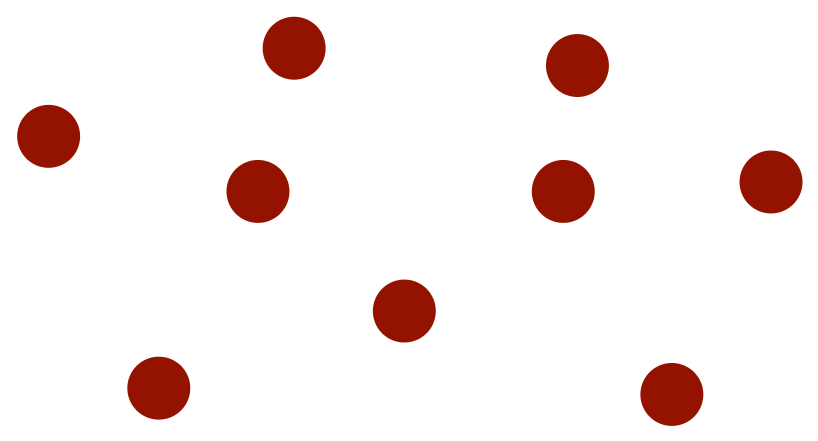
EXTENDED FORMULATIONS

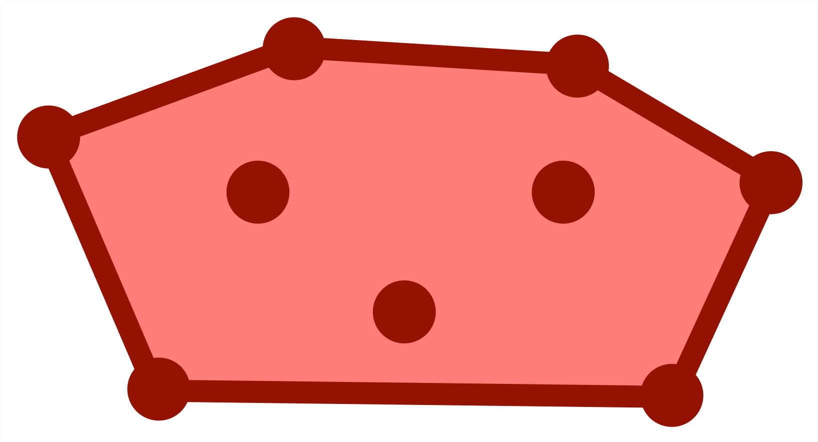
Volker Kaibel (OvGU Magdeburg)

Spring School ISCO 2016

THE CONCEPT

CONVEX HULLS AND LINEAR PROGRAMMING





From points to polytopes

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LP-duality

$$\max\{\langle c,x\rangle \ : \ Ax \leq b, x \in \mathbb{R}^n\} = \min\{\langle b,y\rangle \ : \ A^t y = c, y \in \mathbb{R}^m_+\}$$

From points to polytopes

$\max\{\langle c, x\rangle : x \in X\} = \max\{\langle c, x\rangle : x \in \operatorname{conv}(X)\}$

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LP-algorithms

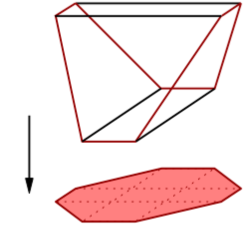
Efficient both in theory and praxis.

Extension of a Polytope $P \subseteq \mathbb{R}^n$:

A polytope $Q \subseteq \mathbb{R}^d$ and a linear projection $p : \mathbb{R}^d \to \mathbb{R}^n$ with

$$P = p(Q).$$

Size: Number of facets of Q



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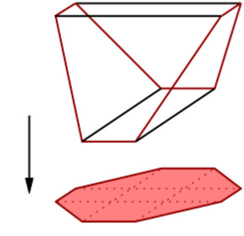

Extended Formulation of P:

Linear description of some extension of P

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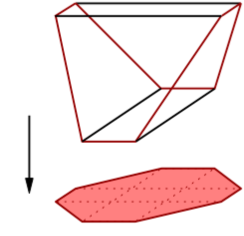

Extension Complexity of P:

$$\mathrm{xc}(P) = \text{smallest size of any extension of } P$$

REPRESENTATIONS AS PROJECTIONS

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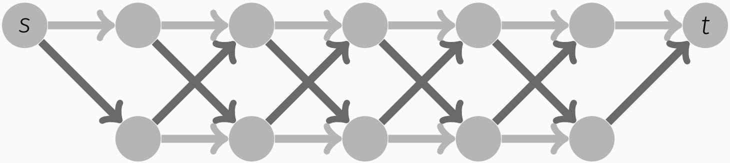
Example (Carr & Konjevod 2004)

$$\text{xc}(\text{conv}\{v \in \{0,1\}^n \;:\; v \text{ has even \# of 1's}\}) \leq 4n - 4$$



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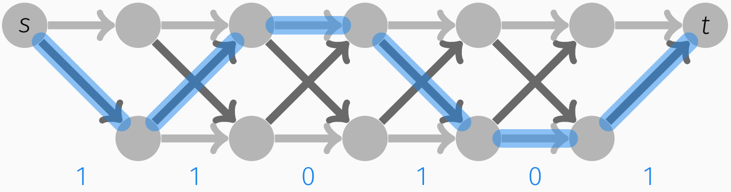

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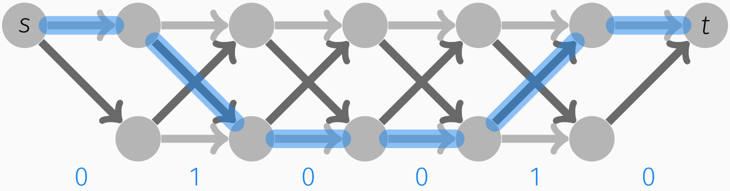

COMPLETION TIMES POLYTOPE

Jobs with processing times $p_1, \ldots, p_n$

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Queyranne 1993

For $0 < p_1 \leq \cdots \leq p_n$: Description by one equation and

$$\sum_{i \in I} p_i x_i \geq \sum_{i=1}^{|I|} p_i \sum_{j=1}^{i} p_j \quad \text{for all } \varnothing \neq I \subseteq [n]$$

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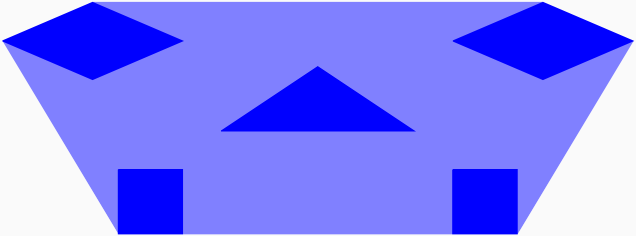
Wolsey 1986

The cube $Q = [0,1]^{\binom{[n]}{2}}$ projects to $\mathrm{P}_{\mathrm{ct}}(p_1, \ldots, p_n)$ via

$$x_i = \sum_{j=1}^{i-1} p_j y_{\{i,j\}} + \sum_{j=i+1}^{n} p_j (1 - y_{\{i,j\}}) \quad \text{for all } i \in [n]\,.$$

- Disjunctive Programming
- Dynamic Programming
- Branched Polyhedral Systems
- Dualization
- Redundant Information
- Reflections

DISJUNCTIVE PROGRAMMING



Balas 1975

For polytopes $P_1, \ldots, P_q \subseteq \mathbb{R}^m$ (with $\dim(P_i) > 0$)

$$\mathrm{xc}(\mathrm{conv}(\bigcup_{i=1}^{q} P_i)) \leq \sum_{i=1}^{q} \mathrm{xc}(P_i)$$

holds.

Matchings with ℓ edges

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Edmonds 1965

$P^\ell_{\text{match}}(n)$ is described by $x \geq \mathbf{0}$, $x(E) = \ell$, and:

$$x(E(S)) \leq \tfrac{|S|-1}{2} \text{ for all } S \subseteq V, 3 \leq |S| \text{ odd}$$

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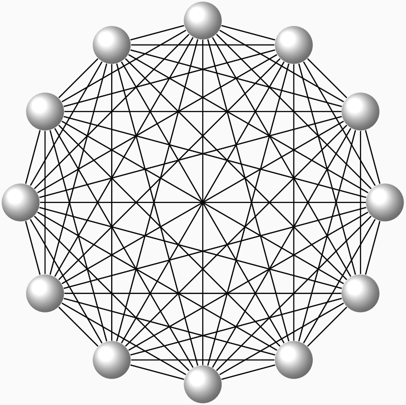
The Strategy

1. Cover by few subproblems.
2. Find small (extended) formulations for subproblems.
3. Take (convex hull of) union.

SPECIAL MATCHINGS



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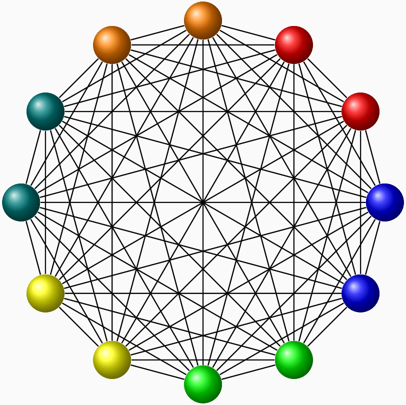
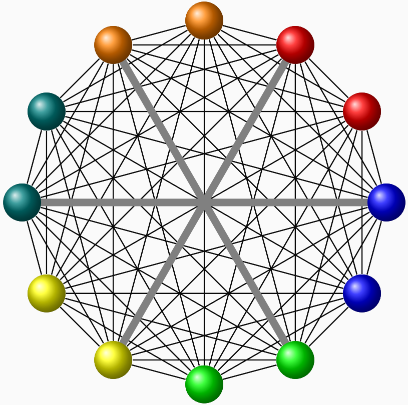
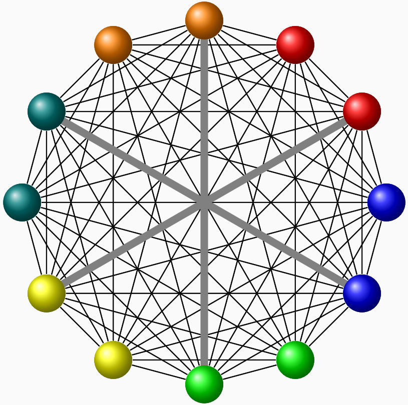




Colorful Matchings

For $V = W_1 \uplus \cdots \uplus W_{2\ell}$, a matching $M \subseteq E$ is colorful if it matches exactly one node from each set W_i.



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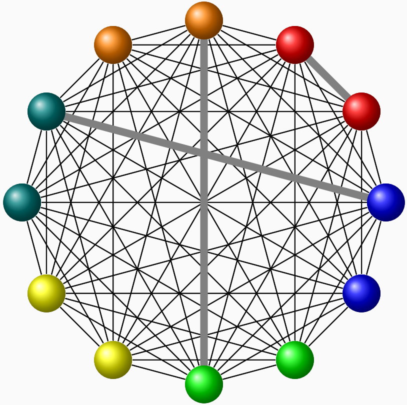

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Linear Description in $\mathbb{R}^E_+$

$$
\begin{aligned}
x(E(W_i)) &= 0 && \text{for all } i \in \{1, \dots, 2\ell\} \\
x(\delta(W_i)) &= 1 && \text{for all } i \in \{1, \dots, 2\ell\} \\
x(E(\cup_{i \in S} W_i)) &\leq (|S| - 1)/2 && \text{for all } S \subseteq \{1, \dots, 2\ell\}, |S| \text{ odd}
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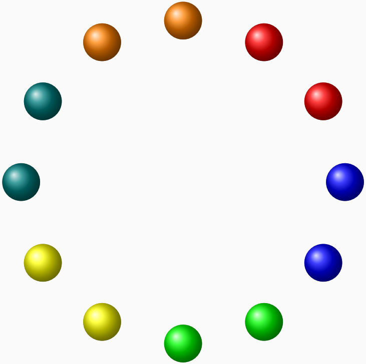

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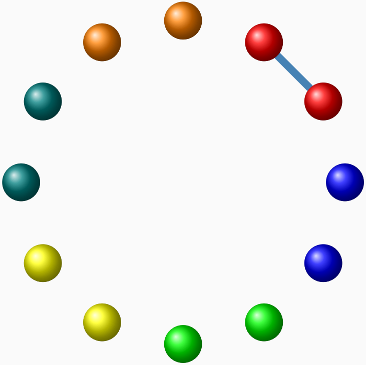

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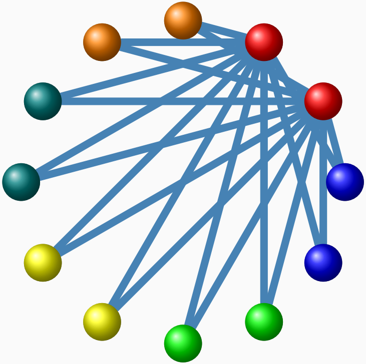

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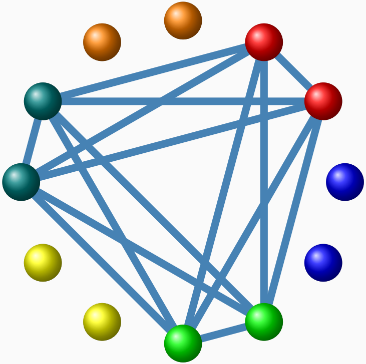
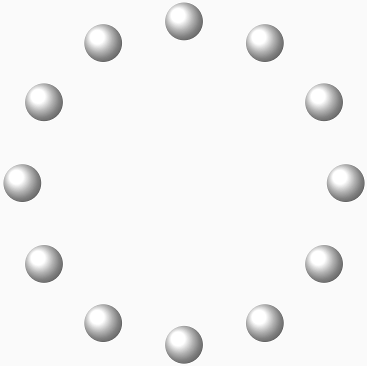


(n,k)-perfect hash function family of size q

$\phi_1, \ldots, \phi_q : [n] \to [k]$ such that for all $W \subseteq [n]$ with $|W| = k$ there is some $i \in [q]$ with ϕ_i bijective on W



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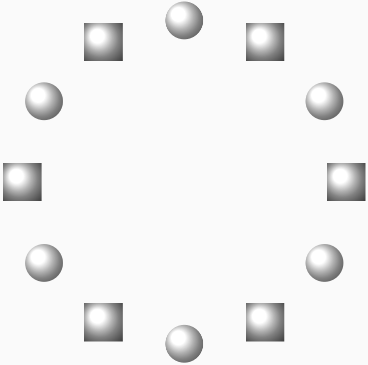

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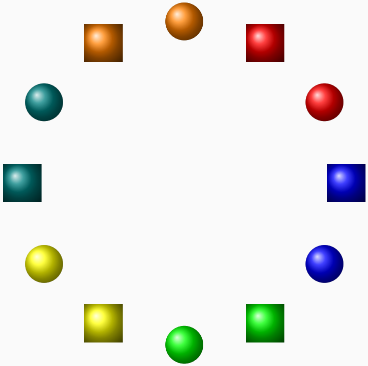

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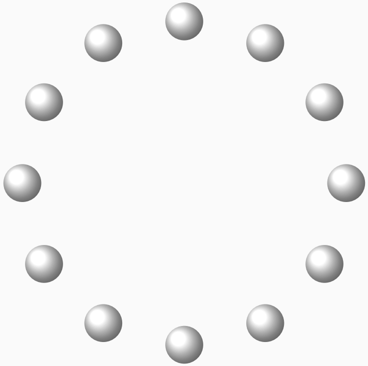

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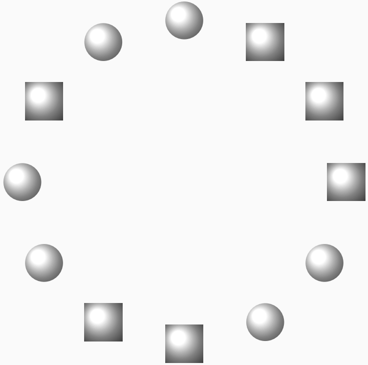

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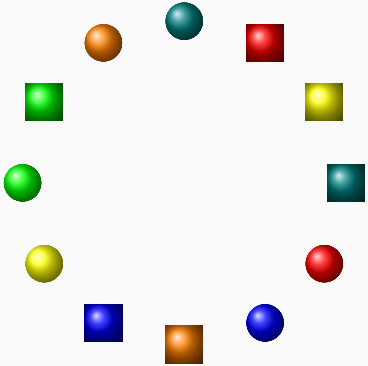

ALON, YUSTER, ZWICK 1995

For n and k there are (n,k)-perfect hash function families of size

$$2^{O(k)}\log(n).$$

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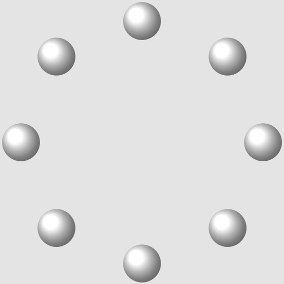
Example: $n = 8$, $k = 2$, size 3



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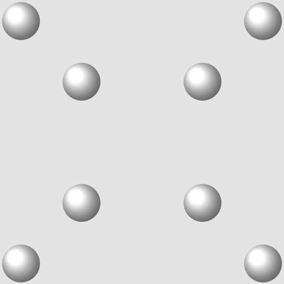

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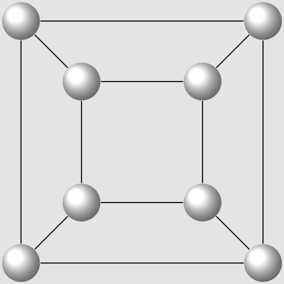

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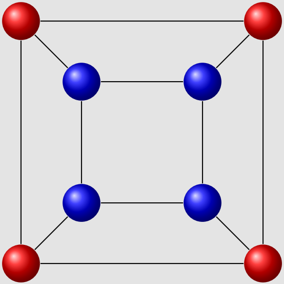

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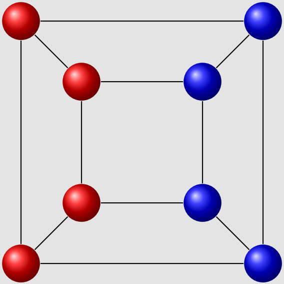

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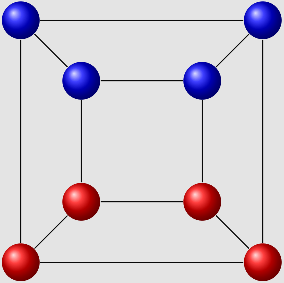

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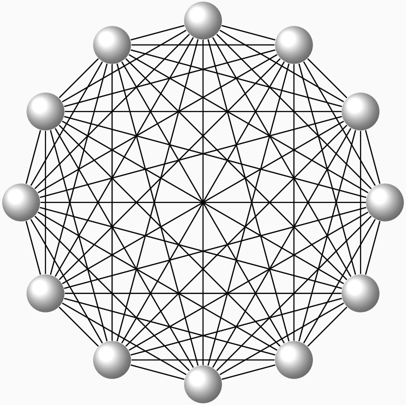
Consequence

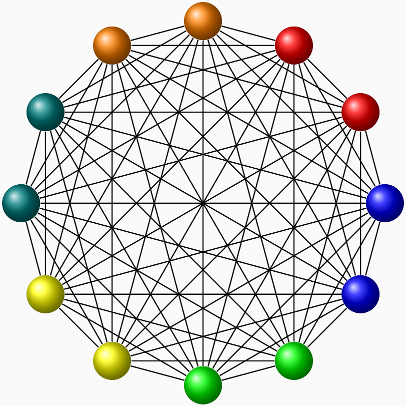
$$\text{xc}(P^{\lfloor \log n \rfloor}_{\text{match}}(n)) \text{ is bounded polynomially in } n.$$

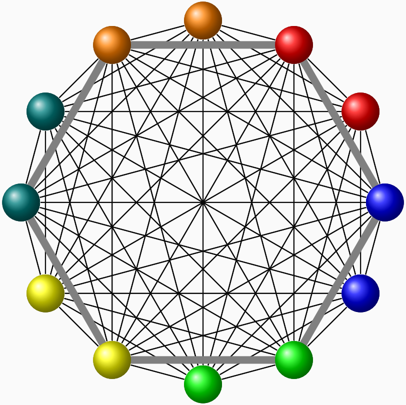
DYNAMIC PROGRAMMING

SPECIAL CYCLES



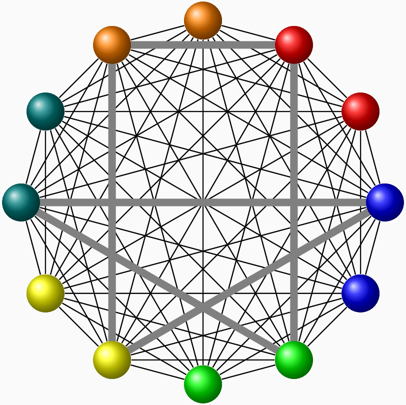
SPECIAL CYCLES





Colorful cycles

For $V = W_1 \uplus \cdots \uplus W_\ell$, a cycle $C \subseteq E$ is colorful if it visits each set W_i exactly ones.



