. \end{array}$$

Corner formulation:

The key idea, introduced by Gomory in the late 1960s, is to drop the nonnegativity restriction on all the basic variables x_i, $i \in B$.

Note that in this relaxation we can drop the constraints $x_i = \bar{b}_i - \sum_{j \in N} \bar{a}_{ij} x_j$ for all $i \in B \cap \{p+1, \ldots, n\}$ since these variables x_i are continuous and only appear in one equation and no other constraint. Therefore from now on we assume that all basic variables in are integer, i.e. $B \subseteq \{1, \ldots, p\}$.

Corner Polyhedron

Therefore the *corner formulation* introduced by Gomory is

$$\begin{array}{rcll} x_i & = & \bar{b}_i - \sum_{j \in N} \bar{a}_{ij} x_j & \text{for } i \in B \\ x_i & \in & \mathbb{Z} & \text{for } i = 1, \ldots, p \\ x_j & \geq & 0 & \text{for } j \in N. \end{array}$$

The convex hull of its feasible solutions is called the *corner polyhedron* relative to the basis B and it is denoted by corner(B).

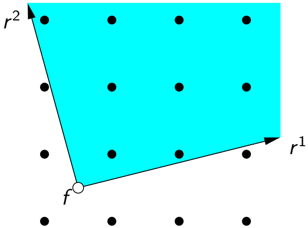
Note Any valid inequality for the corner polyhedron is valid for the initial formulation.

Let $P(B)$ be the linear relaxation of the corner polyhedron.
$P(B)$ is a polyhedron whose vertices and extreme rays are simple to describe and this will be useful in generating valid inequalities for corner(B).

Corner Polyhedron

Relax nonnegativity on basic variables x_i.

Example



Feasible set $\left\{ \begin{pmatrix} x_3 \\ x_4 \end{pmatrix} \in \mathbb{Z}^2 : \right.$

$$\begin{pmatrix} x_3 \\ x_4 \end{pmatrix} = f + r^1 x_1 + r^2 x_2$$

$$\left. \text{where} \quad x_1 \geq 0, x_2 \geq 0 \right\}$$

Restricted to the (x_3, x_4)-space, $P(B)$ is the blue region. The feasible solutions are the integer points in the blue region, and corner(B) is the convex hull of these points.

The Polyhedron $P(B)$

$P(B)$ has a unique vertex $\bar{x}$ where $\bar{x}_i = \bar{b}_i, i \in B,\ x_j = 0, j \in N$. The recession cone of $P(B)$ is

$$\begin{array}{rcll} x_i & = & -\sum_{j \in N} \bar{a}_{ij} x_j & \text{for } i \in B \\ x_j & \geq & 0 & \text{for } j \in N. \end{array}$$

Since the projection of this cone onto $\mathbb{R}^N$ is defined by the inequalities $x_j \geq 0$, $j \in N$, its extreme rays are the vectors satisfying at equality all but one nonnegativity constraints. Thus there are $|N|$ extreme rays, $\bar{r}^j$ for $j \in N$, defined by

$$\bar{r}^j_h = \begin{cases} -\bar{a}_{hj} & \text{if } h \in B, \\ 1 & \text{if } j = h, \\ 0 & \text{if } h \in N \setminus \{j\}. \end{cases}$$

Remark The vectors $\bar{r}^j,\ j \in N$ are linearly independent. Hence $P(B)$ is an $|N|$-dimensional polyhedron whose affine hull is defined by the equations $x_i = \bar{b}_i - \sum_{j \in N} \bar{a}_{ij} x_j$ for $i \in B$.

The Polyhedron $P(B)$

$$\begin{array}{rcll} x_i & = & \bar{b}_i - \sum_{j \in N} \bar{a}_{ij} x_j & \text{for } i \in B \\ x_j & \geq & 0 & \text{for } j \in N. \end{array}$$

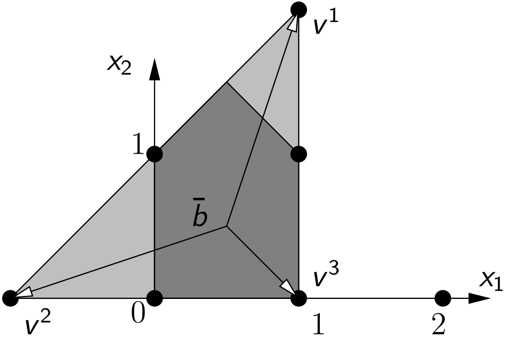
Lemma Assume all data are rational. If the affine hull of $P(B)$ contains a point in $\mathbb{Z}^p \times \mathbb{R}^{n-p}$, then corner$(B)$ is an $|N|$-dimensional polyhedron. Otherwise corner(B) is empty.

Proof left as an exercise.

Corner Polyhedron Example

Consider the pure integer program

$$\begin{array}{rcl} \max \frac{1}{2}x_2 + x_3 & & \\ x_1 + x_2 + x_3 & \leq & 2 \\ x_1 \quad - \frac{1}{2}x_3 & \geq & 0 \\ x_2 - \frac{1}{2}x_3 & \geq & 0 \\ x_1 \quad + \frac{1}{2}x_3 & \leq & 1 \\ -x_1 + x_2 + x_3 & \leq & 1 \\ x_1, x_2, x_3 & \in & \mathbb{Z} \\ x_1, x_2, x_3 & \geq & 0. \end{array}$$

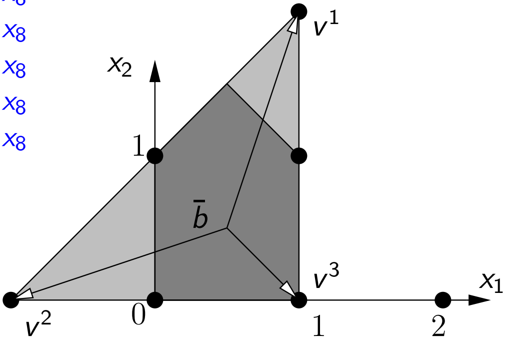


This problem has 4 feasible solutions $(0,0,0)$, $(1,0,0)$, $(0,1,0)$ and $(1,1,0)$, all satisfying $x_3 = 0$. The intersection of the 5 inequalities in the formulation with the plane $x_3 = 0$ is the darker region in the figure.