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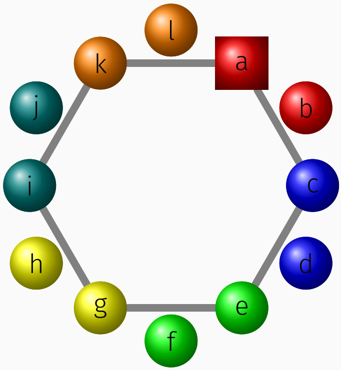
COLORFUL CYCLES WITH PRESCRIBED NODE a



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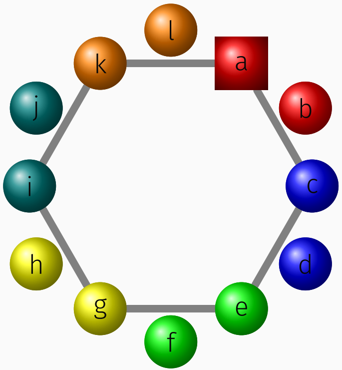

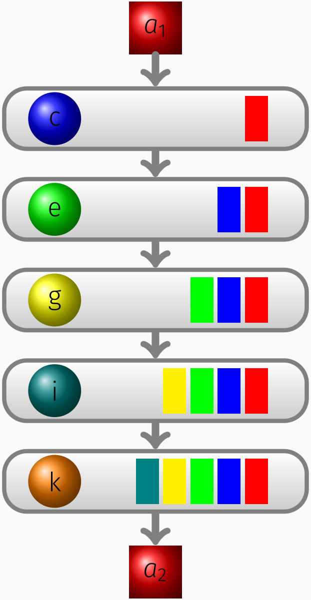

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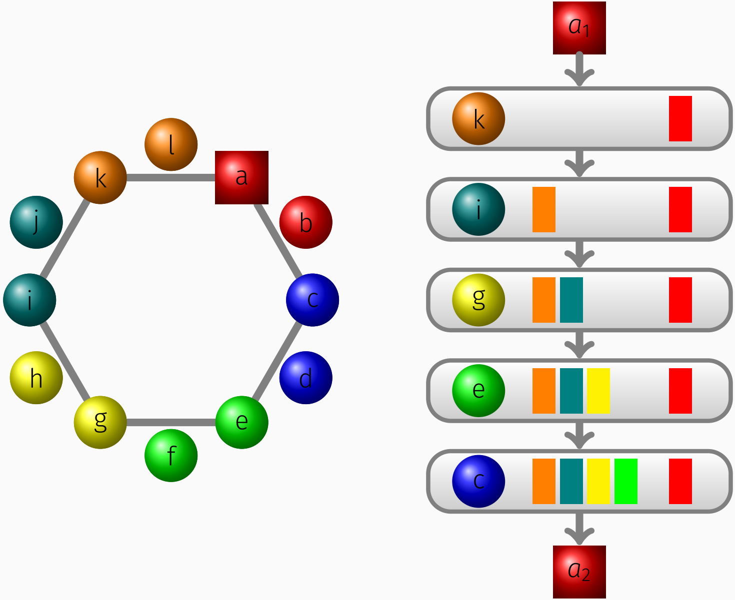

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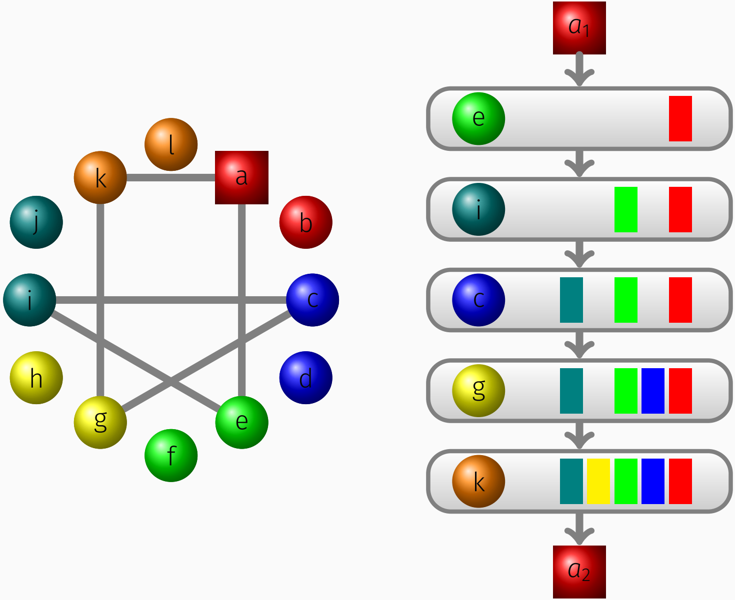

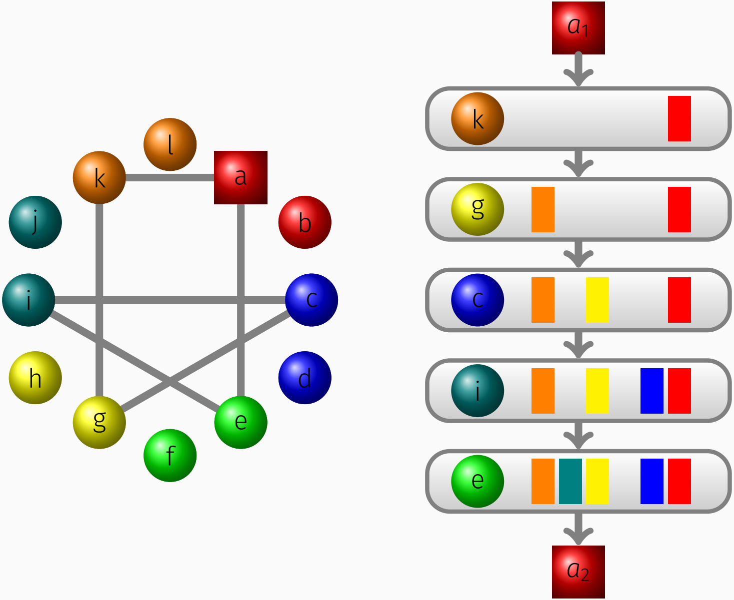
l
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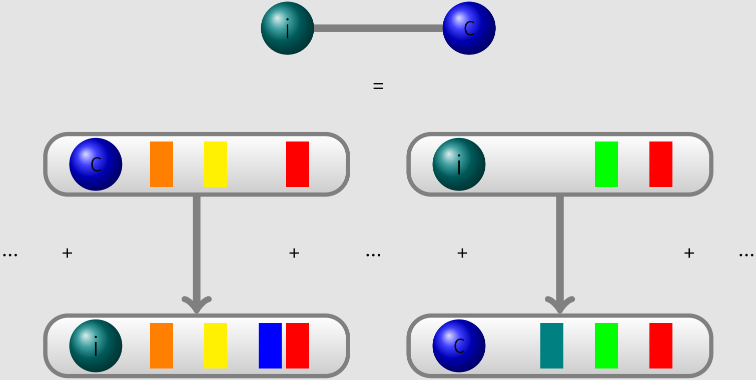
a_2

Extended Formulation via

- a_1-a_2 flows of value one
- and projection



We have seen:

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  colorful cycle polytopes (with prescribed nodes)...

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K, Pashkovich, Theis 2010

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K, Pashkovich, Theis 2010

$$\mathrm{xc}(P^{\ell}_{\mathrm{cycl}}(n)) \leq 2^{O(\ell)} n^3 \log(n)$$

Consequence

$\mathrm{xc}(P^{\lfloor \log n \rfloor}_{\mathrm{cycl}}(n))$ is bounded polynomially in n.

Martin, Rardin, Campbell 1990

The source-to-terminals path polytope of a directed single-tail acyclic hypergraph with a unique source and terminal-disjoint head sets is described by the conservation equations and nonnegativity constraints.

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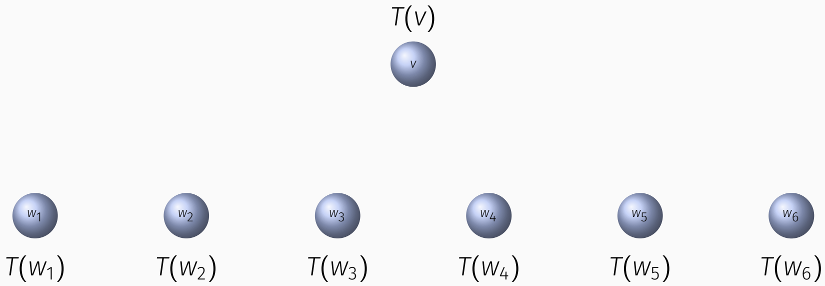

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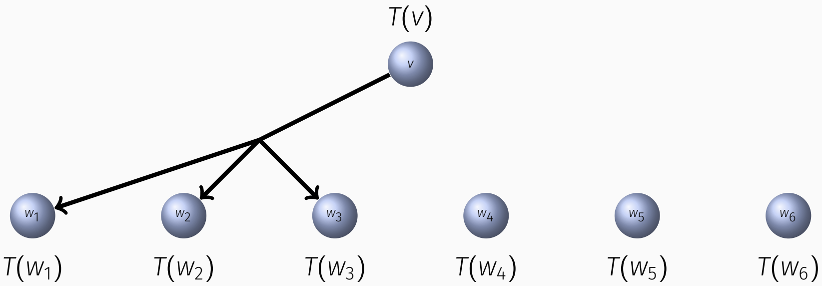

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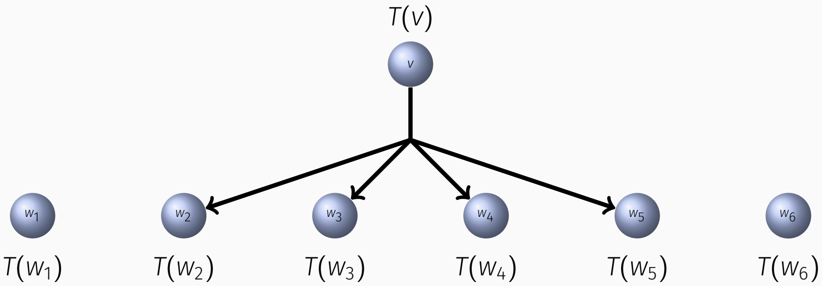

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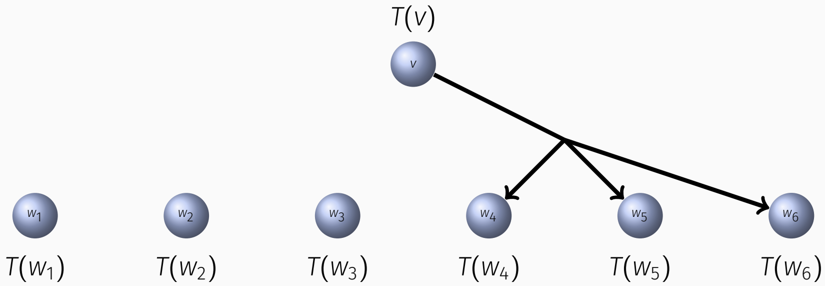
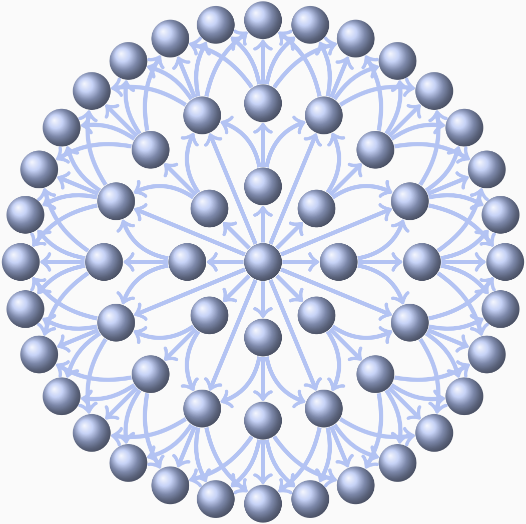
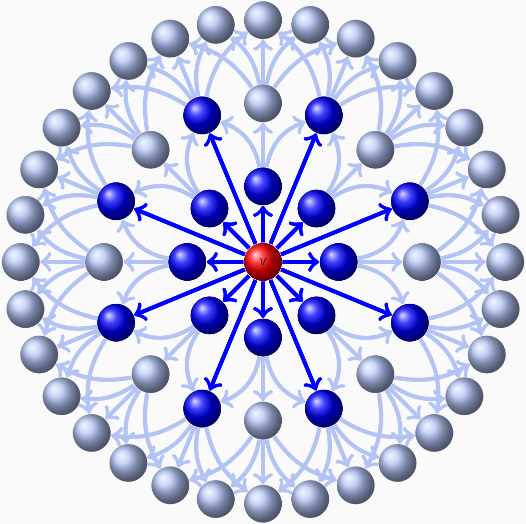


BRANCHED POLYHEDRAL SYSTEMS

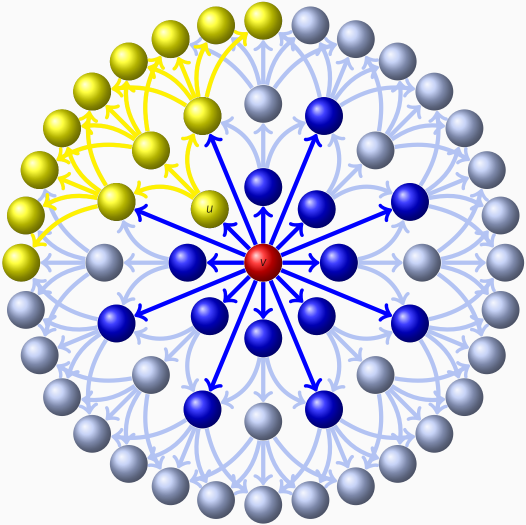
BRANCHED COMBINATORIAL/POLYHEDRAL SYSTEMS





For each non-sink v

$$\mathcal{S}^{(v)} \subseteq 2^{\mathrm{N}^{\mathrm{out}}(v)}$$



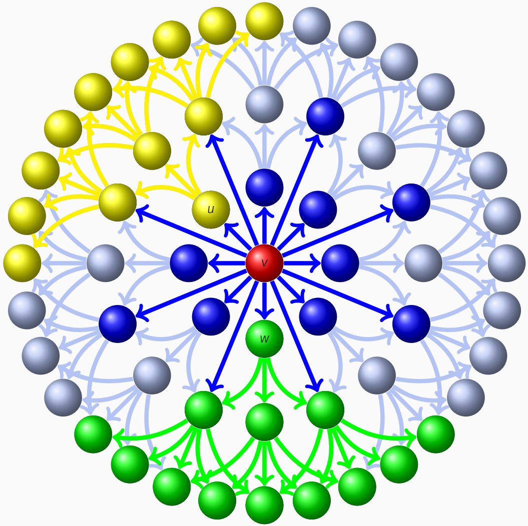
For each non-sink v

$$\mathcal{S}^{(v)} \subseteq 2^{N^{out}(v)}$$

Condition

$S \in \mathcal{S}^{(v)}$, $u, w \in S$, $u \neq w$:

$$R(u) \cap R(w) = \varnothing$$



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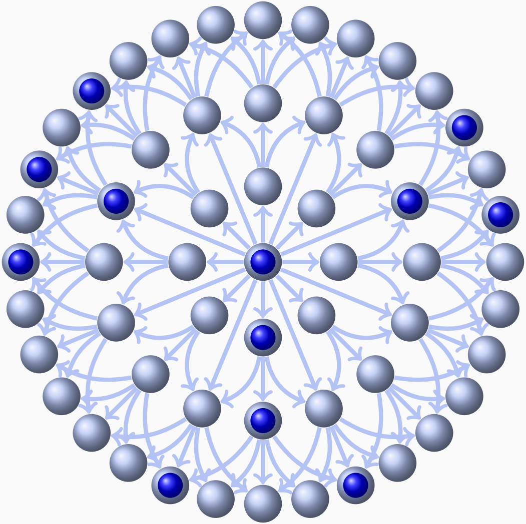
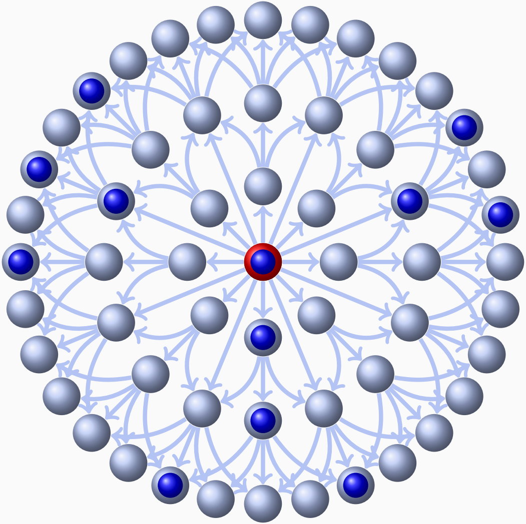


Feasible set $F \subseteq V$



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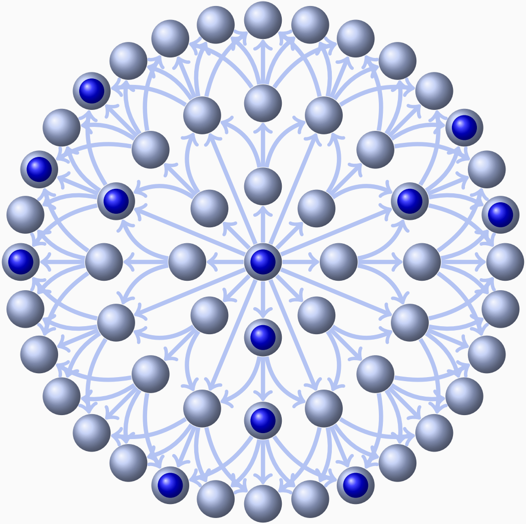

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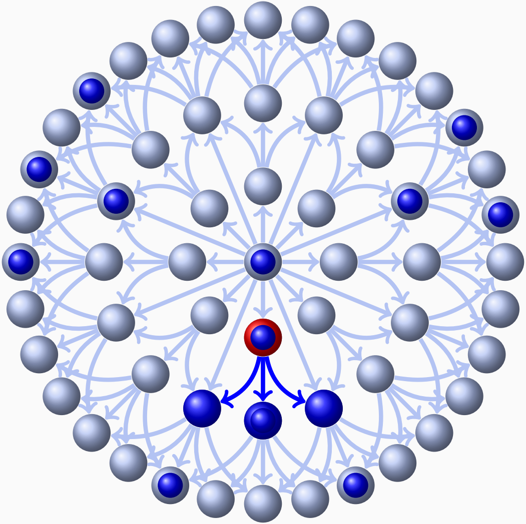

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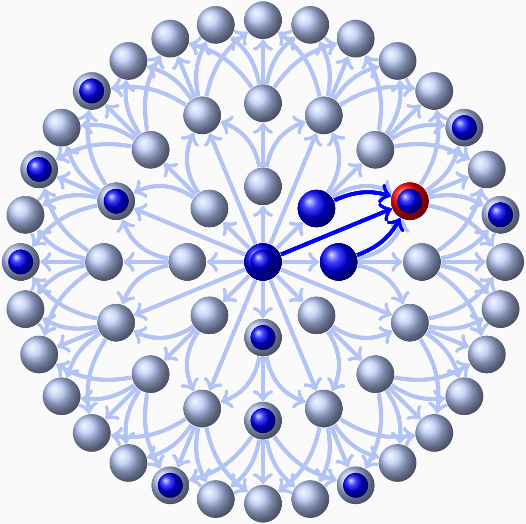
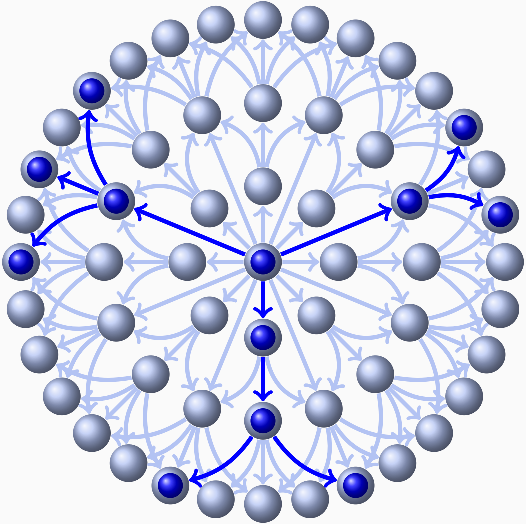
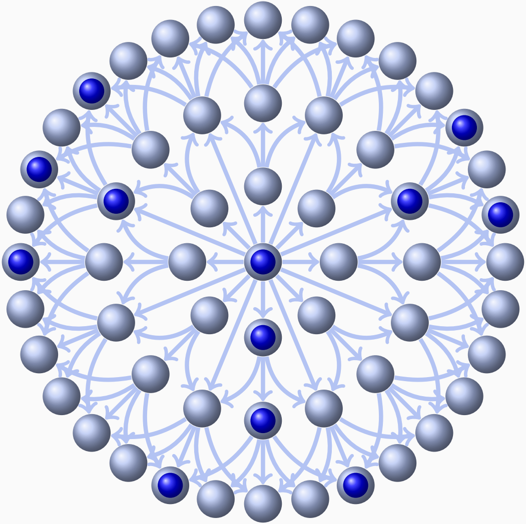
- $s \in F$
- $\forall v \in F$:
  - $v \notin T$: $F \cap \mathrm{N}^{\mathrm{out}}(v) \in \mathcal{S}^{(v)}$