Corner Polyhedron Example

We first write the problem in standard form by introducing continuous slack or surplus variables $x_4, \ldots, x_8$. Solving the LP relaxation, we get

$$
\begin{array}{lcllll}
x_1 & = & \frac{1}{2} & +\frac{1}{4}x_6 & -\frac{3}{4}x_7 & +\frac{1}{4}x_8 \\
x_2 & = & \frac{1}{2} & +\frac{3}{4}x_6 & -\frac{1}{4}x_7 & -\frac{1}{4}x_8 \\
x_3 & = & 1 & -\frac{1}{2}x_6 & -\frac{1}{2}x_7 & -\frac{1}{2}x_8 \\
x_4 & = & 0 & -\frac{1}{2}x_6 & +\frac{3}{2}x_7 & +\frac{1}{2}x_8 \\
x_5 & = & 0 & +\frac{1}{2}x_6 & -\frac{1}{2}x_7 & +\frac{1}{2}x_8
\end{array}
$$

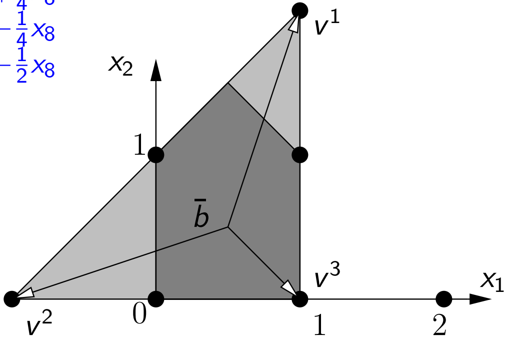


The optimal basic solution is $x_1 = x_2 = \frac{1}{2}$, $x_3 = 1$, $x_4 = \ldots = x_8 = 0$.

Corner Polyhedron Example

Relaxing the nonnegativity of the basic variables and dropping the two constraints relative to the continuous basic variables x_4 and x_5, we obtain the corner formulation:

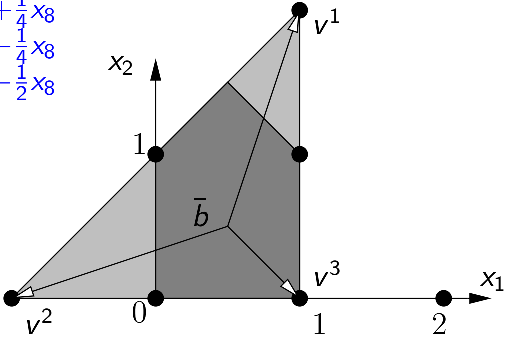
$$
\begin{array}{rcrrrr}
x_1 & = & \frac{1}{2} & +\frac{1}{4}x_6 & -\frac{3}{4}x_7 & +\frac{1}{4}x_8 \\
x_2 & = & \frac{1}{2} & +\frac{3}{4}x_6 & -\frac{1}{4}x_7 & -\frac{1}{4}x_8 \\
x_3 & = & 1 & -\frac{1}{2}x_6 & -\frac{1}{2}x_7 & -\frac{1}{2}x_8 \\
x_1, & x_2, & x_3 & & \in & \mathbb{Z} \\
x_6, & x_7, & x_8 & & \geq & 0.
\end{array}
$$



Corner Polyhedron Example

Let $P(B)$ be the linear relaxation of corner formulation. The projection of $P(B)$ in the space of original variables x_1, x_2, x_3 is a polyhedron with unique vertex $\bar{b} = (\frac{1}{2}, \frac{1}{2}, 1)$. The extreme rays of its recession cone are $v^1 = (\frac{1}{2}, \frac{3}{2}, -1)$, $v^2 = (-\frac{3}{2}, -\frac{1}{2}, -1)$ and $v^3 = (\frac{1}{2}, -\frac{1}{2}, -1)$.

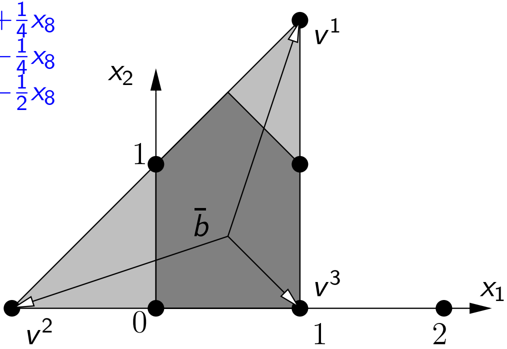
$$
\begin{array}{rcrrrr}
x_1 & = & \frac{1}{2} & +\frac{1}{4}x_6 & -\frac{3}{4}x_7 & +\frac{1}{4}x_8 \\
x_2 & = & \frac{1}{2} & +\frac{3}{4}x_6 & -\frac{1}{4}x_7 & -\frac{1}{4}x_8 \\
x_3 & = & 1 & -\frac{1}{2}x_6 & -\frac{1}{2}x_7 & -\frac{1}{2}x_8 \\
x_6, & x_7, & x_8 & \geq & 0. &
\end{array}
$$



Corner Polyhedron Example

In the figure, the shaded region (both light and dark) is the intersection of $P(B)$ with the plane $x_3 = 0$.

$$
\begin{array}{ccccccc}
x_1 & = & \frac{1}{2} & +\frac{1}{4}x_6 & -\frac{3}{4}x_7 & +\frac{1}{4}x_8 \\
x_2 & = & \frac{1}{2} & +\frac{3}{4}x_6 & -\frac{1}{4}x_7 & -\frac{1}{4}x_8 \\
x_3 & = & 1 & -\frac{1}{2}x_6 & -\frac{1}{2}x_7 & -\frac{1}{2}x_8 \\
x_6, & x_7, & x_8 & & \geq & & 0.
\end{array}
$$



Let P be defined by the inequalities of the initial formulation that are satisfied at equality by the point $\bar{b} = (\frac{1}{2}, \frac{1}{2}, 1)$. The intersection of P with the plane $x_3 = 0$ is the dark shaded region.

Corner Polyhedron

Since variables $x_i, \in B$ are free integer variables, the corner formulation can be reformulated as follows

$$\begin{array}{rcll} \sum_{j\in N} \bar{a}_{ij}x_j & \equiv & \bar{b}_i \mod 1 & \text{for } i \in B \\ x_j & \in & \mathbb{Z} & \text{for } j \in \{1,\ldots,p\} \cap N \\ x_j & \geq & 0 & \text{for } j \in N. \end{array}$$

This point of view was introduced by Gomory and extensively studied by Gomory and Johnson in the 1970's.

Intersection Cuts [Balas 1971]

Inequalities that are valid for the corner polyhedron and that cut off the basic solution $\bar{x}$.

Consider a closed convex set $C \subseteq \mathbb{R}^n$ such that $\bar{x} \in \text{int}(C)$.
Assume that the interior of C contains no point in $\mathbb{Z}^p \times \mathbb{R}^{n-p}$.

For each of the $|N|$ extreme rays of $\text{corner}(B)$, define

$$\alpha_j = \max\{\alpha \geq 0 : \ \bar{x} + \alpha \bar{r}^j \in C\}.$$

Since $\bar{x}$ is in the interior of C, $\alpha_j > 0$.
When the half-line $\{\bar{x} + \alpha \bar{r}^j : \alpha \geq 0\}$ intersects the boundary of C, then α_j is finite, the point $\bar{x} + \alpha_j \bar{r}^j$ belongs to the boundary of C.
When $\bar{r}_j$ belongs the recession cone of C, we have $\alpha_j = +\infty$.
Define $\frac{1}{+\infty} = 0$. The inequality

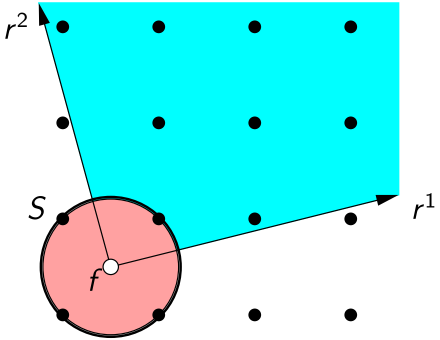
$$\sum_{j \in N} \frac{x_j}{\alpha_j} \geq 1$$

is the *intersection cut* defined by C.

Intersection Cuts

Assume $f \notin \mathbb{Z}^2$.

Want to cut off the basic solution $\begin{pmatrix} x_3 \\ x_4 \end{pmatrix} = f,\ x_1 = 0, x_2 = 0$.



Feasible set $\{\begin{pmatrix} x_3 \\ x_4 \end{pmatrix} \in \mathbb{Z}^2 :$

$$\begin{pmatrix} x_3 \\ x_4 \end{pmatrix} = f + r^1 x_1 + r^2 x_2$$

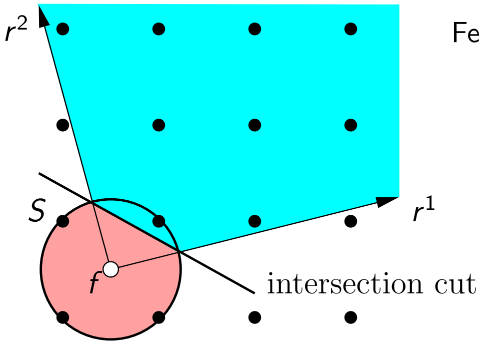
where $x_1 \geq 0, x_2 \geq 0\}$

Any convex set S with $f \in int(S)$ and no integer point in $int(S)$.

Intersection Cuts

Assume $f \notin \mathbb{Z}^2$.

Want to cut off the basic solution $\begin{pmatrix} x_3 \\ x_4 \end{pmatrix} = f,\ x_1 = 0, x_2 = 0$.



Feasible set $\{\begin{pmatrix} x_3 \\ x_4 \end{pmatrix} \in \mathbb{Z}^2 :$

$$\begin{pmatrix} x_3 \\ x_4 \end{pmatrix} = f + r^1 x_1 + r^2 x_2$$

where $x_1 \geq 0, x_2 \geq 0\}$

Any convex set S with $f \in int(S)$ and no integer point in $int(S)$.
Intersection cut is obtained by intersecting the rays with the boundary of S: $\alpha_1 = \frac{1}{4},\ \alpha_2 = \frac{1}{4}$. Thus $4x_1 + 4x_2 \geq 1$.

Intersection Cuts

The *corner formulation* introduced by Gomory is

$$\begin{array}{rcll} x_i & = & \bar{b}_i - \sum_{j\in N} \bar{a}_{ij} x_j & \text{for } i \in B \\ x_i & \in & \mathbb{Z} & \text{for } i = 1,\ldots,p \\ x_j & \geq & 0 & \text{for } j \in N. \end{array}$$

Basic solution $\bar{x}$ where $\bar{x}_i = \bar{b}_i, i \in B,\ x_j = 0, j \in N$.

Assume $B \subseteq \{1,\ldots,p\}$ and $\bar{b} \notin \mathbb{Z}^{|B|}$.

THEOREM Let $C \subset \mathbb{R}^n$ be a closed convex set whose interior contains the point $\bar{x}$ but no point in $\mathbb{Z}^p \times \mathbb{R}^{n-p}$. The intersection cut defined by C is a valid inequality for $\mathrm{corner}(B)$.

Intersection Cuts

THEOREM Let $C \subset \mathbb{R}^n$ be a closed convex set whose interior contains the point $\bar{x}$ but no point in $\mathbb{Z}^p \times \mathbb{R}^{n-p}$. The intersection cut defined by C is a valid inequality for $\mathrm{corner}(B)$.

PROOF The set of points of $P(B)$ cut off by the intersection cut is $S := \{x \in \mathbb{R}^n : x_i = \bar{b}_i - \sum_{j \in N} \bar{a}_{ij} x_j$ for $i = 1, \ldots, q,$ $x_j \geq 0, j \in N, \sum_{j \in N} \frac{x_j}{\alpha_j} < 1\}$. We will show that S is contained in the interior of C. Since the interior of C does not contain a point in $\mathbb{Z}^p \times \mathbb{R}^{n-p}$, the theorem will follow.

Intersection Cuts

PROOF We need to show that $S := \{x \in \mathbb{R}^n : x_i = \bar{b}_i - \sum_{j \in N} \bar{a}_{ij} x_j$ for $i = 1, \ldots, q$, $x_j \geq 0$, $j \in N$, $\sum_{j \in N} \frac{x_j}{\alpha_j} < 1\}$ is contained in the interior of C.