Feasible set $F \subseteq V$

- $s \in F$
- $\forall v \in F$:
  - $v \notin T$: $F \cap \mathrm{N}^{\mathrm{out}}(v) \in \mathcal{S}^{(v)}$
  - $v \neq s$: $F \cap \mathrm{N}^{\mathrm{in}}(v) \neq \varnothing$





0/1-Polytope $P(\mathcal{B})$

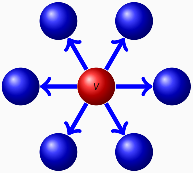
$\text{conv}(\{\chi(F) : F \text{ feasible}\})$

K & Loos 2010

$P(\mathcal{B})$ is described by the following extended formulation:

$$\begin{aligned} x_s &= 1 & & \\ x_v &= y(\delta^{\text{in}}(v)) & & \text{for all } v \neq s \\ A^{(v)} y_{\delta^{\text{out}}(v)} - x_v b^{(v)} &\leq 0 & & \text{for all non-sinks } v \\ x_v &\geq 0 & & \text{for all non-sinks } v \end{aligned}$$

(if $A^{(v)} z \leq b^{(v)}$ describe $P^{(v)} = \text{conv}(\{\chi(S) : S \in \mathcal{S}^{(v)}\})$)



DUALIZATION

$$P = \{x \in \mathbb{R}^n \,:\, \langle a, x \rangle \leq \alpha \quad \forall a \in \mathcal{A}\}$$

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THE METHOD

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$$\begin{aligned} P &= \{x \in \mathbb{R}^n \,:\, \max\{\langle a, x\rangle \,:\, a \in \mathcal{A}\} \;\leq\; \alpha\} \\ &= \{x \in \mathbb{R}^n \,:\, \max\{\langle q(z), x\rangle \,:\, Cz \leq c\} \;\leq\; \alpha\} \\ &= \{x \in \mathbb{R}^n \,:\, \max\{\langle q^\star(x), z\rangle \,:\, Cz \leq c\} \;\leq\; \alpha\} \\ &= \{x \in \mathbb{R}^n \,:\, \langle c, y\rangle \leq \alpha, y^T C = q^\star(x), y \geq \mathbb{O}\} \end{aligned}$$

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Martin 87

$$\text{xc}(P) \leq \text{xc}(\mathcal{A}) + 1$$

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Martin 87

$$\mathrm{xc}(P) \leq \mathrm{xc}(\mathcal{A}) + 1$$

In fact (if $P \neq \varnothing$):

$$|\,\mathrm{xc}(P) - \mathrm{xc}(\mathcal{A})| \leq 1$$

SPANNING FOREST POLYTOPES

Edmonds 71

$$P_{\mathrm{spf}}(G) = \{x \in \mathbb{R}_+^E \ : \ x(E(S)) \leq |S| - 1 \quad \forall \varnothing \neq S \subseteq V\}$$

Edmonds 71

$$P_{spf}(G) = \{x \in \mathbb{R}^E_+ \; : \; x(E(S)) \leq |S| - 1 \quad \forall \emptyset \neq S \subseteq V\}$$

$$\begin{aligned}
&\{x \in \mathbb{R}^E_+ \; : \; x(E(S)) - |S| \leq -1 \quad \forall \emptyset \neq S \subseteq V\} \\
=&\{x \in \mathbb{R}^E_+ \; : \; \langle(\chi(E(S)), \chi(S)), (x, -\mathbf{1})\rangle \leq -1 \quad \forall \emptyset \neq S \subseteq V\} \\
=&\{x \in \mathbb{R}^E_+ \; : \; \langle(\chi(F), \chi(S)), (x, -\mathbf{1})\rangle \leq -1 \quad \forall \emptyset \neq S \subseteq V, F \subseteq E(S)\} \\
=&\{x \in \mathbb{R}^E_+ \; : \; \langle(a, b), (x, -\mathbf{1})\rangle \leq -1 \quad \forall (a, b) \in P^\star_{sub}(G)\}
\end{aligned}$$

Edmonds 71

$$P_{spf}(G) = \{x \in \mathbb{R}^E_+ \,:\, x(E(S)) \leq |S| - 1 \quad \forall \varnothing \neq S \subseteq V\}$$

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Subgraph polytopes

$$\begin{aligned} P_{sub}(G) &:= \mathrm{conv}\{(\chi(F), \chi(S)) \,:\, S \subseteq V, F \subseteq E(S)\} \\ P^\star_{sub}(G) &:= \mathrm{conv}\{(\chi(F), \chi(S)) \,:\, S \subseteq V, F \subseteq E(S), S \neq \varnothing\} \\ P^{v_0}_{sub}(G) &:= \mathrm{conv}\{(\chi(F), \chi(S)) \,:\, S \subseteq V, F \subseteq E(S), v_0 \in S\} \end{aligned}$$

$$P_{\text{sub}}(G) = \{(a, b) \in [0, 1]^E \times [0, 1]^V \,:\, a_e \leq b_v \;\forall v \in V, e \in \delta(v)\}$$

$$P_{\text{sub}}^{v_0}(G) = P_{\text{sub}}(G) \cap \{(a, b) \,:\, b_{v_0} = 1\}$$

$$P_{\text{sub}}^{\star}(G) = \text{conv}\Big(\bigcup_{v_0 \in V} P_{\text{sub}}^{v_0}(G)\Big)$$

$$P_{\text{sub}}(G) = \{(a,b) \in [0,1]^E \times [0,1]^V \,:\, a_e \leq b_v \;\forall v \in V, e \in \delta(v)\}$$

$$P^{v_0}_{\text{sub}}(G) = P_{\text{sub}}(G) \cap \{(a,b) \,:\, b_{v_0} = 1\}$$

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Martin 87

$$\text{xc}(P_{\text{spf}}(G)) \leq \text{xc}(P^{\star}_{\text{sub}}(G)) + O(|E|) \leq O(|V| \cdot |E|)$$

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Conforti, K, Walter, Weltge 14

$$P^{\star}_{\mathrm{sub}}(G) = \{(a,b) \in P_{\mathrm{sub}}(G) \,:\, a(E(T)) - b(T) \le -1 \;\forall T \subseteq E \text{ spf}\}$$

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Conforti, K, Walter, Weltge 14

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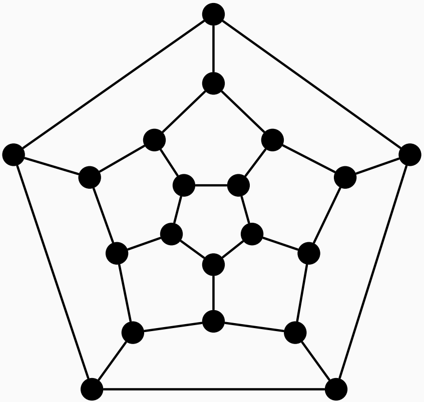
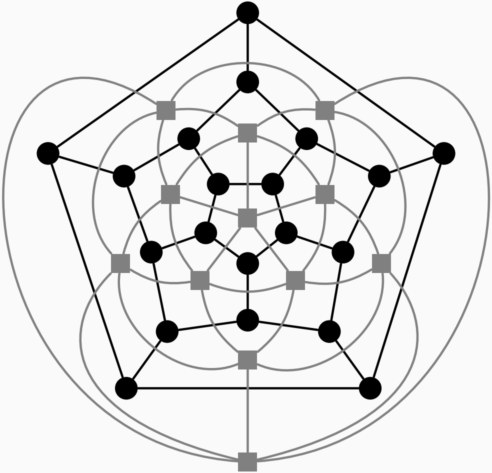
$$|\,\mathrm{xc}(P_{\mathrm{spf}}(G)) - \mathrm{xc}(P^{\star}_{\mathrm{sub}}(G))| \le O(|V| + |E|)$$

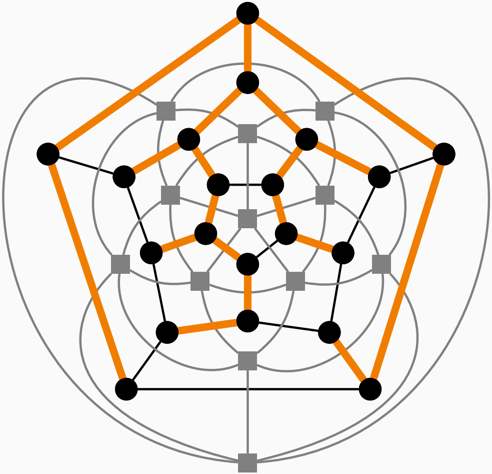
REDUNDANT INFORMATION

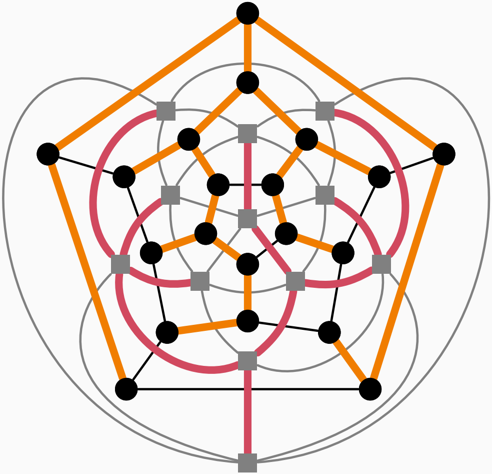


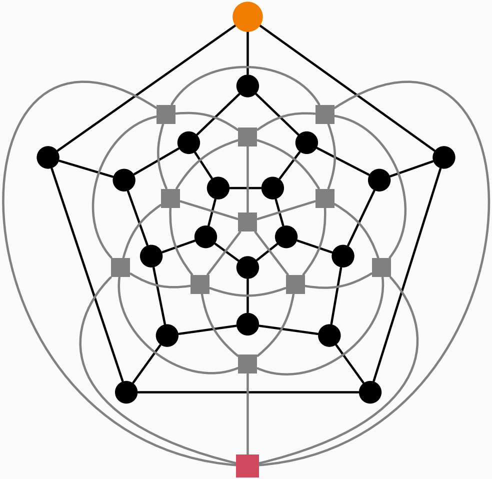
SPANNING TREES IN PLANAR GRAPHS



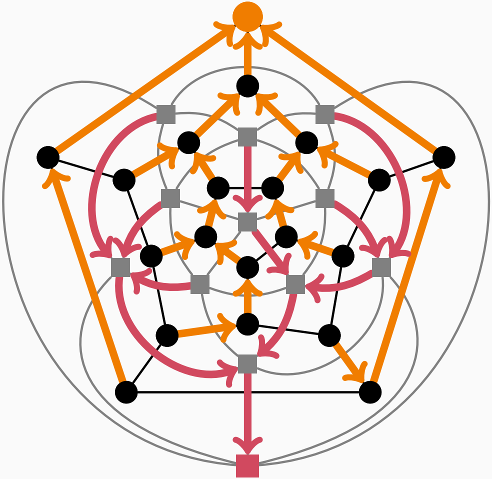
SPANNING TREES IN PLANAR GRAPHS

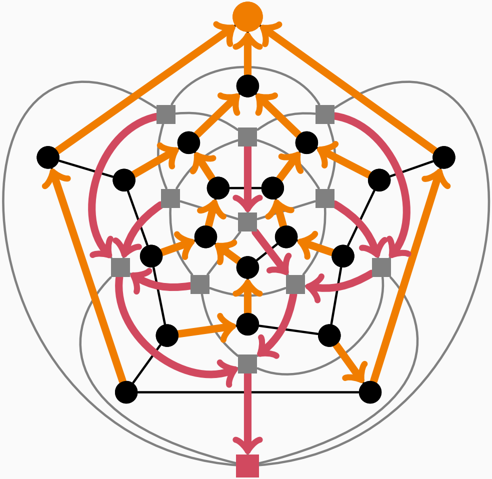




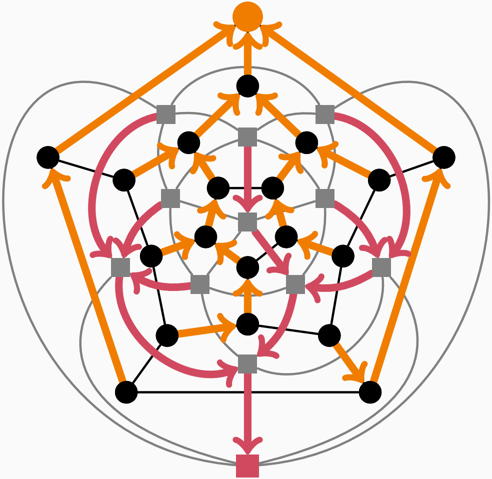


SPANNING TREES IN PLANAR GRAPHS



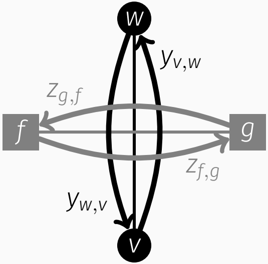


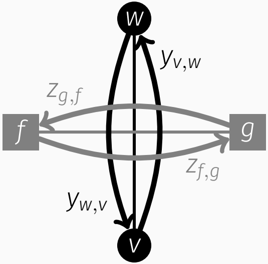
$$P_{\mathrm{arb}}(G) = \mathrm{conv}\{(\chi(\vec{T}), \chi(\vec{T^\star})) \;:\; T \text{ spanning tree of } G\}$$



$$P_{\mathrm{arb}}(G) = \operatorname{conv}\{(\chi(\vec{T}), \chi(\vec{T^\star})) \ : \ T \text{ spanning tree of } G\}$$
$$P_{\mathrm{spt}}(G) = p(P_{\mathrm{arb}}(G)) \text{ with linear projection } p$$

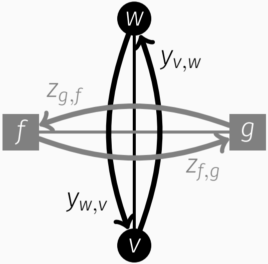
LINEAR DESCRIPTION OF $P_{\text{arb}}(\cdot)$





Williams 2001

$$
\begin{aligned}
y_{v,w} + y_{w,v} + z_{f,g} + z_{g,f} &= 1 \quad \forall \{v, w\} \in E \\
\textstyle\sum_w y_{v,w} &= 1 \quad \forall v \neq \text{root} \\
\textstyle\sum_g z_{f,g} &= 1 \quad \forall w \neq \text{root} \\
y_{v,w}, y_{w,v}, z_{f,g}, z_{g,f} &\geq 0 \quad \forall \{v, w\} \in E
\end{aligned}
$$



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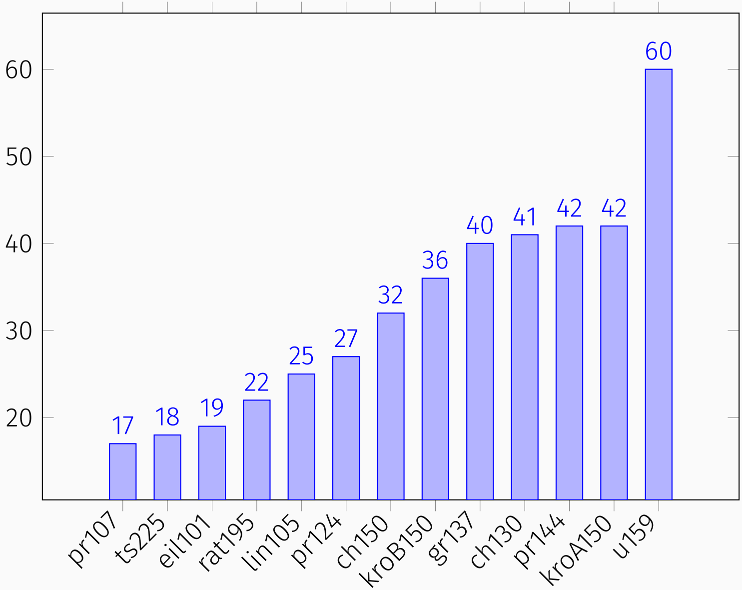
Thus $\mathrm{xc}(P_{spt}(G)) \leq O(n)$ for planar G on n nodes.

SPEED-UPS FOR DEGREE ≤ 3 [K & SORGATZ 2012]



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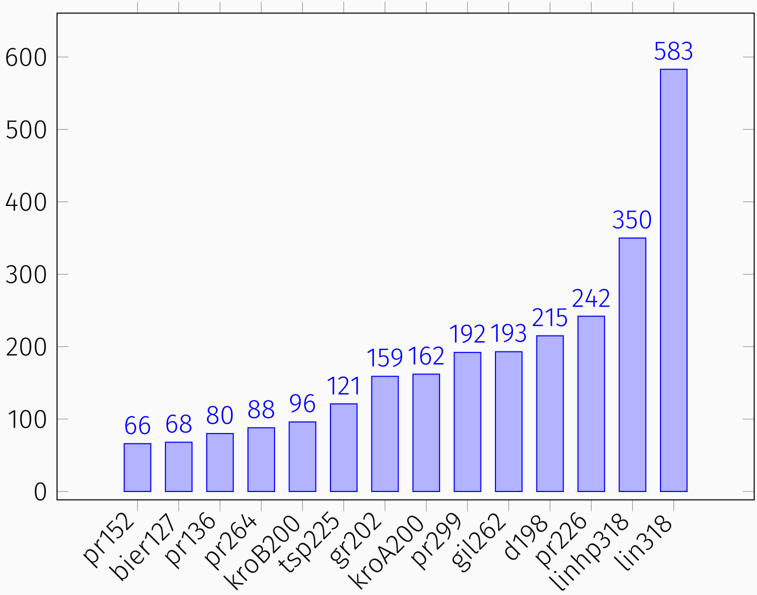
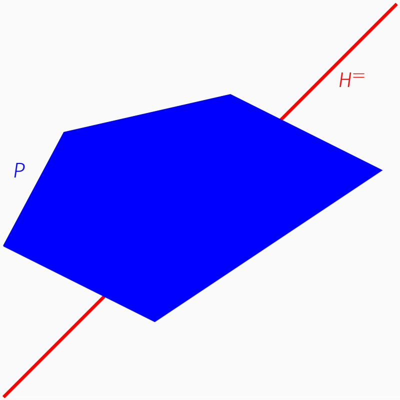


REFLECTIONS

THE REFLECTION OPERATION



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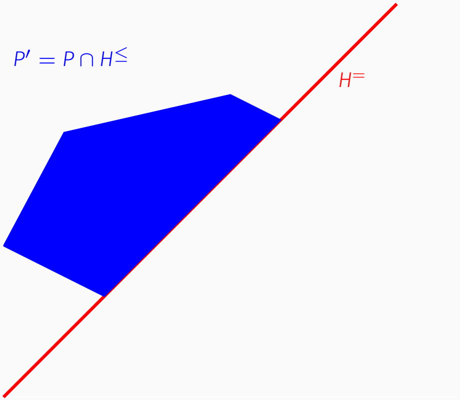
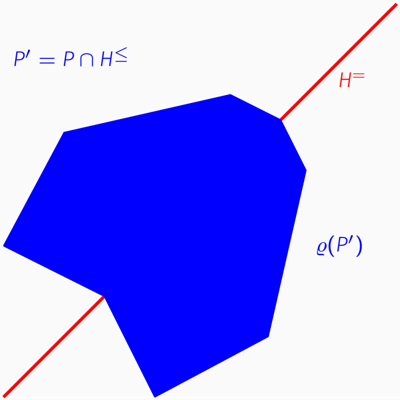
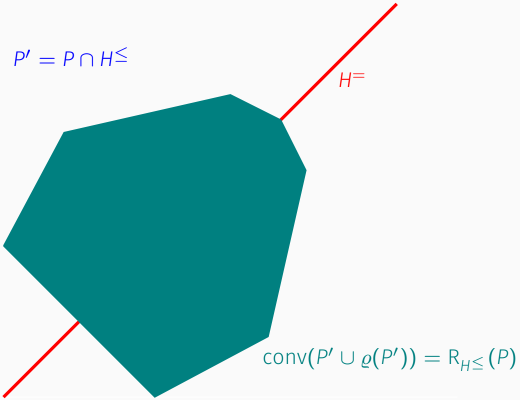
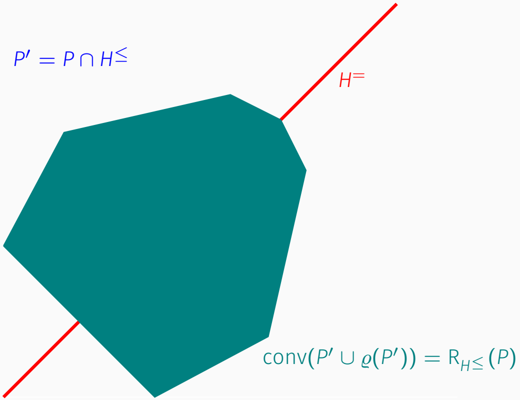