Consider the polyhedron $\bar{S} := \{x \in \mathbb{R}^n : x_i = \bar{b}_i - \sum_{j \in N} \bar{a}_{ij} x_j$ for $i = 1, \ldots, q$, $x_j \geq 0$, $j \in N$, $\sum_{j \in N} \frac{x_j}{\alpha_j} \leq 1\}$.
$\bar{S}$ is a $|N|$-dimensional polyhedron with vertices $\bar{x}$ and $\bar{x} + \alpha_j \bar{r}^j$ for α_j finite, and extreme rays $\bar{r}_j$ for $\alpha_j = +\infty$.
Since the vertices of $\bar{S}$ that lie on the hyperplane $\{x \in \mathbb{R}^n : \sum_{j \in N} \frac{x_j}{\alpha_j} = 1\}$ are the points $\bar{x} + \alpha_j \bar{r}^j$ for α_j finite, every point in S can be expressed as a convex combination of points in the segments $\{\bar{x} + \alpha \bar{r}^j,\ 0 \leq \alpha < 1\}$ for α_j finite, plus a conic combination of extreme rays $\bar{r}_j$, for $\alpha_j = +\infty$.
Since the interior of C contains the segments $\{\bar{x} + \alpha \bar{r}^j,\ 0 \leq \alpha < 1\}$ for α_j finite, and the rays $\bar{r}_j$ for $\alpha_j = +\infty$ belong to the recession cone of C, the set S is contained in the interior of C. □

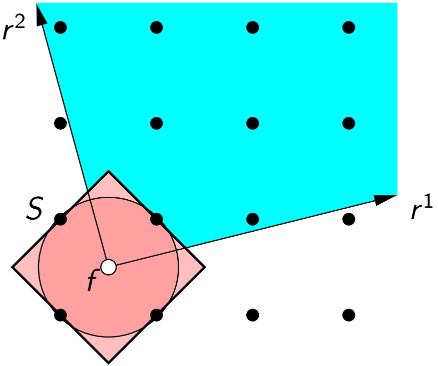
Intersection Cuts

Let $K \subseteq \mathbb{R}^n$ be a closed, convex set with the origin in its interior. A standard concept in convex analysis (Minkowski, Rockafellar) is that of *gauge* (sometimes called Minkowski function), which is the function γ_K defined by

$$\gamma_K(r) = \inf\{t > 0 \,:\, r \in tK\}, \qquad \text{for } r \in \mathbb{R}^n.$$

For a scalar $t > 0$, the set tK is a scaled version of K, namely $\{y = tx : \ x \in K\}$. In words, the gauge $\gamma_K(r)$ is the smallest factor t such that the scaled set tK contains the point r. The coefficients α_j of the intersection cut can be expressed in terms of the gauge of $C - \bar{x}$, namely $\frac{1}{\alpha_t} = \gamma_{C-\bar{x}}(\bar{r}_j)$. Intersection cuts can therefore be written as $\sum_{j \in N} \gamma_{C-\bar{x}}(\bar{r}_j) x_j \geq 1$.

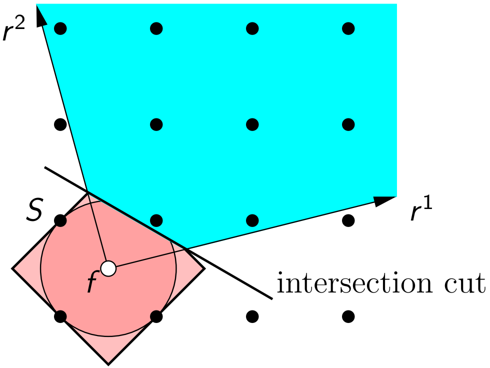
A Better Intersection Cut for our Example



Bigger convex set:

Octahedron $f \in int(S)$ with no integral point in $int(S)$.

A Better Intersection Cut for our Example



Bigger convex set:

Octahedron $f \in int(S)$ with no integral point in $int(S)$.

Better cut: $\alpha_1 = \frac{1}{3},\ \alpha_2 = \frac{1}{3}$. Thus $3x_1 + 3x_2 \geq 1$.

Intersection Cuts

- Let C_1, C_2 be two closed convex sets whose interiors contain $\bar{x}$ but no point of $\mathbb{Z}^p \times \mathbb{R}^{n-p}$. If $C_1 \subset C_2$, then the intersection cut relative to C_2 dominates the intersection cut relative to C_1 for all $x \in \mathbb{R}^n$ such that $x_j \geq 0$, $j \in N$.

- A closed convex set C whose interior contains $\bar{x}$ but no point of $\mathbb{Z}^p \times \mathbb{R}^{n-p}$ is *maximal* if C is not strictly contained in a closed convex set with the same properties. Any closed convex set whose interior contains $\bar{x}$ but no point of $\mathbb{Z}^p \times \mathbb{R}^{n-p}$ is contained in a maximal such set.

- One way of constructing a closed convex set C whose interior contains $\bar{x}$ but no point of $\mathbb{Z}^p \times \mathbb{R}^{n-p}$ is the following.
A set $K \subset \mathbb{R}^p$ that contains no point of $\mathbb{Z}^p$ in its interior is called *$\mathbb{Z}^p$-free*. In the space $\mathbb{R}^p$, construct a $\mathbb{Z}^p$-free closed convex set K whose interior contains the projection of $\bar{x}$. The cylinder $C = K \times \mathbb{R}^{n-p}$ is a closed convex set whose interior contains $\bar{x}$ but no point of $\mathbb{Z}^p \times \mathbb{R}^{n-p}$.

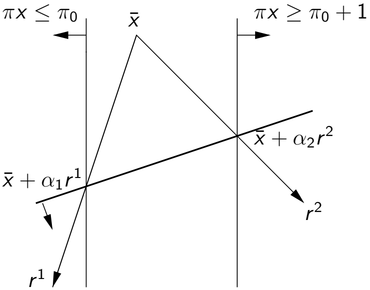
EXAMPLE: Intersection Cuts from Split Disjunctions

Consider a split disjunction $\pi x \leq \pi_0$ or $\pi x \geq \pi_0 + 1$, where $\pi \in \mathbb{Z}^p \times \{0\}^{n-p}$ and $\pi_0 \in \mathbb{Z}$.
$K := \{x \in \mathbb{R}^p \, : \, \pi_0 \leq \sum_{j=1}^{p} \pi_j x_j \leq \pi_0 + 1\}$ is $\mathbb{Z}^p$-free and convex, while the set $C := K \times \mathbb{R}^{n-p}$ is $\{x \in \mathbb{R}^n \, : \, \pi_0 \leq \pi x \leq \pi_0 + 1\}$.
Assume $\bar{x} \in \mathrm{int}(C)$.

Define $\epsilon := \pi \bar{x} - \pi_0$. We have $0 < \epsilon < 1$. For $j \in N$, define:

$$\alpha_j := \begin{cases} -\frac{\epsilon}{\pi \bar{r}^j} & \text{if } \pi \bar{r}^j < 0, \\ \frac{1-\epsilon}{\pi \bar{r}^j} & \text{if } \pi \bar{r}^j > 0, \\ +\infty & \text{otherwise,} \end{cases}$$

where $\bar{r}^j$ are the rays of $P(B)$.



Intersection cut $\quad \sum_{j \in N} \frac{x_j}{\alpha_j} \geq 1$.

Gomory Mixed Integer Cuts from the Tableau

Let $x_i, i \in B$ be a basic integer variable, and suppose $\bar{x}_i = \bar{b}_i$ is fractional. We define $\pi_0 := \lfloor \bar{x}_i \rfloor$, and for $j = 1, \ldots, p$,

$$\pi_j := \begin{cases} \lfloor \bar{a}_{ij} \rfloor & \text{if } j \in N \text{ and } f_j \leq f_0, \\ \lceil \bar{a}_{ij} \rceil & \text{if } j \in N \text{ and } f_j > f_0, \\ 1 & \text{if } j = i, \\ 0 & \text{otherwise.} \end{cases}$$

For $j = p+1, \ldots, n$, we define $\pi_j := 0$.

Next we derive the intersection cut from the split disjunction $\pi x \leq \pi_0$ or $\pi x \geq \pi_0 + 1$ as shown in the previous slide. We need to compute $\alpha_j,\ j \in N$ using our formula:

$$\alpha_j := \begin{cases} -\frac{\epsilon}{\pi \bar{r}^j} & \text{if } \pi \bar{r}^j < 0, \\ \frac{1-\epsilon}{\pi \bar{r}^j} & \text{if } \pi \bar{r}^j > 0, \\ +\infty & \text{otherwise,} \end{cases}$$

where $\bar{r}^j$ are the rays of $P(B)$.

Gomory Mixed Integer Cuts from the Tableau

Let $f_0 = \bar{b}_i - \lfloor \bar{b}_i \rfloor$ and $f_j = \bar{a}_{ij} - \lfloor \bar{a}_{ij} \rfloor$. We have

$$\epsilon = \pi\bar{x} - \pi_0 = \sum_{h \in B} \pi_h \bar{x}_h - \pi_0 = \bar{x}_i - \lfloor \bar{x}_i \rfloor = f_0.$$

Let $j \in N$. We have $\pi\bar{r}^j = \pi_j \bar{r}^j_j + \pi_i \bar{r}^j_i$ since $\bar{r}^j_h = 0$ for all $h \in N \setminus \{j\}$ and $\pi_h = 0$ for all $h \in B \setminus \{i\}$. Therefore

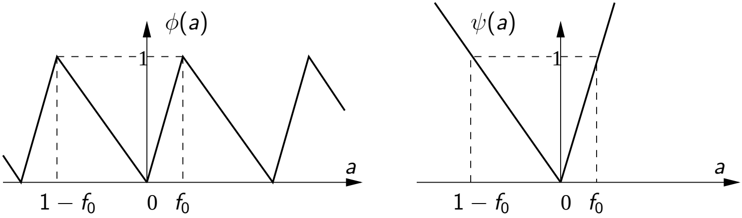
$$\pi\bar{r}^j = \begin{cases} \lfloor \bar{a}_{ij} \rfloor - \bar{a}_{ij} \quad = -f_j & \text{if } 1 \le j \le p \text{ and } f_j \le f_0, \\ \lceil \bar{a}_{ij} \rceil - \bar{a}_{ij} \quad = 1 - f_j & \text{if } 1 \le j \le p \text{ and } f_j > f_0, \\ -\bar{a}_{ij} & \text{if } j \ge p+1. \end{cases}$$

Now α_j follows. Therefore the intersection cut associated with the split disjunction $\pi x \le \pi_0$ or $\pi x \ge \pi_0 + 1$ is

$$\sum_{\substack{j \in N,\, j \le p \\ f_j \le f_0}} \frac{f_j}{f_0} x_j + \sum_{\substack{j \in N,\, j \le p \\ f_j > f_0}} \frac{1 - f_j}{1 - f_0} x_j + \sum_{\substack{p+1 \le j \le n \\ \bar{a}_{ij} > 0}} \frac{\bar{a}_{ij}}{f_0} x_j - \sum_{\substack{p+1 \le j \le n \\ \bar{a}_{ij} < 0}} \frac{\bar{a}_{ij}}{1 - f_0} x_j \ge 1.$$

This is the GMI cut, since $f_i = 0$ for $i \in B$.

Gomory Functions



The Gomory formula looks complicated, and it may help to think of it as an inequality of the form

$$\sum_{j=1}^{p} \phi(\bar{a}_{ij})x_j + \sum_{j=p+1}^{n} \psi(\bar{a}_{ij})x_j \geq 1$$

where the functions ϕ and ψ, are

$$\phi(a) := \min\{\frac{f}{f_0}, \frac{1-f}{1-f_0}\} \quad \text{and} \quad \psi(a) := \max\{\frac{a}{f_0}, \frac{-a}{1-f_0}\}.$$

with $f = a - \lfloor a \rfloor$.

Intersection Cuts from Splits

This example shows that Gomory Mixed Integer cuts from the tableau are intersection cuts from split disjunctions.

A natural question is whether the same statement is true for Gomory Mixed Integer cuts derived from any linear combinations of the equality constraints in the initial formulation.

A theorem of Nemhauser and Wolsey shows that this family of cuts is exactly the family of split cuts.

So the question is whether every split cut can be derived as an intersection cut from a split disjunction. This was answered positively by Andersen, Cornuéjols and Li 2005. They show that split cuts are intersection cuts relative to some basis, where the $(\mathbb{Z}^p \times \mathbb{R}^{n-p})$-free convex set defining the intersection cut is a split.

Split Rank of Intersection Cuts

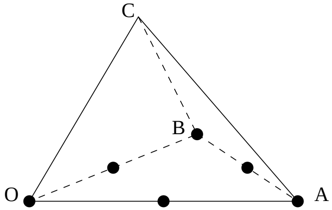
Intersection cuts can have arbitrarily large split rank (Cook, Kannan and Schrijver 1990).

Consider the polytope

$P := \{(x_1, x_2, y) \in \mathbb{R}^3_+ : \ x_1 \ \geq y,$
$x_2 \ \geq y, \ x_1 + x_2 + 2y \leq 2\}$,

and let

$S = \{(x_1, x_2, y) \in P \ : \ x_1, x_2 \in \mathbb{Z}\}$.



The corner polyhedron $\mathrm{corner}(B)$ is the convex hull of the points satisfying

$$\begin{aligned} x_1 &= \tfrac{1}{2} + \tfrac{3}{4}s_1 - \tfrac{1}{4}s_2 - \tfrac{1}{4}s_3 \\ x_2 &= \tfrac{1}{2} - \tfrac{1}{4}s_1 + \tfrac{3}{4}s_2 - \tfrac{1}{4}s_3 \\ & \quad s_1, s_2, s_3 \geq 0 \\ & \quad x_1, x_2 \in \mathbb{Z} \end{aligned}$$

Split Rank of Intersection Cuts

Let K be the triangle $\mathrm{conv}\{(0,0),(2,0),(0,2)\}$, and $C = K \times \mathbb{R}^3$. Since K is lattice-free, C defines an intersection cut.