$P' = P \cap H^{\leq}$
$H^{=}$
$\varrho(P')$

THE REFLECTION OPERATION





▷ $\mathrm{R}_{H^{\leq}}(P) = \{x + \lambda a \;:\; x \in P, \langle a, x\rangle \leq \langle a, x + \lambda a\rangle \leq 2b - \langle a, x\rangle\}$

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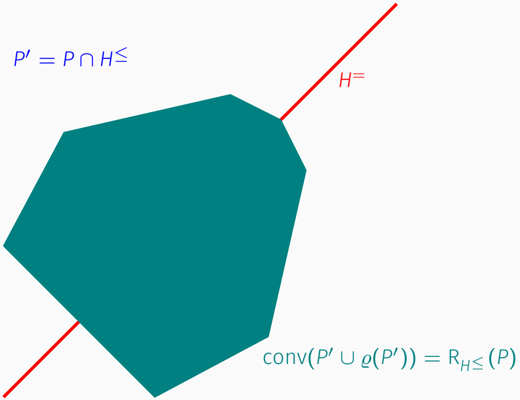

- $\mathrm{R}_{H^{\leq}}(P) = \{x + \lambda a \,:\, x \in P, \langle a, x\rangle \leq \langle a, x + \lambda a\rangle \leq 2b - \langle a, x\rangle\}$
- Thus: $\mathrm{xc}(\mathrm{R}_{H^{\leq}}(P)) \leq \mathrm{xc}(P) + 2$

Consequence

For each sequence $H_1^{\leq}, \ldots, H_r^{\leq} \subseteq \mathbb{R}^n$ of halfspaces and for each polytope $P \subseteq \mathbb{R}^n$, the polytope

$$\mathcal{R}_{H_1^{\leq},\ldots,H_r^{\leq}}(P) = \mathrm{R}_{H_r^{\leq}}(\mathrm{R}_{H_{r-1}^{\leq}}(\ldots \mathrm{R}_{H_1^{\leq}}(P) \ldots))$$

satisfies

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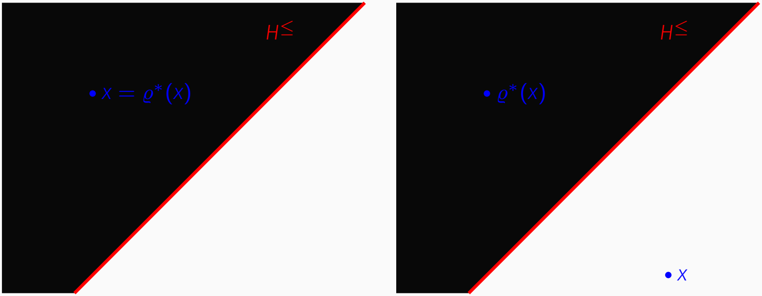
Task for target polytope Q

Find (and describe) P, design sequence $H_1^{\leq}, \ldots, H_r^{\leq}$, and prove

$$Q = \mathcal{R}_{H_r^{\leq},\ldots,H_1^{\leq}}(P)\,.$$

Define $\varrho^\star : \mathbb{R}^n \to \mathbb{R}^n$ via $\varrho^*(x) = \begin{cases} x & \text{if } x \in H^{\leq} \\ \varrho(x) & \text{otherwise} \end{cases}$.



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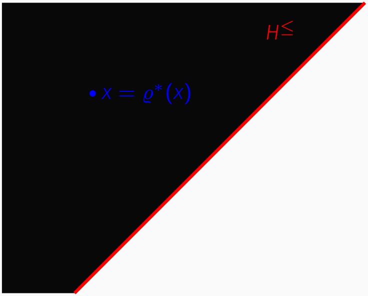

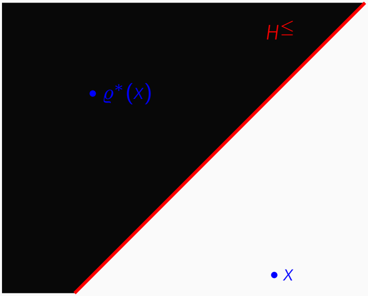

$$\varrho^*(x) \in P \quad \Rightarrow \quad x \in \mathrm{R}_{H^{\leq}}(P)$$

K & Pashkovich 11

Let

- $Q = \text{conv}(W)$ be some (target) polytope,
- $H_1^{\leq}, \dots, H_r^{\leq} \subseteq \mathbb{R}^n$ be a sequence of halfspaces, and
- ϱ_i (and $\varrho_i^{\star}$) the associated (conditional) reflections.

K & Pashkovich 11

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K & Pashkovich 11

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K & Pashkovich 11

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2. $\varrho_1^\star(\varrho_2^\star(\cdots(\varrho_r^\star(w)\cdots)) \in P$ for all $w \in W$.

Finite Reflection Group G

A *finite* group generated by a (finite) family $\varrho^{H_i} : \mathbb{R}^n \to \mathbb{R}^n$ ($i \in I$) of reflections at hyperplanes $\mathbf{0} \in H_i \subseteq \mathbb{R}^n$.

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Coxeter-Arrangement of G

The set of all hyperplanes $\mathbf{0} \in H \subseteq \mathbb{R}^n$ with $\varrho^H \in G$.

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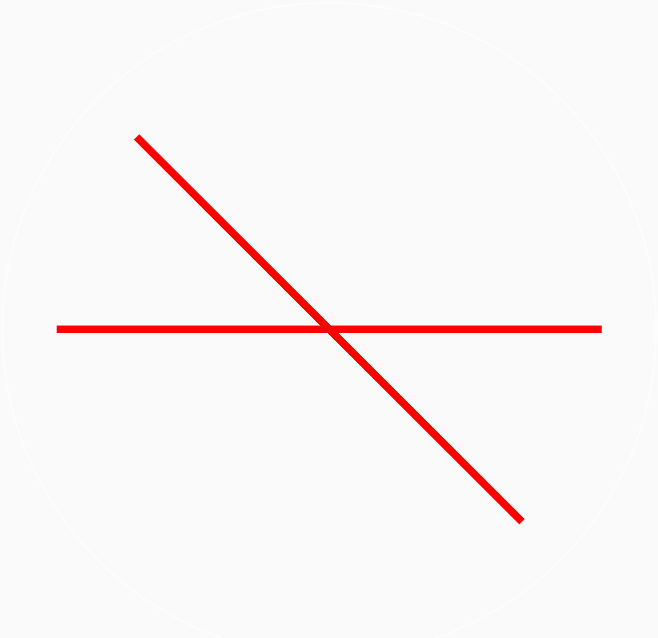
G-Permutahedron of polytope P in one region

$$\mathrm{P}^G_{\mathrm{perm}}(P) = \mathrm{conv}(\bigcup_{g \in G} g.P)$$

EXAMPLE: $I_2(4)$



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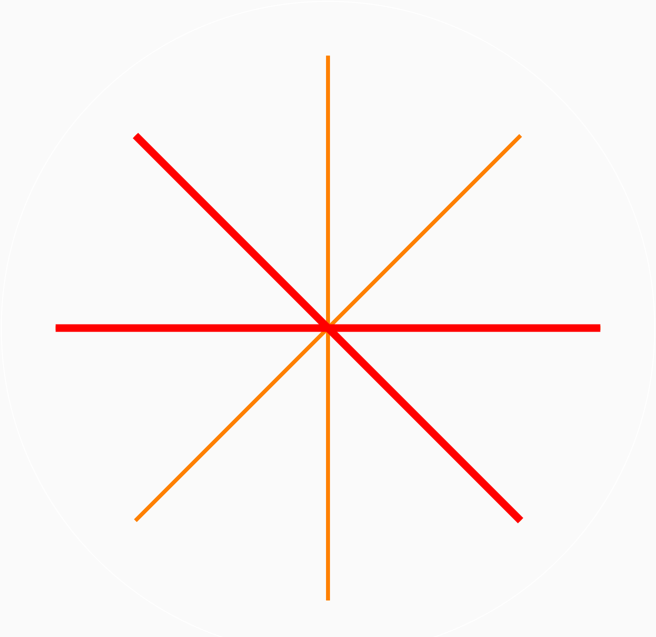

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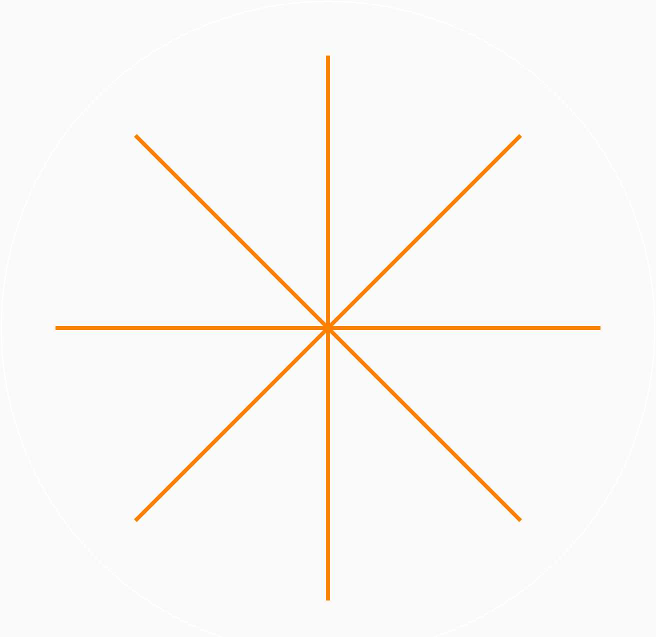

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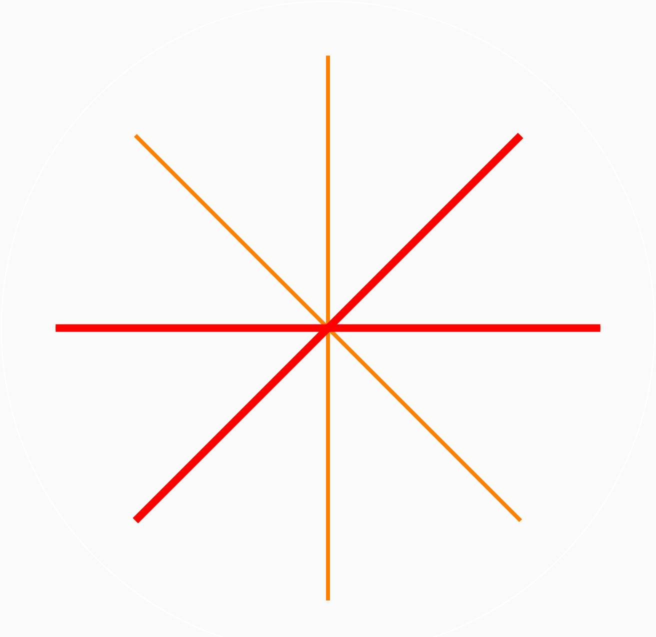

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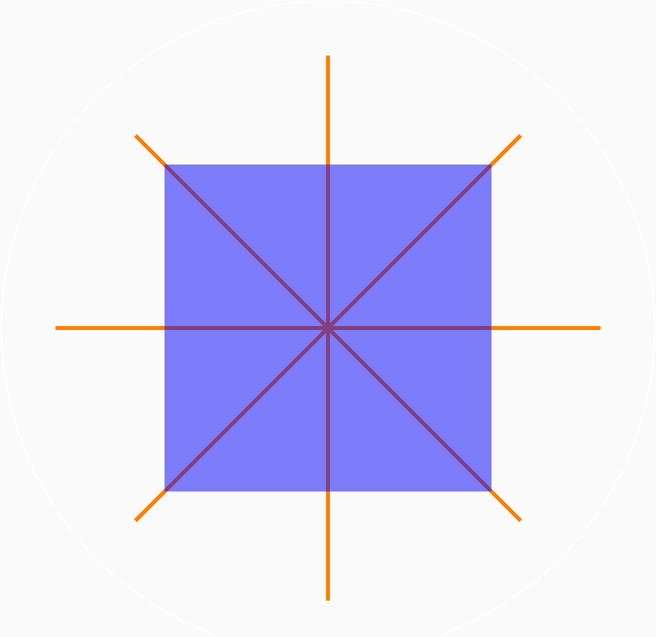

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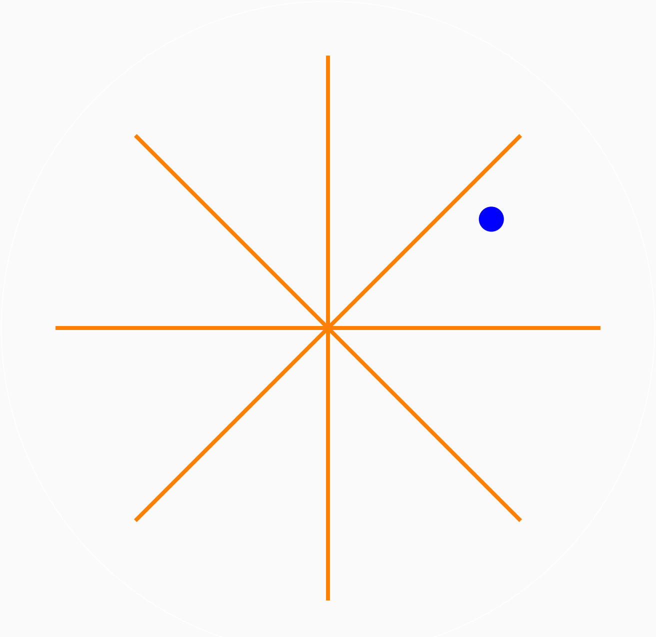

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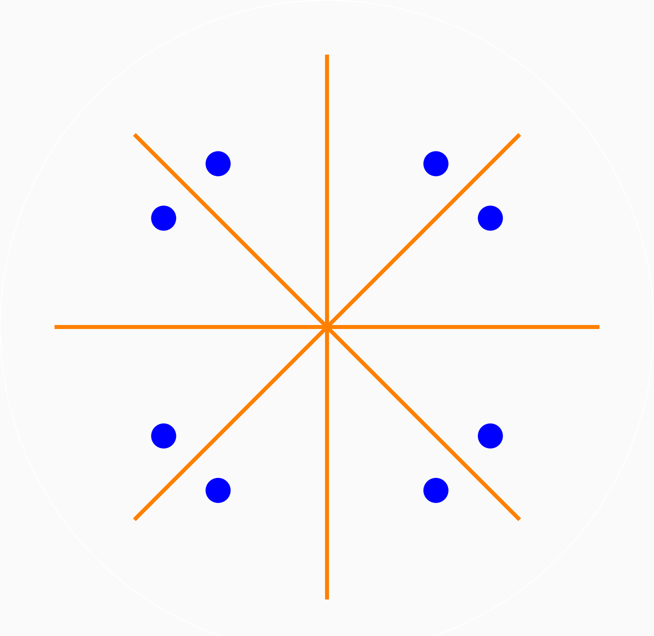

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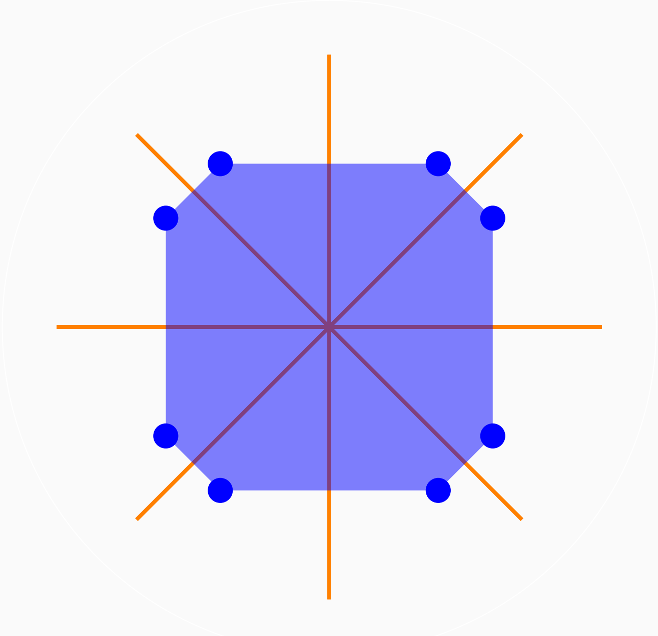
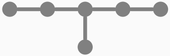
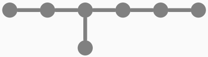


CLASSIFICATION OF IRREDUCIBLE REFLECTION GROUPS

Name	Dynkin Diagram	Regular Polytope
$I_2(m)$		m-gon
A_{n-1}		$(n-1)$-simplex
B_n		n-cube, n-cross polytope
D_n		
E_6		
E_7		
E_8		
F_4		24-cell
H_3		dodecahedron, icosahedron
H_4		120-cell, 600-cell

The group

- $H_\varphi = \mathrm{H}^{=}((-\sin\varphi, \cos\varphi), 0)$, $H_\varphi^{\leq} = \mathrm{H}^{\leq}((-\sin\varphi, \cos\varphi), 0)$

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If P lies in $\mathrm{FR}_{\mathrm{I}_2(m)}$:

$$\mathrm{P}_{\mathrm{perm}}^{\mathrm{I}_2(m)}(P) = \mathcal{R}_{H_{r\pi/m}^{\leq},\ldots,H_{4\pi/m}^{\leq},H_{2\pi/m}^{\leq},H_{\pi/m}^{\leq}}(P)$$

(with $r = \lceil \log(m) \rceil$)

THE REFLECTION GROUP $\mathrm{I}_2(m)$

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Thus we have:

$$\mathrm{xc}(\mathrm{P}_{\mathrm{perm}}^{I_2(m)}(P)) \leq \mathrm{xc}(P) + 2\lceil \log(m) \rceil + 2$$

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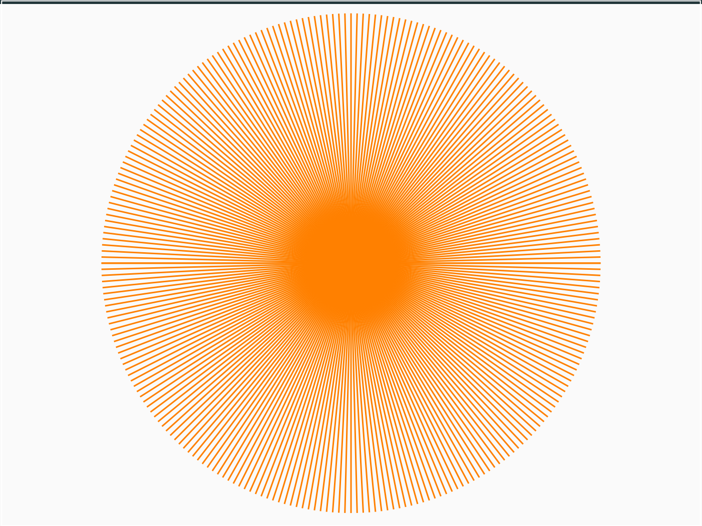

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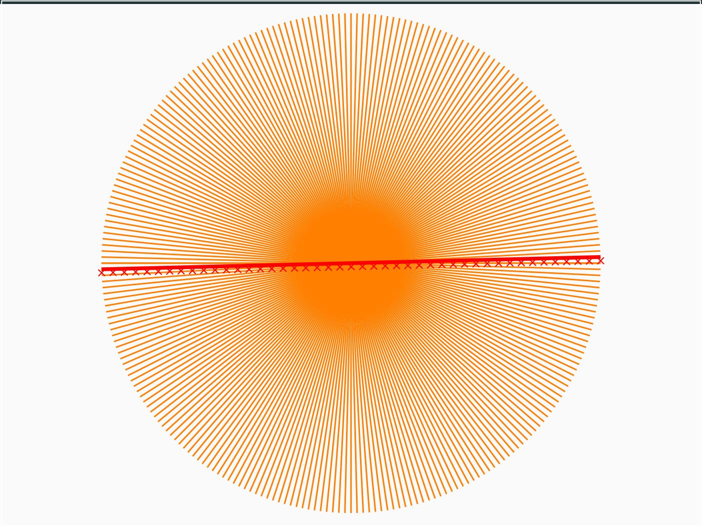