One can verify that this intersection cut is $\frac{1}{2}s_1 + \frac{1}{2}s_2 + \frac{1}{2}s_3 \geq 1$.

Since $y = \frac{1}{2} - \frac{1}{4}s_1 - \frac{1}{4}s_2 - \frac{1}{4}s_3$, the intersection cut is equivalent to $y \leq 0$. Adding this single inequality to the initial formulation, we obtain $\mathrm{conv}(S)$.

Cook, Kannan and Schrijver showed that $y \leq 0$ does not have finite split rank.

Dey and Louveaux 2009 study the split rank of intersection cuts for problems with two integer variables. Surprisingly, they show that all intersection cuts have finite split rank except for the ones defined by lattice-free triangles with integral vertices and an integral point in the middle of each side. The triangle K defined above is of this type.

Intersection Cuts $\equiv$ Corner Polyhedron

We showed earlier that intersection cuts are valid for $\mathrm{corner}(B)$. The following theorem provides a converse statement.
We assume here that $\mathrm{corner}(B)$ is nonempty. Therefore $\mathrm{corner}(B)$ has dimension $|N|$.

Inequalities $\sum_{j \in N} \gamma_j x_j \geq \gamma_0$ with $\gamma_j \geq 0, j \in N$ and $\gamma_0 \leq 0$ are implied by the nonnegativity constraints $x_j \geq 0,\ j \in N$ and will be called *trivial*.

Every nontrivial valid inequality for $\mathrm{corner}(B)$ can be written in the form $\sum_{j \in N} \gamma_j x_j \geq 1$ with $\gamma_j \geq 0, j \in N$. (Exercise).

We say that such an inequality is *minimal* if there is no other valid inequality $\sum_{j \in N} \gamma'_j x_j \geq 1$ for $\mathrm{corner}(B)$ such that $\gamma'_j \leq \gamma_j$ for all $j \in N$, and the inequality is strict for at least one index $j \in N$.

THEOREM Let $\sum_{j \in N} \gamma_j x_j \geq 1$ be a nontrivial minimal valid inequality for $\mathrm{corner}(B)$ with rational coefficients. Then $\sum_{j \in N} \gamma_j x_j \geq 1$ is an intersection cut.

Intersection Cuts $\equiv$ Corner Polyhedron

PROOF

Consider the polyhedron

$$S = \{x \in \mathbb{R}^n : \begin{array}{l} \sum_{j\in N} \gamma_j x_j \leq 1, \\ x_j \geq 0 \text{ for } j \in N, \\ x_i = \bar{b}_i - \sum_{j\in N} \bar{a}_{ij} x_j \text{ for } i \in B\}. \end{array}$$

(1) No face of S containing $\bar{x}$ has a point of $\mathbb{Z}^p \times \mathbb{R}^{n-p}$ in its relative interior. (Exercise)

Let $\tilde{S} = S + L$ where $L = \{0\}^p \times \mathbb{R}^{n-p}$.
Since S is a rational polyhedron and the lineality space of $\tilde{S}$ contains L, $\tilde{S} = \{x \in \mathbb{R}^n : \sum_{j=1}^p c_j^i x_j \leq d_i,\ i = 1, \ldots, t\}$ for some integral vectors $c^1, \ldots, c^t \in \mathbb{Z}^p$ and $d_1, \ldots, d_t \in \mathbb{Z}$. (Indeed $\{x \in \mathbb{R}^p : \sum_{j=1}^p c_j^i x_j \leq d_i,\ i = 1, \ldots, t\}$ is the projection of S onto $\mathbb{R}^p$.)

Intersection Cuts $\equiv$ Corner Polyhedron

(2) No face of $\tilde{S}$ containing $\bar{x}$ has a point of $\mathbb{Z}^p \times \mathbb{R}^{n-p}$ in its relative interior.

Proof of (2)

Let $\tilde{F}$ be a face of $\tilde{S}$ and let $F = S \cap \tilde{F}$.

Then F is a face of S and $\tilde{F} = F + L$.

Therefore $\mathrm{relint}(\tilde{F}) = \mathrm{relint}(F) + L$ since L is in the lineality space of $\tilde{F}$.

Assume $\tilde{F}$ contains $\bar{x}$.

Since $\bar{x}$ belongs to S and $F = S \cap \tilde{F}$, we have $\bar{x} \in F$.

Suppose there exists $\tilde{x} \in \mathrm{relint}(\tilde{F}) \cap \mathbb{Z}^p \times \mathbb{R}^{n-p}$.

Then $\tilde{x} + L$ is contained in $\mathrm{relint}(\tilde{F}) \cap (\mathbb{Z}^p \times \mathbb{R}^{n-p})$.

Since $\mathrm{relint}(\tilde{F}) = \mathrm{relint}(F) + L$, we have $\tilde{x} + L$ contains a point in $\mathrm{relint}(F) \cap (\mathbb{Z}^p \times \mathbb{R}^{n-p})$, a contradiction to (1). This proves (2).

Intersection Cuts $\equiv$ Corner Polyhedron

(3) There exists a $\mathbb{Z}^p$-free convex set $K \subset \mathbb{R}^p$ such that the cylinder $C := K \times \mathbb{R}^{n-p}$ contains $\bar{x}$ in its interior and $\tilde{S} \subseteq C$.

Proof of (3): Assume, without loss of generality, that $\bar{x}$ satisfies at equality the first h constraints defining $\tilde{S}$ (possibly $h = 0$), and none of the other constraints. That is

$$\begin{array}{lcll} \sum_{j=1}^{p} c_j^i \bar{x}_j & = & d_i & i = 1, \dots, h; \\ \sum_{j=1}^{p} c_j^i \bar{x}_j & < & d_i & i = h+1, \dots, t. \end{array}$$

Define $d_i' = d_i + 1$ for $i = 1, \dots, h$ and $d_i' = d_i$ for $i = h+1, \dots, t$, and let $K = \{x \in \mathbb{R}^p : \sum_{j=1}^{p} c_j^i x_j \leq d_i', \text{ for } i = 1, \dots, t\}$ and $C = K \times \mathbb{R}^{n-p}$.

Note that $C = \{x \in \mathbb{R}^n : \sum_{j=1}^{p} c_j^i x_j \leq d_i', \text{ for } i = 1, \dots, t\}$. By construction, $\bar{x}$ is in the interior of C and $\tilde{S} \subseteq C$.

(2) implies that K is $\mathbb{Z}^p$-free. (Exercise)

This proves (3).

Intersection Cuts $\equiv$ Corner Polyhedron

For $h \in N$, let β_h be the largest β such that $\bar{x} + \beta \bar{r}^h$ is in S.
Since $1 = \sum_{j \in N} \gamma_j (\bar{x}_j + \beta_h \bar{r}^h_j) = \gamma_h \beta_h$, then $\gamma_h = \frac{1}{\beta_h}$.

Let α_j be the largest scalar such that $\bar{x} + \alpha_j \bar{r}^j$ is in C, $j \in N$.
Since $S \subseteq \tilde{S} \subseteq C$, $\alpha_j \geq \beta_j$ for every $j \in N$, hence $\gamma_j \geq \frac{1}{\alpha_j}$.
Therefore the intersection cut defined by C, namely $\sum_{j \in N} x_j / \alpha_j \geq 1$, dominates the inequality $\sum_{j \in N} \gamma_j x_j \geq 1$.
Since the latter is minimal, $\gamma_j = \frac{1}{\alpha_j}$, $j \in N$.

$\square$

COROLLARY Every nontrivial facet defining inequality for $\mathrm{corner}(B)$ is an intersection cut.

Exercises

EXERCISE 1 Assume all data are rational. Show that if the affine hull of $P(B)$ contains a point in $\mathbb{Z}^p \times \mathbb{R}^{n-p}$, then $\text{corner}(B)$ is an $|N|$-dimensional polyhedron.

EXERCISE 2 Show that every nontrivial valid inequality for $\mathrm{corner}(B)$ can be written in the form $\sum_{j \in N} \gamma_j x_j \geq 1$ with $\gamma_j \geq 0, j \in N$.

EXERCISE 3 (In the proof that every nontrivial minimal valid inequality for $\mathrm{corner}(B)$ with rational coefficients is an intersection cut.)
Show that no face of S containing $\bar{x}$ has a point of $\mathbb{Z}^p \times \mathbb{R}^{n-p}$ in its relative interior.

EXERCISE 4 (In the proof that every nontrivial minimal valid inequality for $\mathrm{corner}(B)$ with rational coefficients is an intersection cut.)
Show that K is $\mathbb{Z}^p$-free.

Maximal lattice-free convex sets

As observed earlier, the best possible intersection cuts are the ones defined by full-dimensional *maximal* $(\mathbb{Z}^p \times \mathbb{R}^{n-p})$*-free convex sets* in $\mathbb{R}^n$, that is, full-dimensional subsets of $\mathbb{R}^n$ that are convex, their interior contains no point in $\mathbb{Z}^p \times \mathbb{R}^{n-p}$, and are inclusionwise maximal with the above two properties.

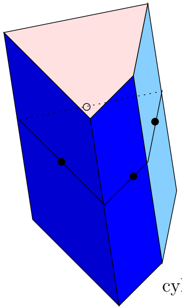
LEMMA Let C be a full-dimensional maximal $(\mathbb{Z}^p \times \mathbb{R}^{n-p})$-free convex set and let K be its projection onto $\mathbb{R}^p$. Then K is a full-dimensional maximal $\mathbb{Z}^p$-free convex set and $C = K \times \mathbb{R}^{n-p}$.

PROOF Since C is a $\mathbb{Z}^p \times \mathbb{R}^{n-p}$-free convex set, its projection K is a $\mathbb{Z}^p$-free convex set. Let K' be a maximal $\mathbb{Z}^p$-free convex set containing K. Then the set $K' \times \mathbb{R}^{n-p}$ is a $(\mathbb{Z}^p \times \mathbb{R}^{n-p})$-free convex set. Furthermore $C \subseteq K \times \mathbb{R}^{n-p} \subseteq K' \times \mathbb{R}^{n-p}$. Since C is maximal, these three sets coincide and the result follows. $\square$

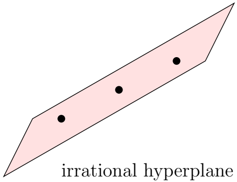
Lovász' Theorem

THEOREM

A set $K \subset \mathbb{R}^p$ is a full-dimensional maximal $\mathbb{Z}^p$-free convex set if and only if
K is a polyhedron of the form $K = P + L$
where P is a polytope, L is a rational linear space,
$\dim(P) + \dim(L) = p$,
K does not contain any point of $\mathbb{Z}^p$ in its interior and there is a point of $\mathbb{Z}^p$ in the relative interior of each facet of K.



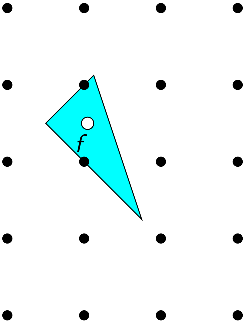
cylinder



irrational hyperplane

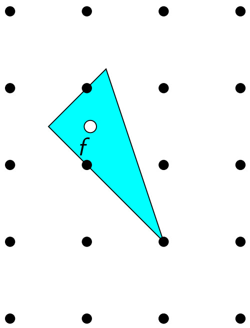
Maximal Lattice-Free Convex Set

- Lattice-free convex set contains no integral point in its interior



Maximal Lattice-Free Convex Set

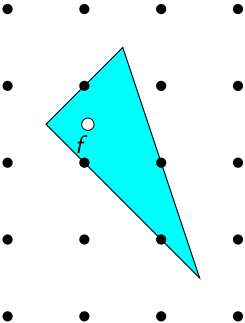
- Lattice-free convex set contains no integral point in its interior



Maximal: each edge contains an integral point in its relative interior.

Maximal Lattice-Free Convex Set

- Lattice-free convex set contains no integral point in its interior



Maximal: each edge contains an integral point in its relative interior.
In the plane: it is a strip, a triangle or a quadrilateral.

Theorem [Lovász 1989]

A maximal lattice-free convex set in the plane (x_1, x_2) is one of the following:

i) Irrational line $ax_1 + bx_2 = c$ with a/b irrational;

ii) A strip $c \leq ax_1 + bx_2 \leq c + 1$
with a, b coprime integers, c integer;

iii) A triangle with an integral point in the relative interior of each edge;

iv) A quadrilateral containing exactly four integral points, one in the relative interior of each edge; The four integral points are vertices of a parallelogram of area 1.

$\mathbb{Z}^p$-free Convex Sets and Valid Functions

Let $B \in \mathbb{R}^p$ be a lattice-free convex set with $f \in \mathrm{int}(B)$.

Define the function $\psi_B : \mathbb{R}^p \to \mathbb{R}$ as follows.