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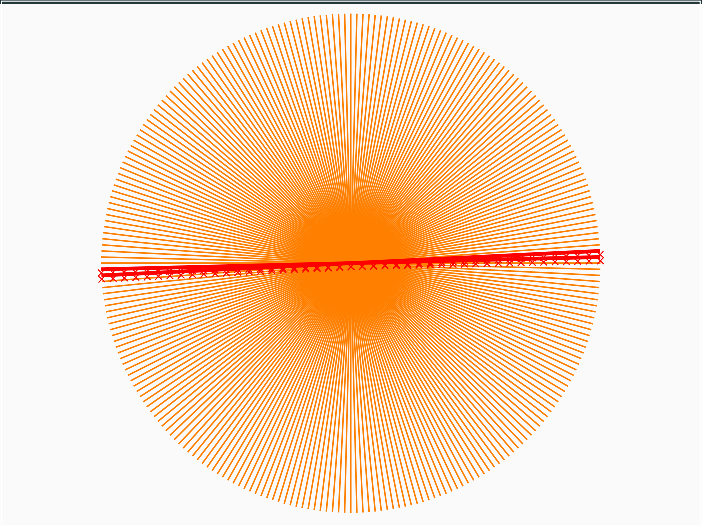

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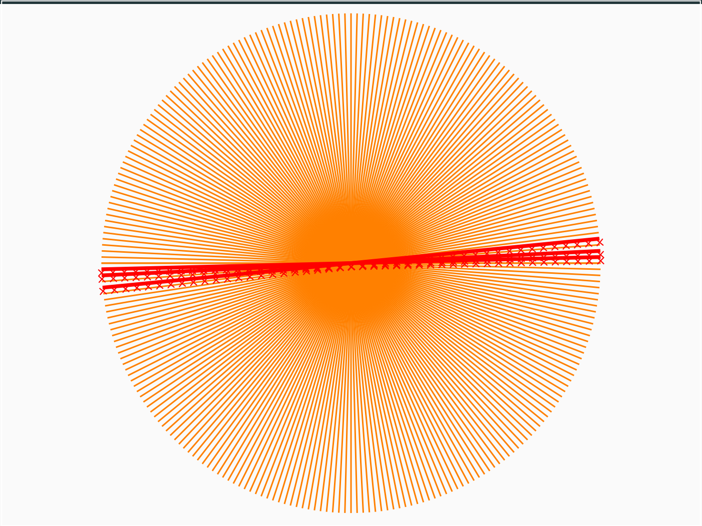

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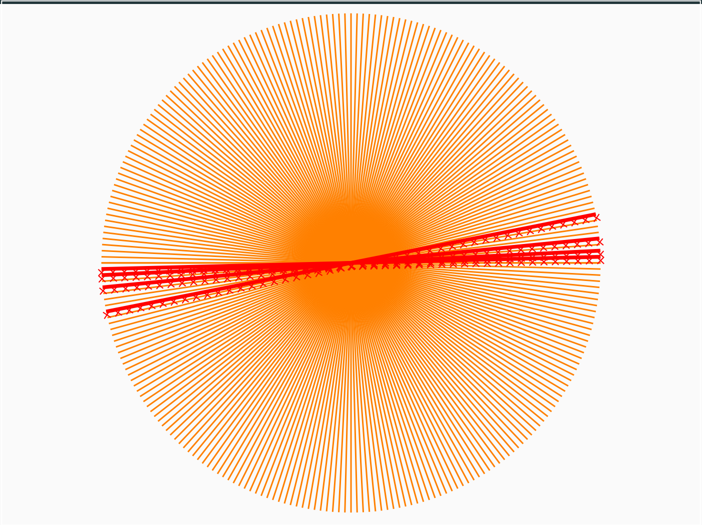

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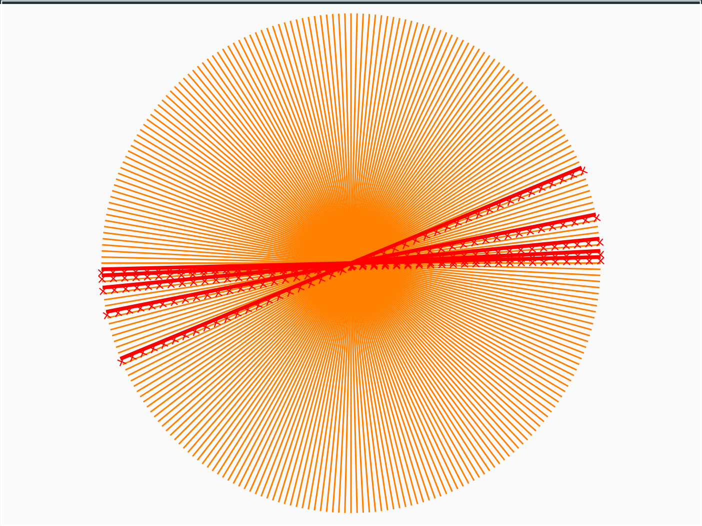

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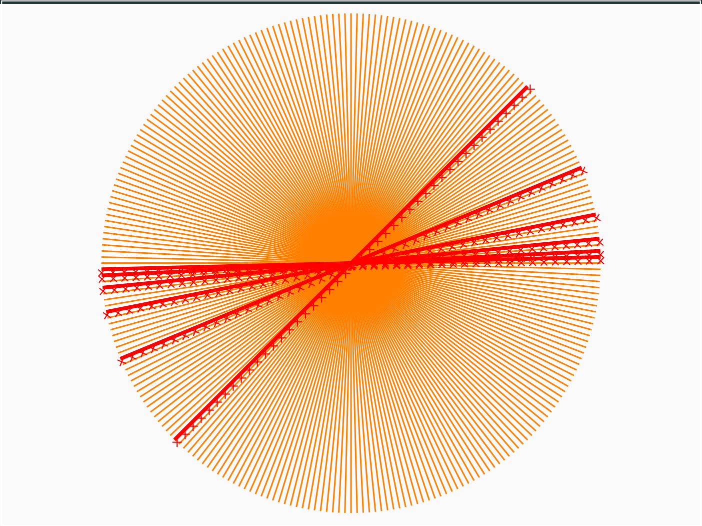

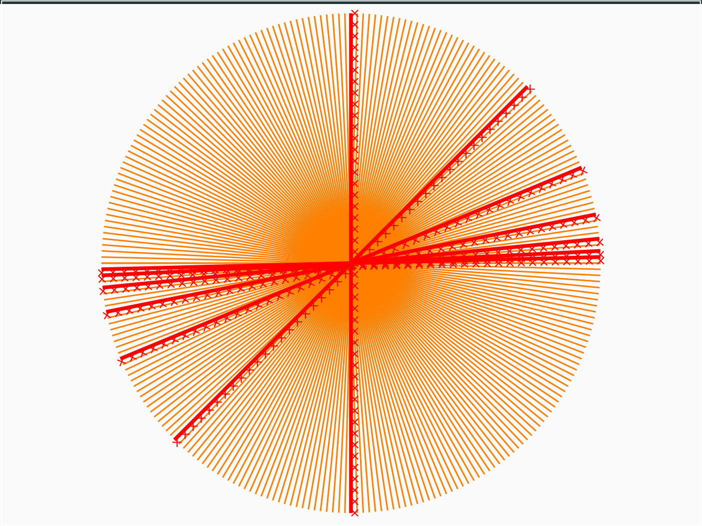

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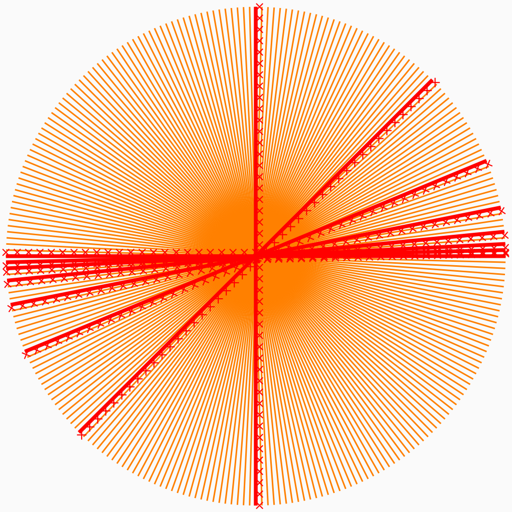

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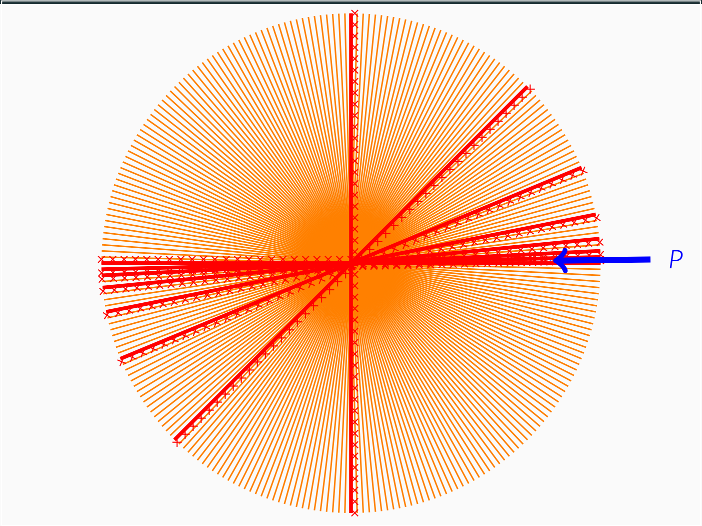

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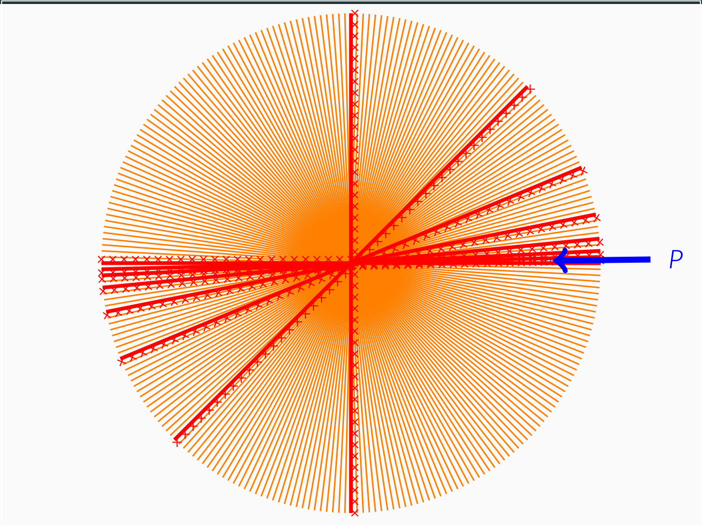

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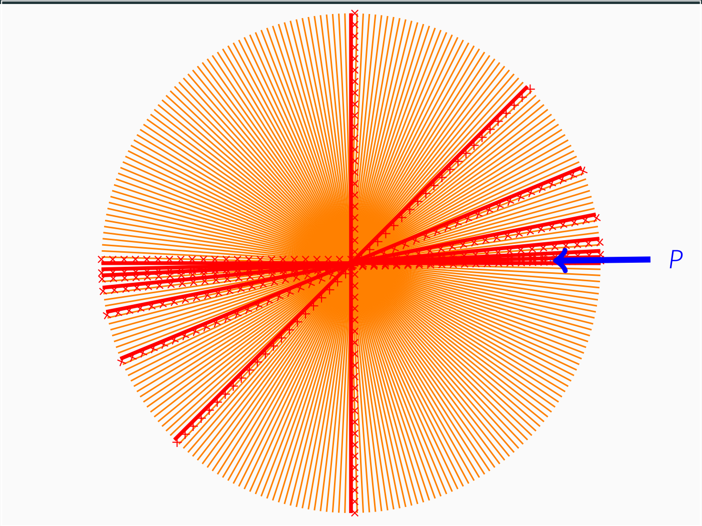

$\mathcal{R}_{H^{\leq}_{\pi/128}}(P)$

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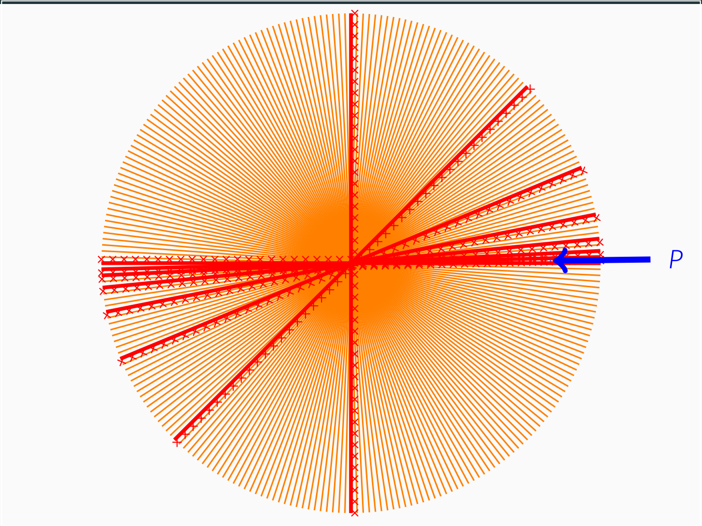

$\mathcal{R}_{H^{\leq}_{2\pi/128}, H^{\leq}_{\pi/128}}(P)$

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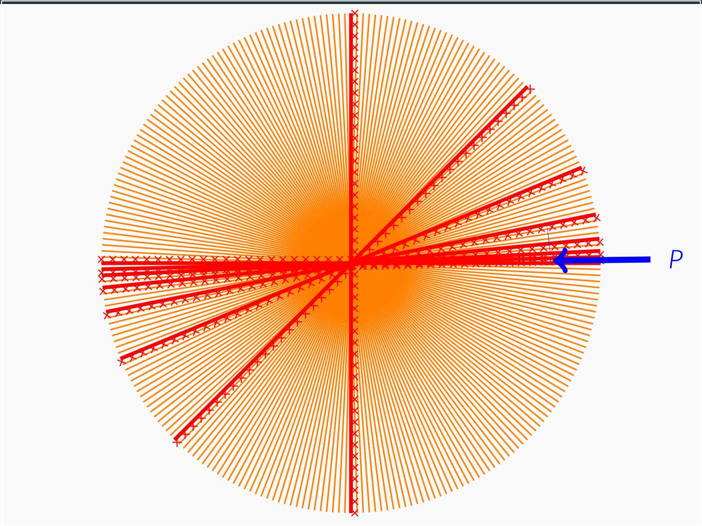

$\mathcal{R}_{H^{\leq}_{4\pi/128}, H^{\leq}_{2\pi/128}, H^{\leq}_{\pi/128}}(P)$

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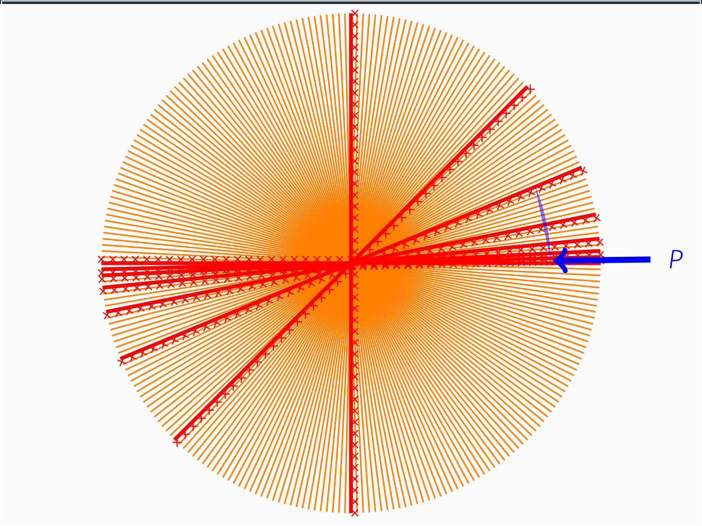

$\mathcal{R}_{H^{\leq}_{8\pi/128}, H^{\leq}_{4\pi/128}, H^{\leq}_{2\pi/128}, H^{\leq}_{\pi/128}}(P)$

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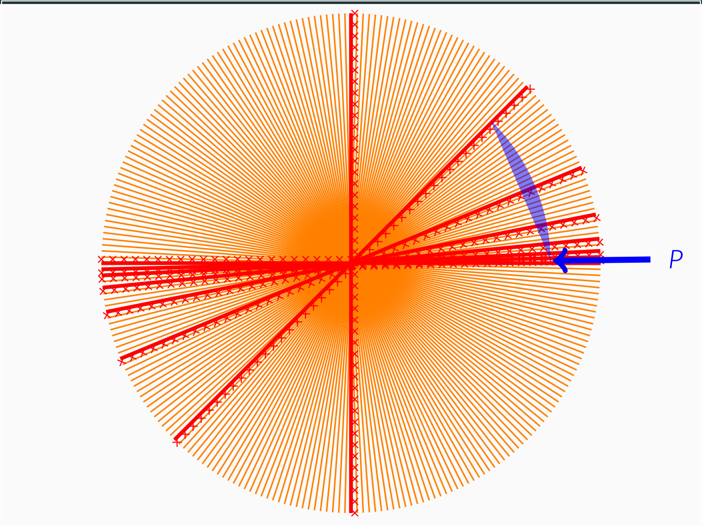

$\mathcal{R}_{H^{\leq}_{16\pi/128}, H^{\leq}_{8\pi/128}, H^{\leq}_{4\pi/128}, H^{\leq}_{2\pi/128}, H^{\leq}_{\pi/128}}(P)$

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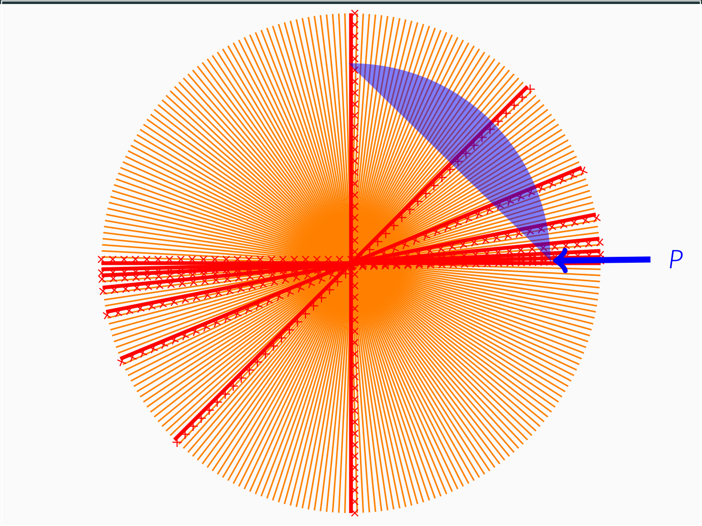

$\mathcal{R}_{H^{\leq}_{32\pi/128}, H^{\leq}_{16\pi/128}, H^{\leq}_{8\pi/128}, H^{\leq}_{4\pi/128}, H^{\leq}_{2\pi/128}, H^{\leq}_{\pi/128}}(P)$

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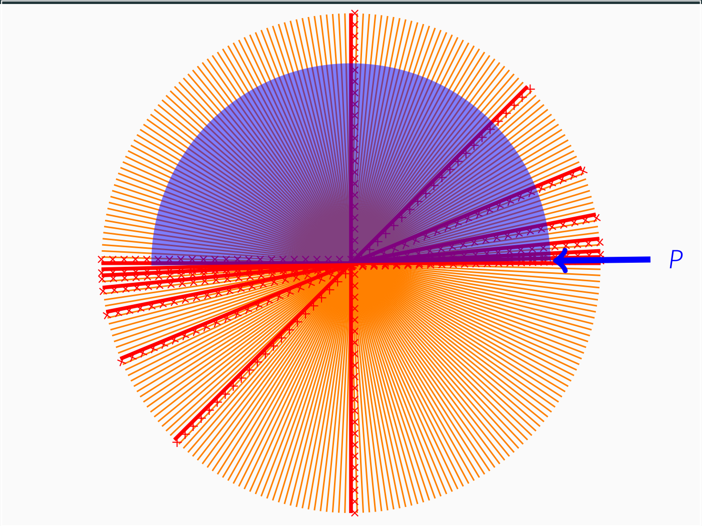

$$\mathcal{R}_{H^{\leq}_{64\pi/128},H^{\leq}_{32\pi/128},H^{\leq}_{16\pi/128},H^{\leq}_{8\pi/128},H^{\leq}_{4\pi/128},H^{\leq}_{2\pi/128},H^{\leq}_{\pi/128}}(P)$$

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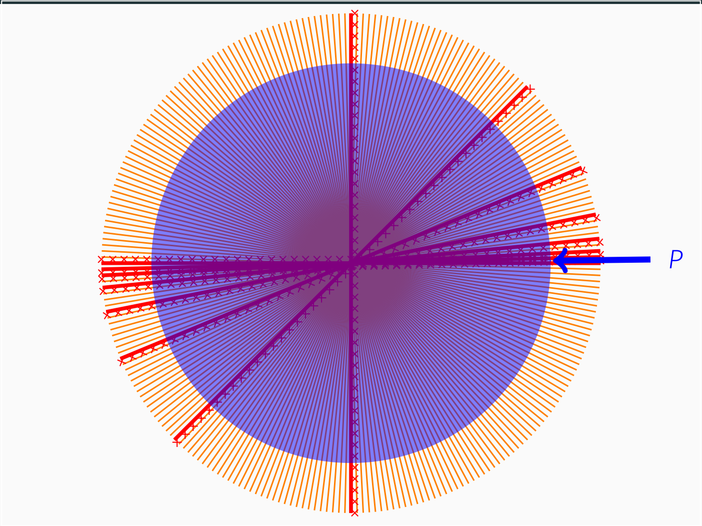

$\mathcal{R}_{H^{\leq}_{128\pi/128}, H^{\leq}_{64\pi/128}, H^{\leq}_{32\pi/128}, H^{\leq}_{16\pi/128}, H^{\leq}_{8\pi/128}, H^{\leq}_{4\pi/128}, H^{\leq}_{2\pi/128}, H^{\leq}_{\pi/128}}(P)$

The group

- $H_{k,\ell} = \mathsf{H}^{=}(\mathbb{e}_k - \mathbb{e}_\ell, 0)$, $H^{\leq}_{k,\ell} = \mathsf{H}^{\leq}(\mathbb{e}_k - \mathbb{e}_\ell, 0)$ $\subseteq \mathbb{R}^n$

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Conditional reflections

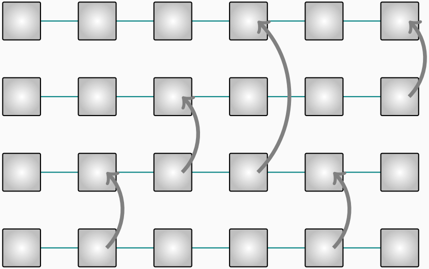
$$\tau^{k,\ell}_{>}(y) = \begin{cases} \tau^{k,\ell}(y) & \text{if } y_k > y_\ell \\ y & \text{otherwise} \end{cases}$$

Sorting Network

Sequence $(k_1, \ell_1), \ldots, (k_r, \ell_r)$ with

$$\tau_>^{k_1,\ell_1} \circ \cdots \circ \tau_>^{k_r,\ell_r}(y) = y^{(\mathrm{sort})}$$

for all $y \in \mathbb{R}^n$.