- Set $\psi_B(r) = 0$ for any vector $r \in \mathbb{R}^p$ in the recession cone of B.
- For any $r \in \mathbb{R}^p$ that is not in the recession cone of B, set $\psi_B(r) = \frac{1}{\alpha}$ where $\alpha > 0$ is s.t. $f + \alpha r$ is on the boundary of B.

The inequality $\sum_{j \in N} \psi_B(r^j) x_j \geq 1$ is valid for

$$\begin{array}{rcll} x_i & = & f_i - \sum_{j \in N} r_i^j x_j & \text{for } i \in B \\ x_i & \in & \mathbb{Z} & \text{for } i \in B \\ x_j & \geq & 0 & \text{for } j \in N. \end{array} \tag{1}$$

where all the nonbasic variables are continuous.

DEFINITION A function ψ that defines a valid inequality $\sum_{j \in N} \psi(r^j) x_j \geq 1$ to (1) for *any choice of data* $r^j, j \in N$ is called a valid function.

Minimal Valid Functions

A valid function $\psi : \mathbb{R}^p \to \mathbb{R}_+$ is *minimal* if there is no other valid function ψ' such that $\psi' \leq \psi$.

THEOREM(Borozan and Cornuejols 2009)

If $\psi : \mathbb{R}^p \to \mathbb{R}_+$ is a minimal valid function, then ψ is

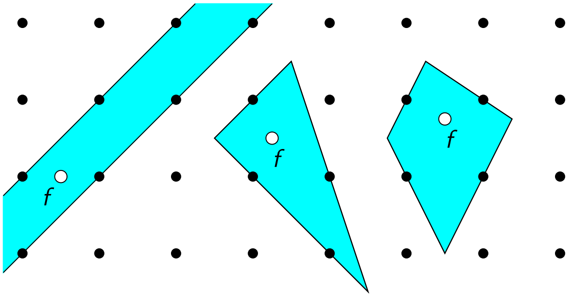
- nonnegative
- piecewise linear
- positively homogeneous
- and convex.

Furthermore $B_\psi := \{x \in \mathbb{R}^p : \ \psi(x - f) \leq 1\}$ is a maximal $\mathbb{Z}^p$-free convex set containing f in its interior.
Conversely, any maximal $\mathbb{Z}^p$-free convex set B containing f in its interior gives rise to a minimal valid function ψ.

Positively homogeneous means $\psi(\lambda r) = \lambda \psi(r)$ for any scalar $\lambda \in \mathbb{R}_+$ and $r \in \mathbb{R}^p$.

Maximal Lattice-Free Sets in the Plane

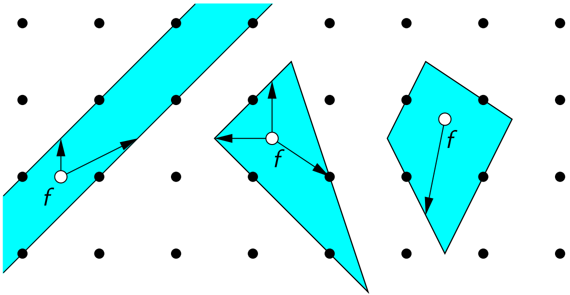
Split, triangles and quadrilaterals



generate valid inequalities $\sum_{j \in N} \psi(r^j) x_j \geq 1$.

Maximal Lattice-Free Sets in the Plane

Split, triangles and quadrilaterals



generate valid inequalities $\sum_{j \in N} \psi(r^j) x_j \geq 1$.

The corresponding valid functions ψ are simple to describe.

Proof of Lovaśz' Theorem

Let $K \subset \mathbb{R}^p$ be a maximal $\mathbb{Z}^p$-free convex set.

We prove the theorem under the assumption that K is a bounded set. We need to show that K is a polytope and that each of its facets has an integer point in its relative interior.

Since we assume K bounded, there exist l, u in $\mathbb{Z}^p$ such that K is contained in the box $B = \{x \in \mathbb{R}^p \;:\; l_i \leq x_i \leq u_i,\; i = 1 \dots p\}$.

Since K is a convex set, for each $y \in B \cap \mathbb{Z}^p$, there exists an half-space $\{x \in \mathbb{R}^p :\; a_y x \leq b_y\}$ containing K such that $a_y y = b_y$ (separation theorem for convex sets).

Since B is a bounded set, $B \cap \mathbb{Z}^p$ is a finite set. Therefore $P = \{x \in \mathbb{R}^p :\; l_i \leq x_i \leq u_i,\; i = 1 \dots p,\; a_y x \leq b_y,\; y \in B \cap \mathbb{Z}^p\}$ is a polytope.

By construction P is $\mathbb{Z}^p$-free and $K \subseteq P$.

Therefore $K = P$ by maximality of K.

Proof of Lovaśz' Theorem

We now show that each facet of K contains an integer point in its relative interior.
Suppose, by contradiction, that facet F_t of K does not contain a point of $\mathbb{Z}^p$ in its relative interior.
Let $a_t x \leq b_t$ be the inequality defining F_t.
Given $\varepsilon > 0$, let K' be the polyhedron defined by the same inequalities that define K except the inequality $\alpha_t x \leq \beta_t$ that has been substituted with the inequality $\alpha_t x \leq \beta_t + \varepsilon$.
Since the recession cones of K and K' coincide, K' is a polytope.
Since K is a maximal $\mathbb{Z}^p$-free convex set and $K \subset K'$, K' contains points of $\mathbb{Z}^p$ in its interior.
Since K' is a polytope, the number of points in $K' \cap \mathbb{Z}^p$ is finite.
Hence there exists one such point minimizing $\alpha_t x$, say z.
Let K'' be the polytope defined by the same inequalities that define K except the inequality $\alpha_t x \leq \beta_t$ that has been substituted with the inequality $\alpha_t x \leq \alpha_t z$.
By construction, K'' does not contain any point of $\mathbb{Z}^p$ in its interior and properly contains K, contradicting the maximality of K. $\square$

Bound on the Number of Facets of Maximal $\mathbb{Z}^p$-Free Polyhedra

Doignon 1973, Bell 1977 and Scarf 1977 show the following.

THEOREM Any full-dimensional maximal lattice-free convex set $K \subseteq \mathbb{R}^p$ has at most 2^p facets.

PROOF By Lovász' theorem, each facet F contains an integral point x^F in its relative interior. If there are more than 2^p facets, then two integral points x^F and $x^{F'}$ must be congruent modulo 2. Now their middle point $\frac{1}{2}(x^F + x^{F'})$ is integral and it is in the interior of K, contradicting the fact that K is lattice-free.