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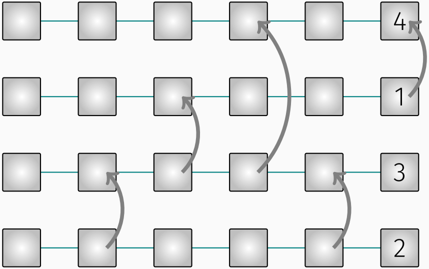

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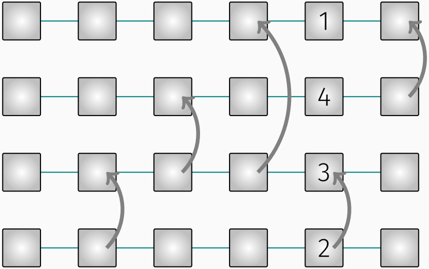

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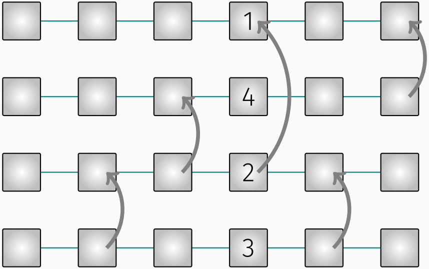

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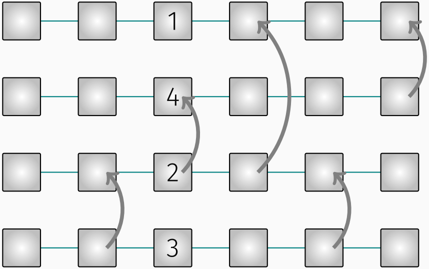

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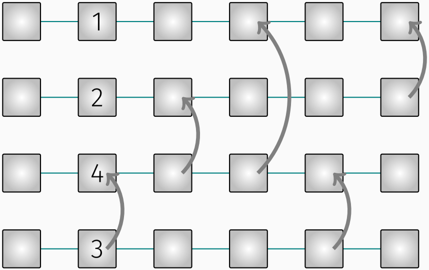

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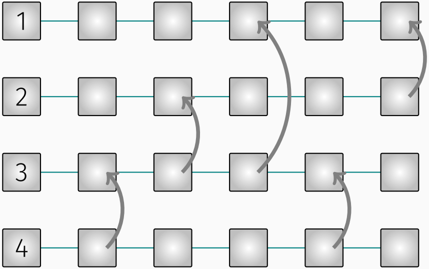

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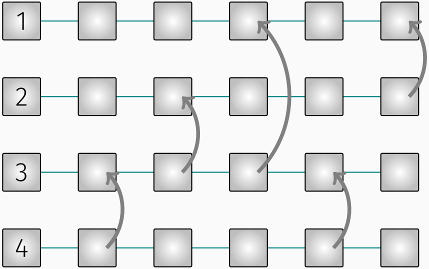

Ajtai, Komlós & Szemerédi 1983

There are sorting networks of size $r = O(n \log n)$.

RESULTS FOR A_{n-1}

If P lies in $\mathsf{FR}_{\mathsf{A}_{n-1}}$:

For each sorting network $(k_1, \ell_1), \ldots, (k_r, \ell_r)$, we have

$$\mathsf{P}^{\mathsf{A}_{n-1}}_{\mathrm{perm}}(P) = \mathcal{R}_{H^{\leq}_{k_r,\ell_r},,\ldots,H^{\leq}_{k_1,\ell_1}}(P).$$

If P lies in $\mathrm{FR}_{A_{n-1}}$:

For each sorting network $(k_1, \ell_1), \ldots, (k_r, \ell_r)$, we have

$$P^{A_{n-1}}_{\mathrm{perm}}(P) = \mathcal{R}_{H^{\leq}_{k_r,\ell_r},,\ldots,H^{\leq}_{k_1,\ell_1}}(P).$$

Thus we have:

$$\mathrm{xc}(P^{A_{n-1}}_{\mathrm{perm}}(P)) \leq \mathrm{xc}(P) + O(n \log(n))$$

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Thus we have:

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Goemans 2009

$$\mathrm{xc}(P_{\mathrm{perm}}^{n}) \leq O(n \log(n))$$

K & Pashkovich 11

If

- G is a finite reflection group on $\mathbb{R}^n$ and
- $P \subseteq \mathbb{R}^n$ is a polytope in one region of G,

K & Pashkovich 11

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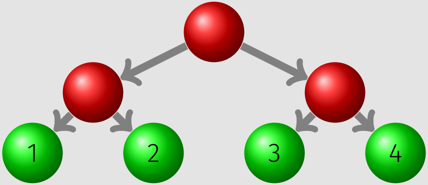
then we have:

$$\mathrm{xc}(\mathrm{P}^G_{\mathrm{perm}}(P)) \leq \mathrm{xc}(P) + \mathrm{O}(\log m + n \log n)$$

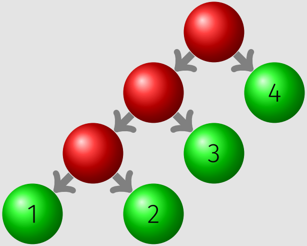
(where m is the largest number such that $\mathrm{I}_2(m)$ is a factor of G).

The set V^n_{huff} of Huffman vectors ($n = 4$)



$(2, 2, 2, 2)$

The set V^n_{huff} of Huffman vectors ($n = 4$)



$(3, 3, 2, 1)$

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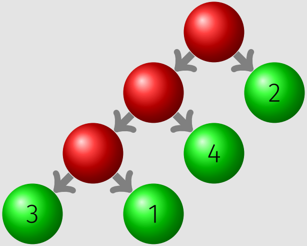

$(3, 1, 3, 2)$

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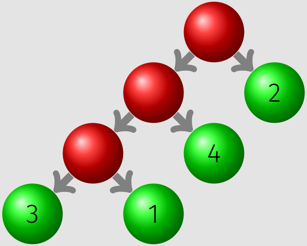

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$$P^n_{\text{huff}} = \text{conv}(V^n_{\text{huff}})$$

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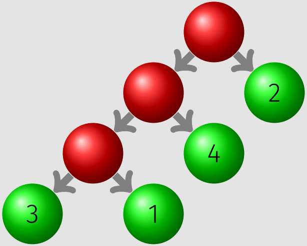

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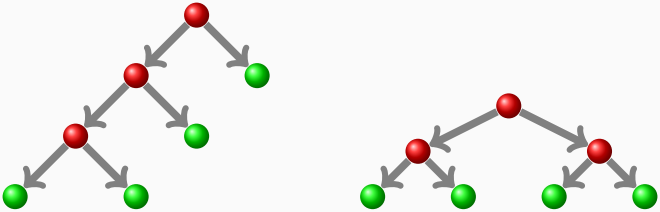
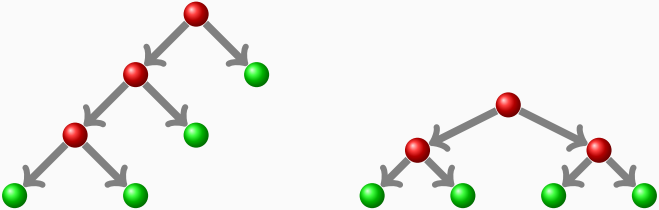
$$P^n_{\text{huff}} = \text{conv}(V^n_{\text{huff}})$$

Nguyen, Nguyen, & Maurras 10

P^n_{huff} has at least $2^{\Omega(n \log n)}$ facets.

AN EXTENDED FORMULATION OF SIZE $O(n^2)$





1. For $\gamma \in \mathfrak{S}(n)$: $\gamma . V^n_{\mathrm{huff}} = V^n_{\mathrm{huff}}$.

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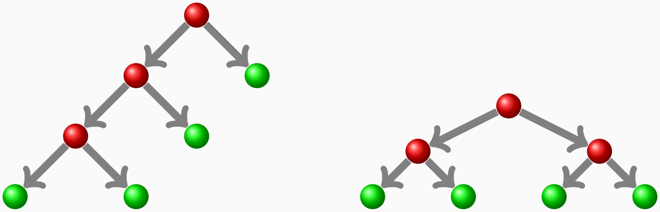

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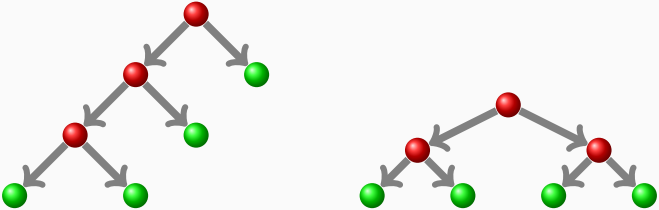
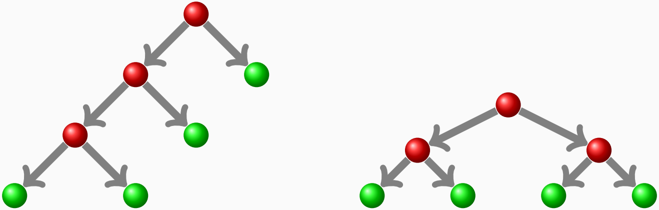
2. For $v \in \mathrm{V}^n_{\mathrm{huff}}$:



1. For $\gamma \in \mathfrak{S}(n)$: $\gamma . V^n_{\text{huff}} = V^n_{\text{huff}}$.
2. For $v \in V^n_{\text{huff}}$:
   2.1 There are $i \neq j$ with $v_i = v_j = \max\{v_k : k \in [n]\}$.



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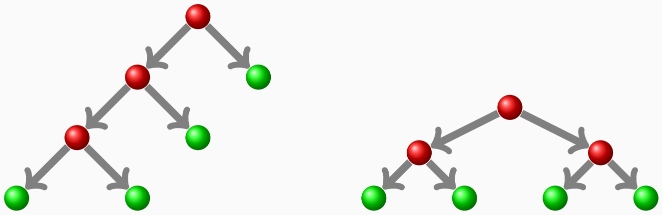
   2.2 $(v_1, \ldots, v_{i-1}, v_i - 1, v_{i+1}, \ldots, v_{j-1}, v_{j+1}, \ldots, v_n) \in \mathrm{V}^{n-1}_{\mathrm{huff}}$



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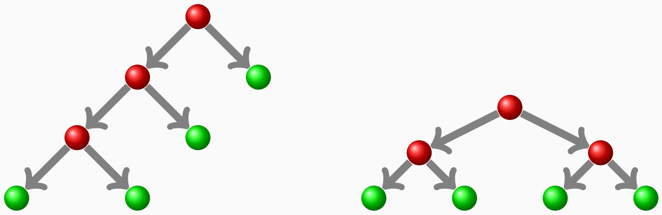
3. For $w' \in V_{\text{huff}}^{n-1}$: $\varphi(w') = (w'_1, \ldots, w'_{n-2}, w'_{n-1} + 1, w'_{n-1} + 1) \in V_{\text{huff}}^{n}$.



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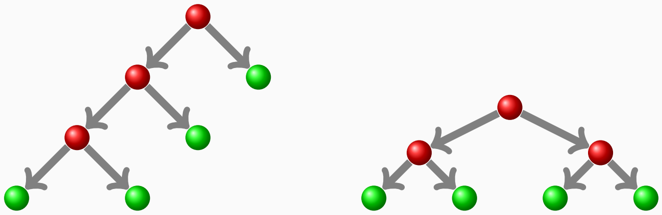
AN EXTENDED FORMULATION OF SIZE $O(n^2)$



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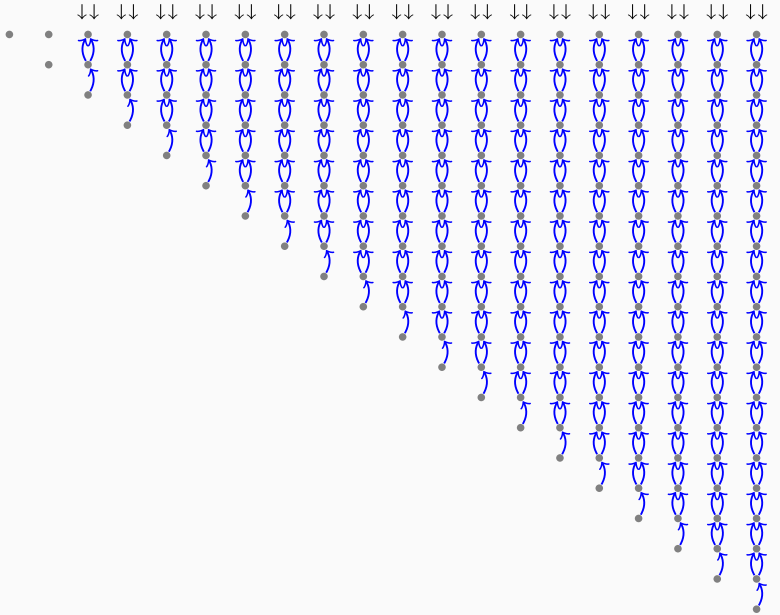
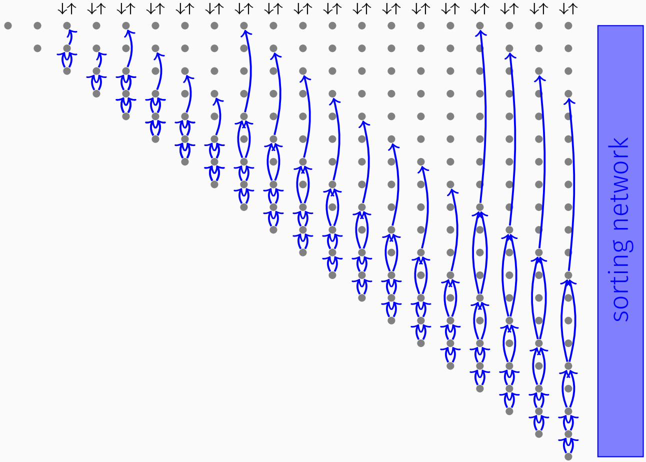
A first extended formulation

$$P^n_{\text{huff}} = \mathcal{R}_{H^{\leq}_{1,2}, \ldots, H^{\leq}_{n-1,n}, H^{\leq}_{1,2}, \ldots, H^{\leq}_{n-2,n-1}}(\varphi(P^{n-1}_{\text{huff}})).$$

SCHEMATIC VIEW ON THE CONSTRUCTION



A BETTER CONSTRUCTION



K & Pashkovich 11

$$\mathrm{xc}(P^n_{\mathrm{huff}}) \leq O(n \log n)$$

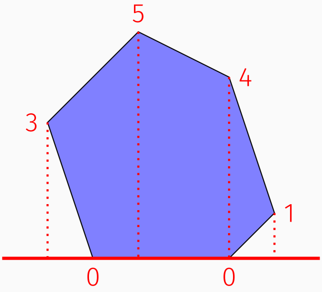
SLACK MATRICES

Definition

For a polytope $P = \text{conv}(V) = \{x \in \mathbb{R}^n \ : \ Wx \leq w\}$ the slack matrix w.r.t. the representation (V, W, w) is:

$$S(V, W, w) = (w, -W) \cdot \begin{pmatrix} \mathbf{1}^T \\ V \end{pmatrix} = (w, \ldots, w) - WV \geq \mathbf{0}$$

(entry (i, j): slack left by j-th generating point in i-th inequality)

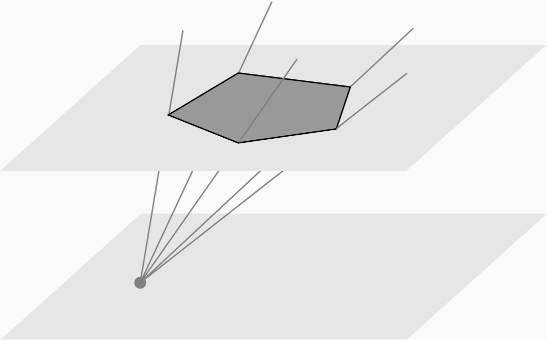


Definition

The *homogenization* of a polytope $P = \{x \in \mathbb{R}^n \,:\, Wx \leq w\}$ is:

$$\text{homog}(P) = \text{cone}(\{1\} \times P) = \{(x_0, x) \in \mathbb{R} \times \mathbb{R}^n \,:\, Wx \leq x_0 w\}$$

$\rightsquigarrow$ pointed (polyhedral) cone



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For $K = \{x \in \mathbb{R}^n \;:\; Ax \geq \mathbf{0}\} = B \cdot \mathbb{R}^k_+$: the *slack matrix* w.r.t. (A, B) is

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- $K = \mathrm{homog}(P)$: $S(V, W, w) = S(A, B)$ with $A = (w, -W)$, $B = \binom{\mathbf{1}^T}{V}$

Lemma

If $S \in \mathbb{R}_+^{m \times k}$ is a slack matrix of some cone K:

$$S \cdot \mathbb{R}_+^k = S \cdot \mathbb{R}^k \cap \mathbb{R}_+^m$$

A PROPERTY OF SLACK MATRICES

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K is pointed: $A \cdot K = S \cdot \mathbb{R}_+^k = S \cdot \mathbb{R}^k \cap \mathbb{R}_+^m$ is isomorphic to K (via $x \mapsto Ax$).

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Thus: $\mathrm{xc}(K) \leq q$

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Thus: With $T = \begin{pmatrix} t_1^T \\ \vdots \\ t_p^T \end{pmatrix}$: $P - T \in \mathbb{R}^{q \times p}_+$ and $S = (P - T)M$

A RESULT OF YANNAKAKIS' (1991)

Definition

The *nonnegative rank* $\mathrm{rank}_+(M)$ of $M \in \mathbb{R}_+^{m\times k}$ is the smallest r such that there are $U \in \mathbb{R}_+^{m\times r}$, $V \in \mathbb{R}_+^{r\times k}$ with $M = UV$.

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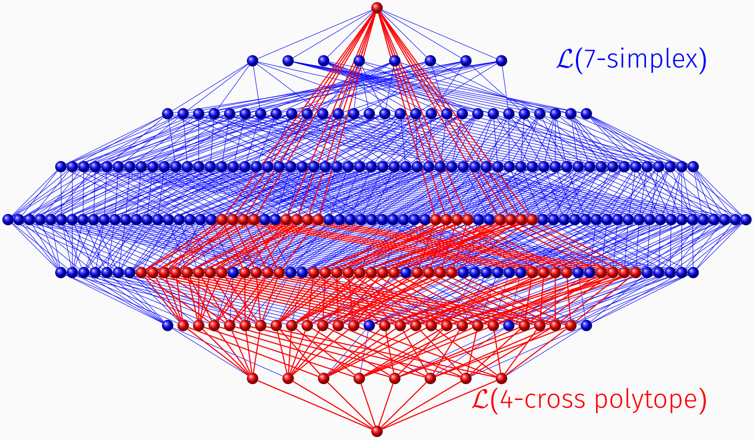
Appearances in:

- ▷ Communication Complexity
- ▷ Stochastic/Statistics
- ▷ Quantum Mechanics
- ▷ Algebraic Complexity

NON-INCIDENCES

Observation

If $P = p(Q)$ with a linear map p then $j : \mathcal{L}(P) \to \mathcal{L}(Q)$ with $F \mapsto p^{-1}(F) \cap Q$ is a lattice-embedding.



For a polytope P and two subsets $\mathcal{F}_1, \mathcal{F}_2 \subseteq \mathcal{L}(P)$:

Non-incidence graph: bipartite graph on $\mathcal{F}_1$ and $\mathcal{F}_2$ with

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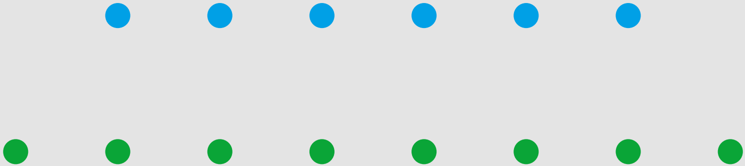
Example: $P =$ 3-cube, $\mathcal{F}_1 = \{\text{vertices}\}$, $\mathcal{F}_2 = \{\text{facets}\}$



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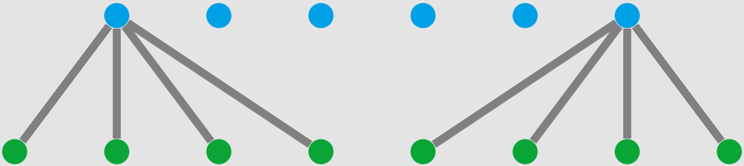

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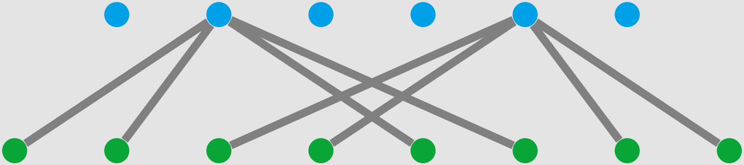

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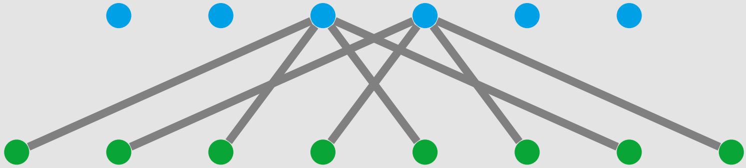

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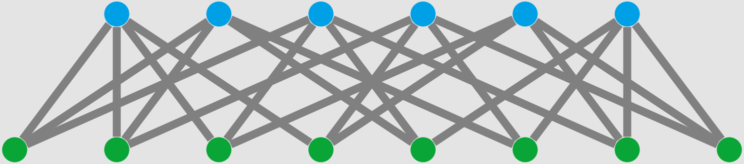

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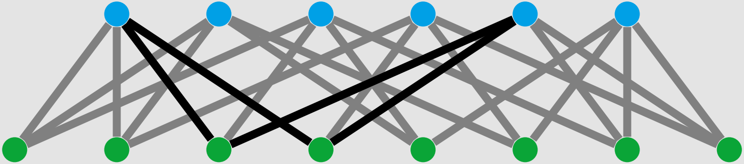

Biclique: complete bipartite subgraph

Lemma

If Q is an extension of P then the facets of Q induce a biclique-cover for every non-incidence graph of P.

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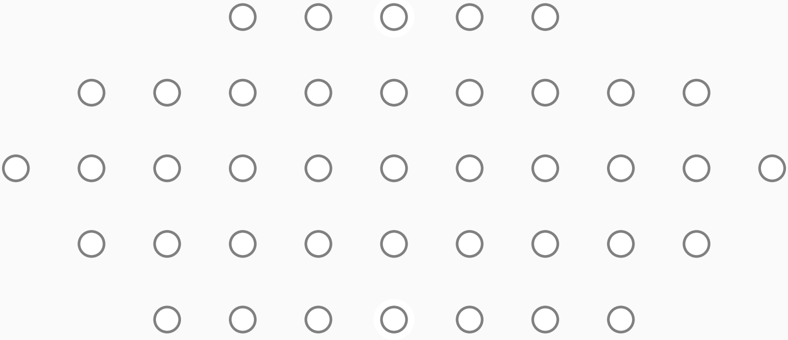

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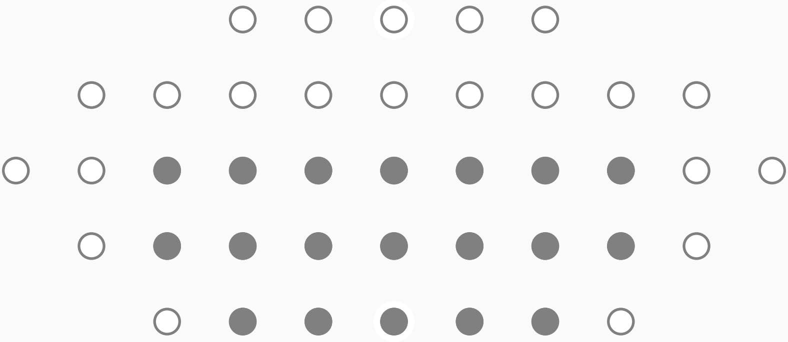

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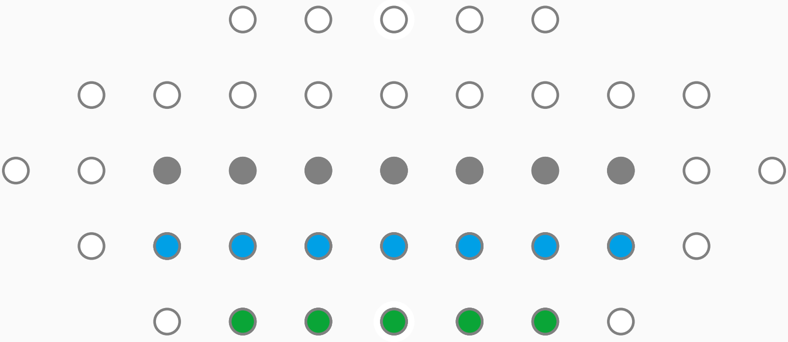

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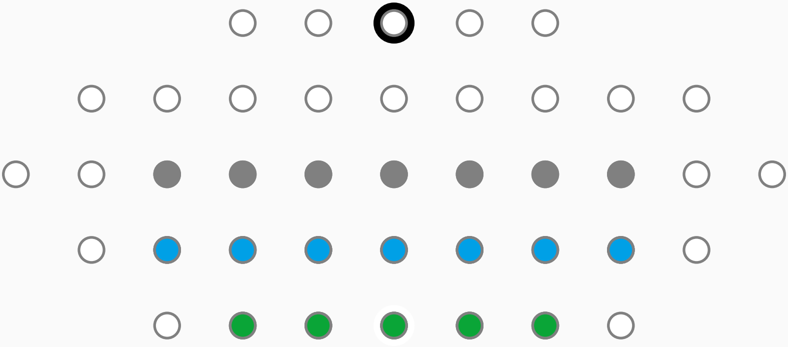

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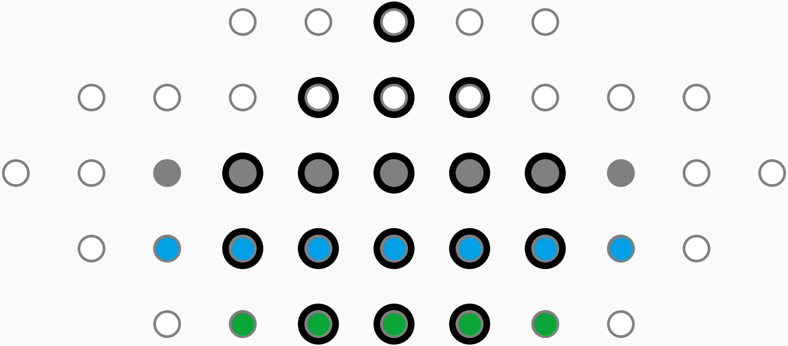

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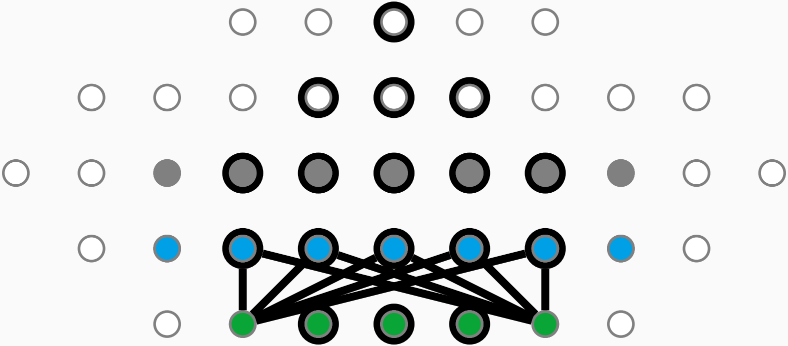

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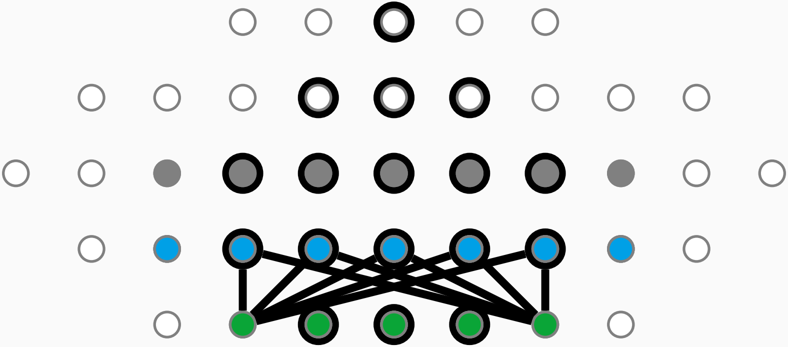

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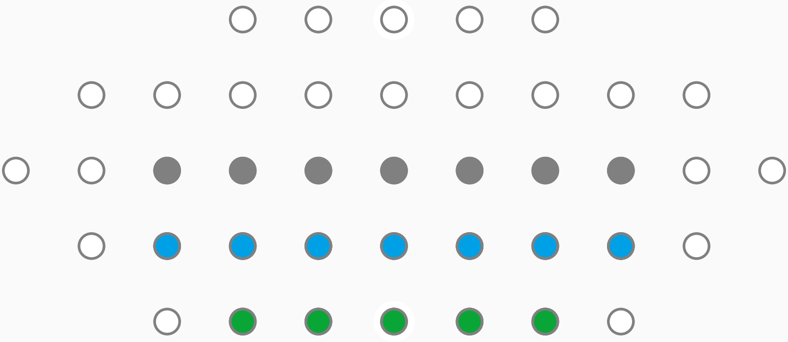

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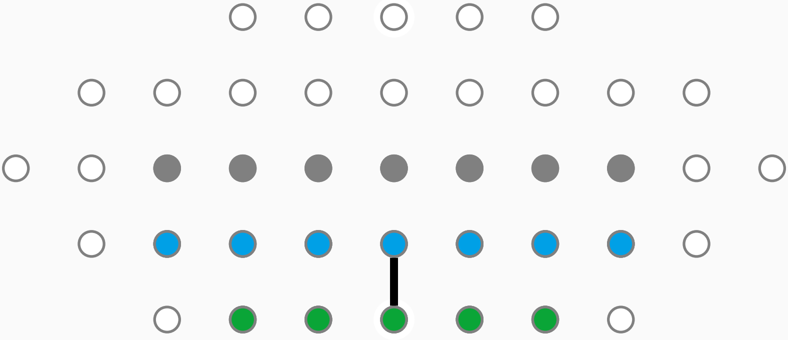

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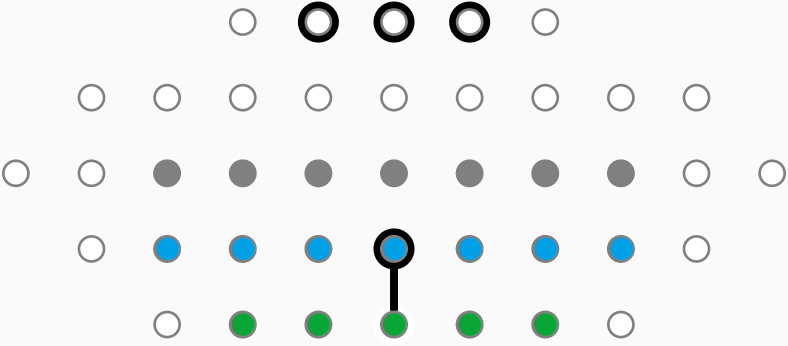

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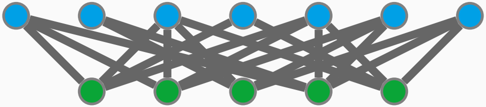
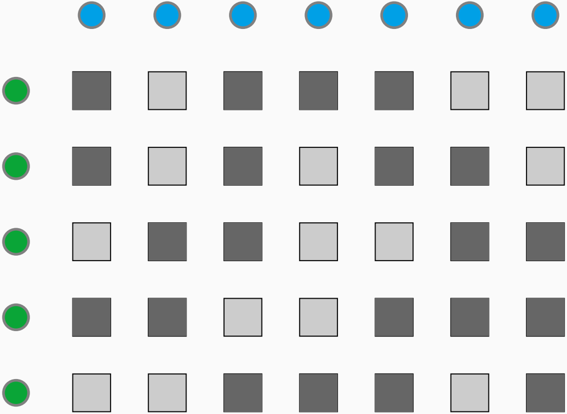
For every non-incidence graph G of P:

$$\mathrm{bc}(G) \leq \mathrm{xc}(P)$$

BICLIQUES VS. RECTANGLES





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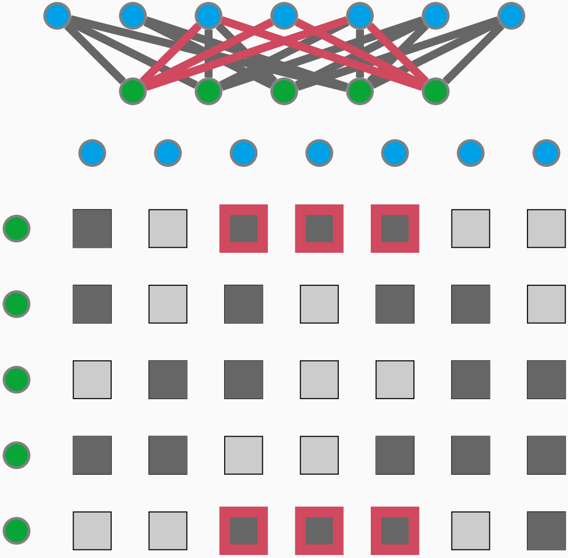

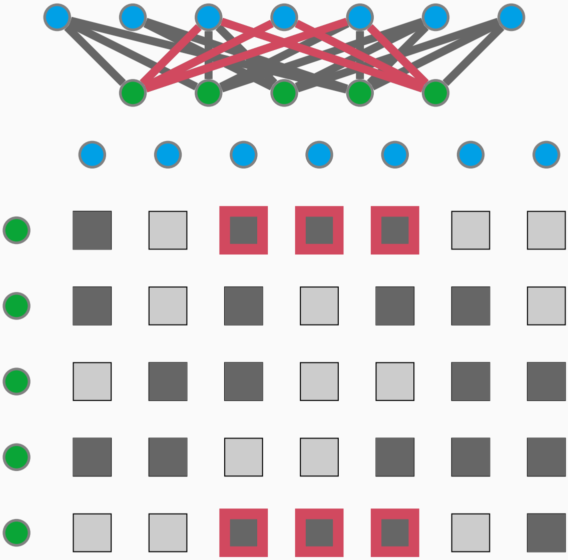

$$\mathrm{rc}(M) := \mathrm{bc}(G)$$

CORRELATION POLYTOPE

Definition

For $b \subseteq [n]$: define $y^b \in \{0,1\}^{n \times n}$ via $y^b_{ij} = 1 \Leftrightarrow i, j \in b$

Example: $n = 6$, $b = \{1, 4, 5\}$

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Thus:

$\mathrm{xc}(\mathrm{P}_{\mathrm{corr}}(n))$ non-polynomial $\Longrightarrow$ $\mathrm{xc}(\mathrm{P}_{\mathrm{tsp}}(\cdot))$, …non-polynomial

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SOME OF ITS FACES

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- $M(n)$: Corresponding non-incidence matrix (of size $2^n \times 2^n$)

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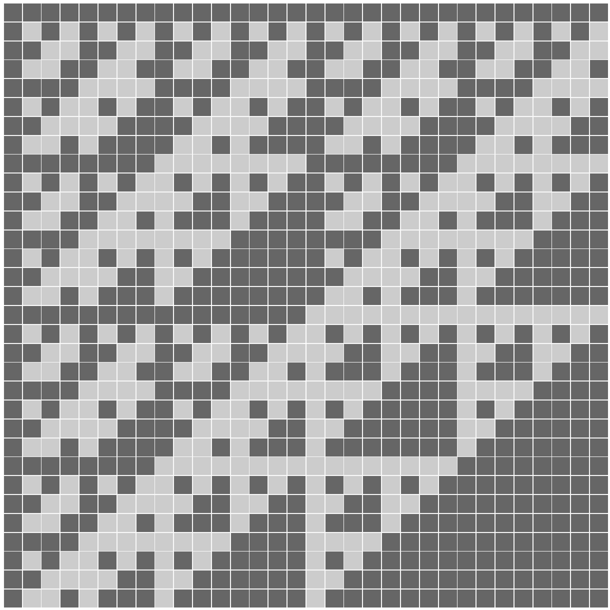
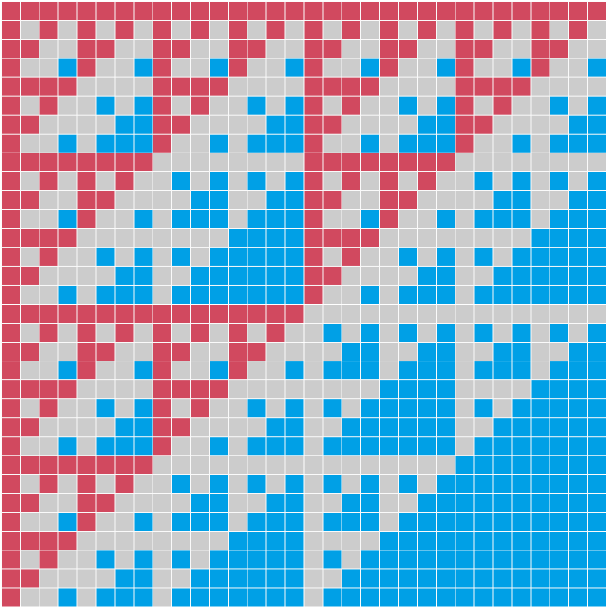
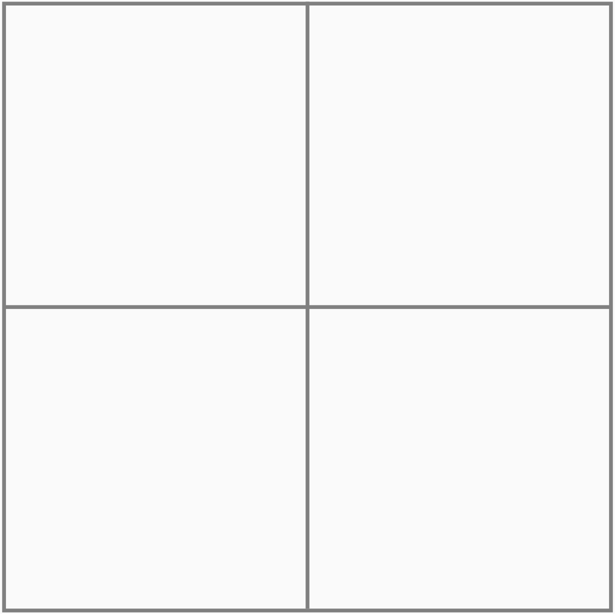
THE NON-INCIDENCE MATRIX $m(n)$



TWO TYPES OF NON-INCIDENCES: $em(n)$





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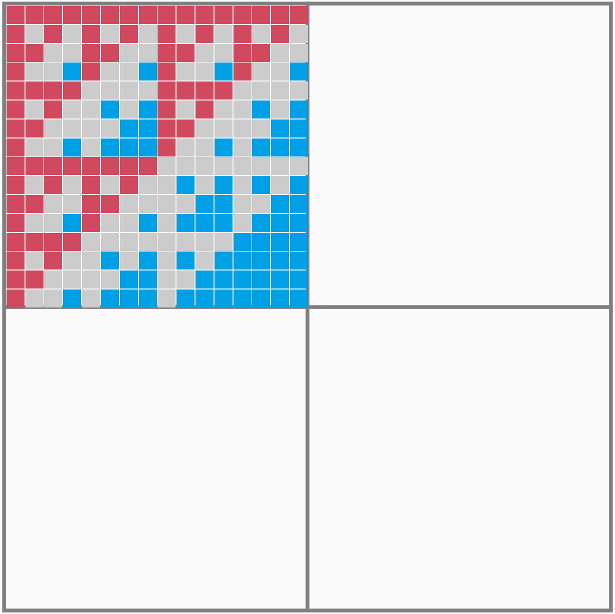

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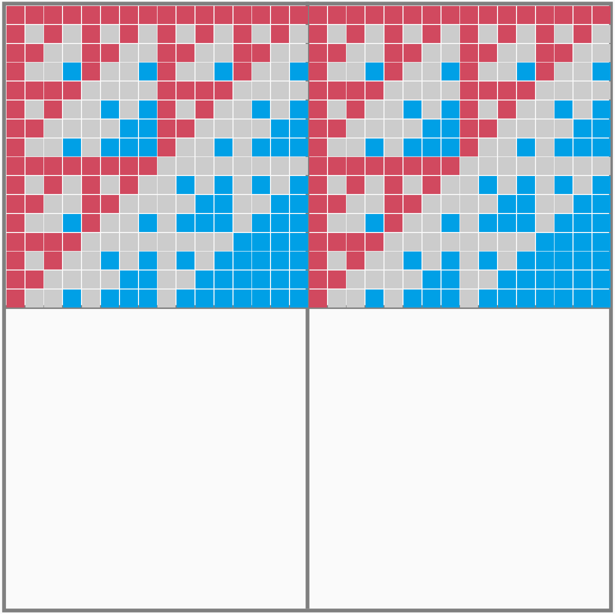

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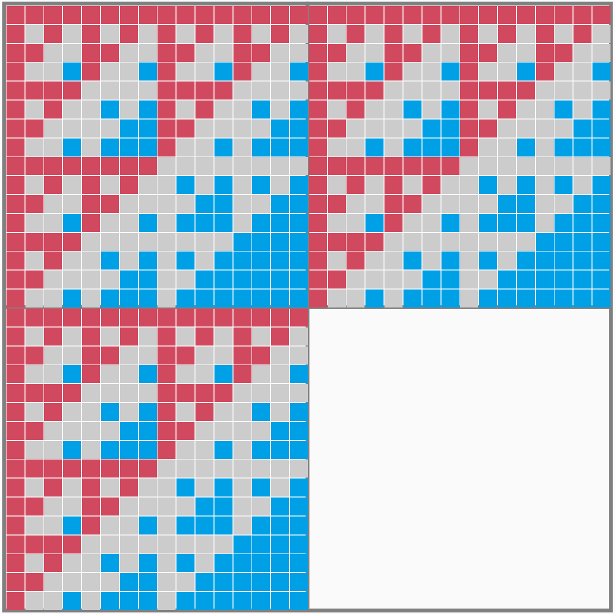

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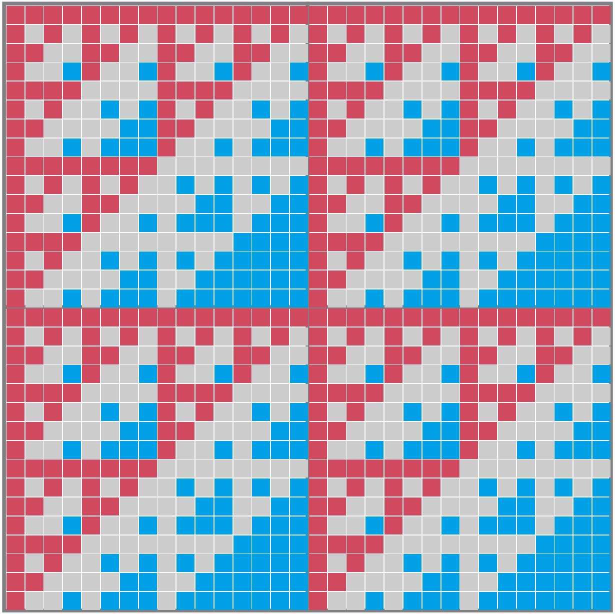

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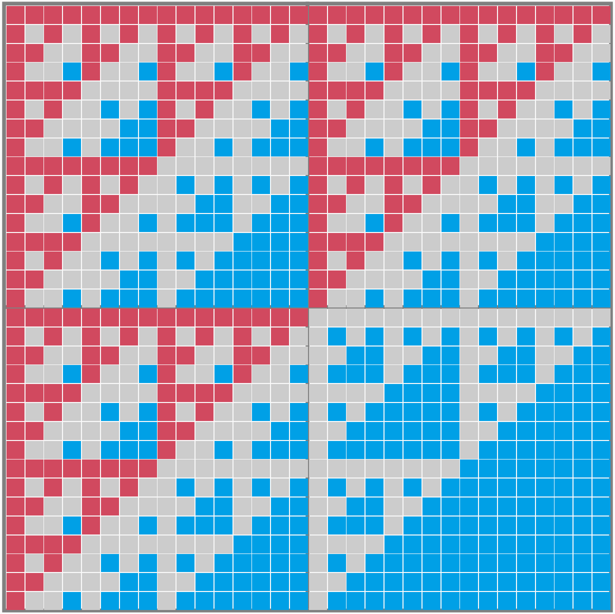

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- ▷ ■ (red): $|a \cap b| = 0$
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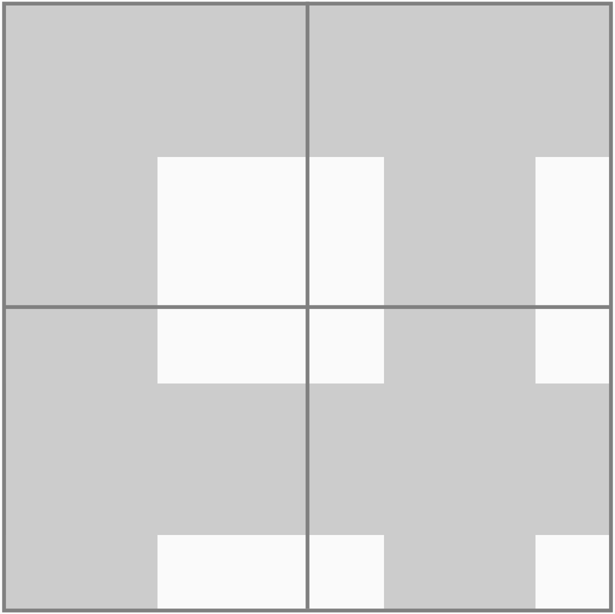
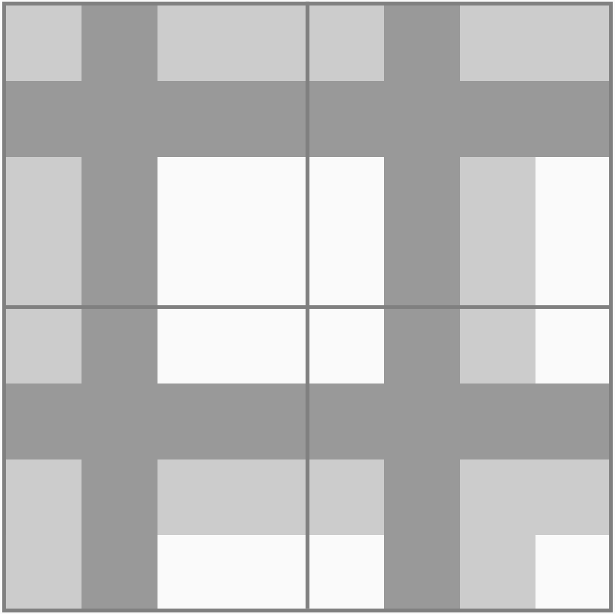
The goal: show $\varrho(n) \leq 2^n$.



BOUNDING RED'S IN GREY-FREE RECTANGLES



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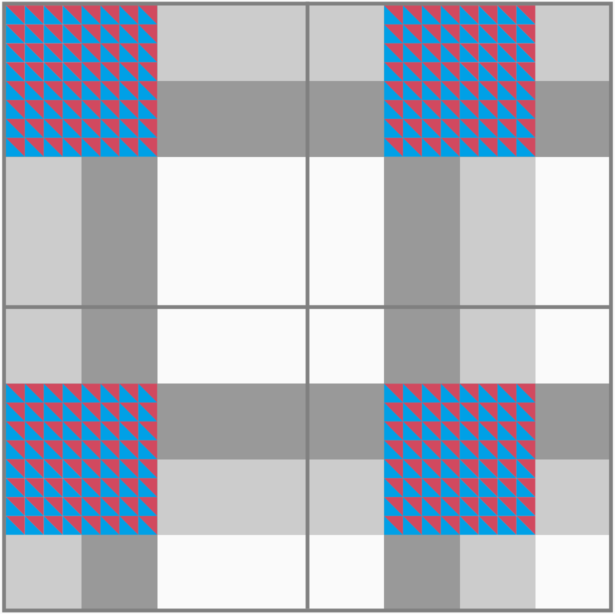

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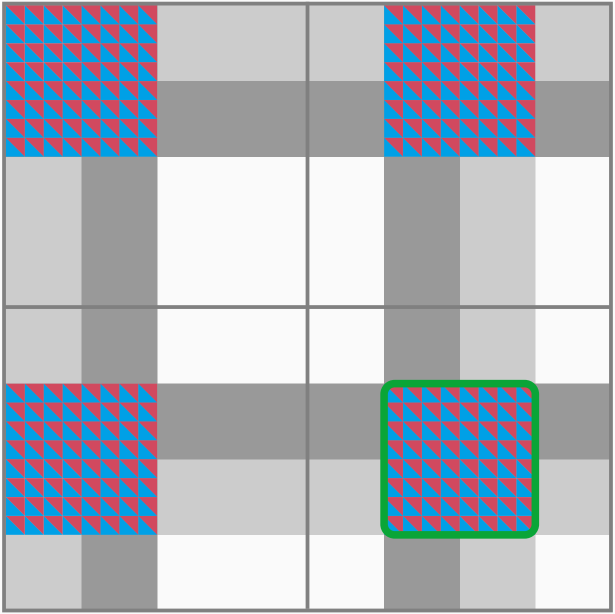

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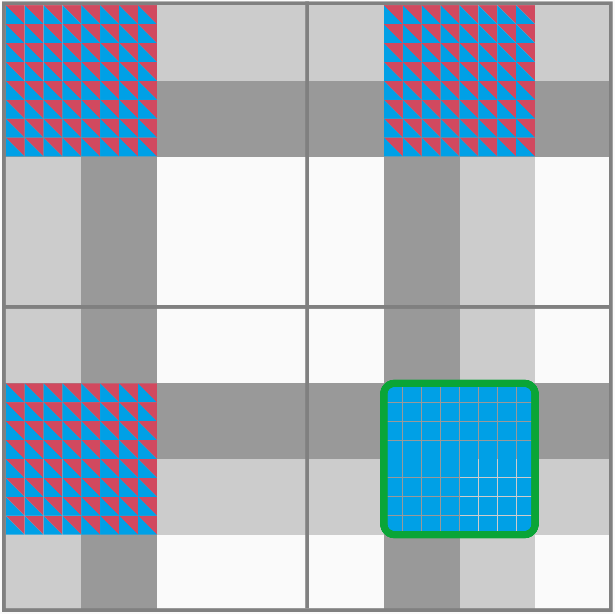

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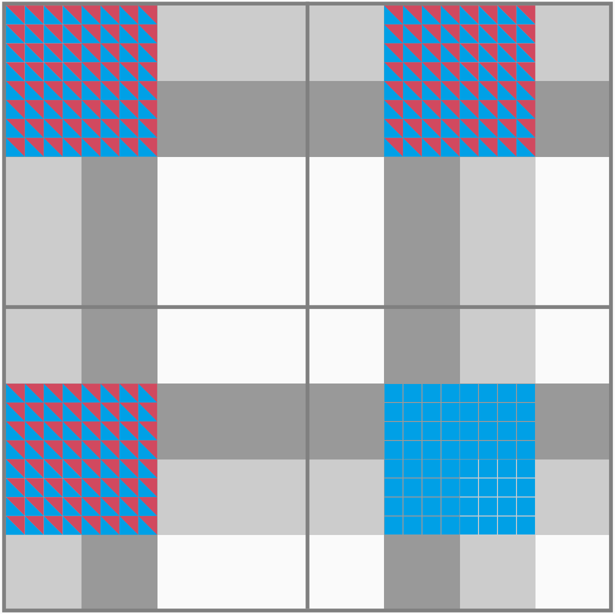

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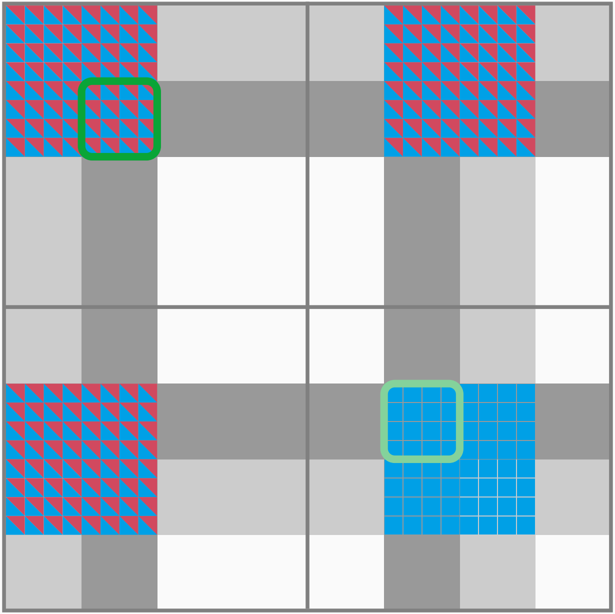

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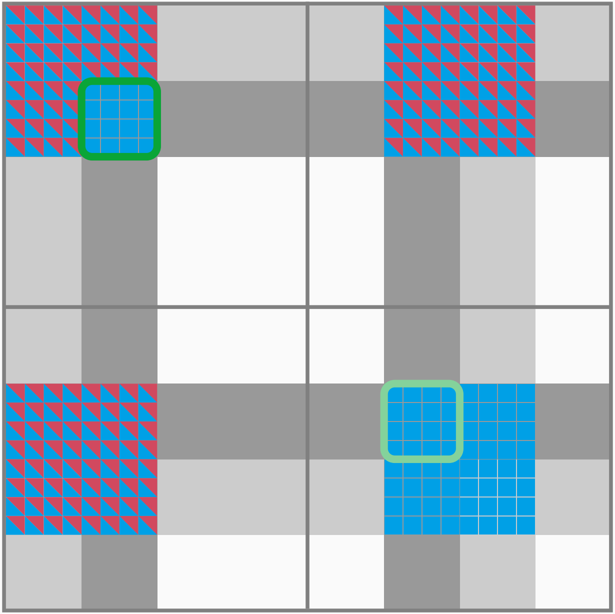

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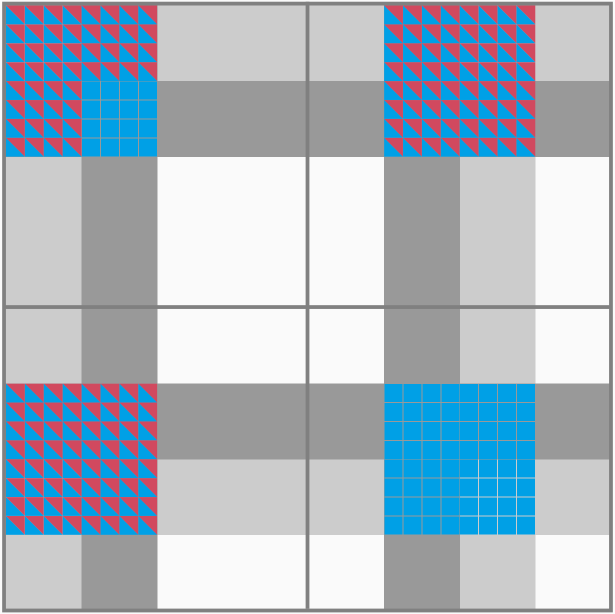

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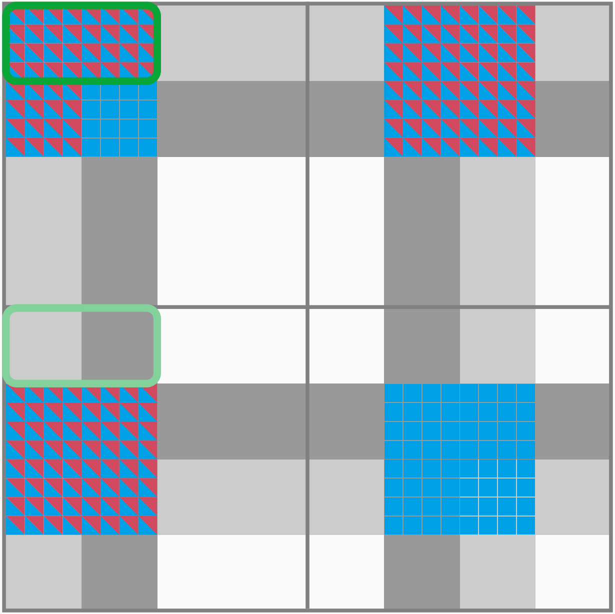
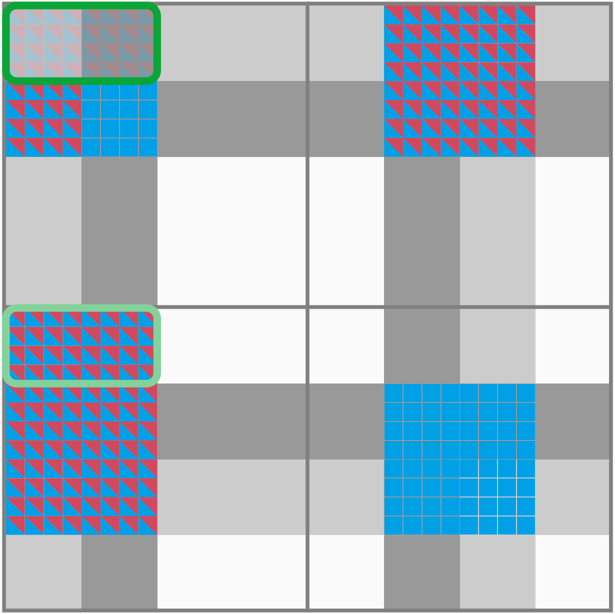
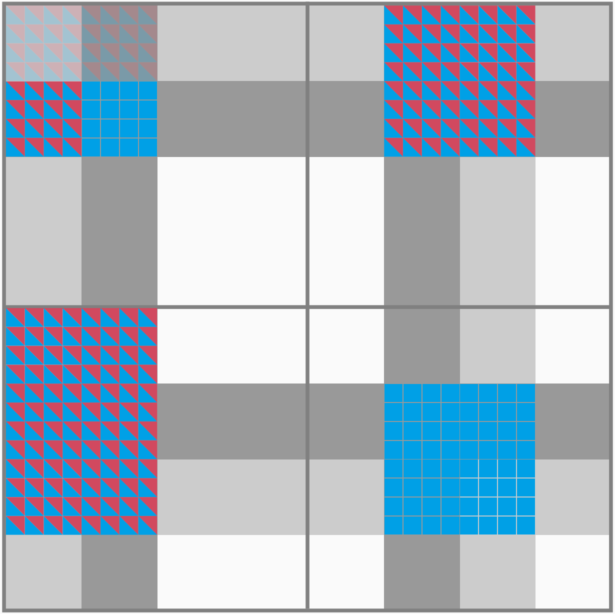


BOUNDING RED'S IN GREY-FREE RECTANGLES





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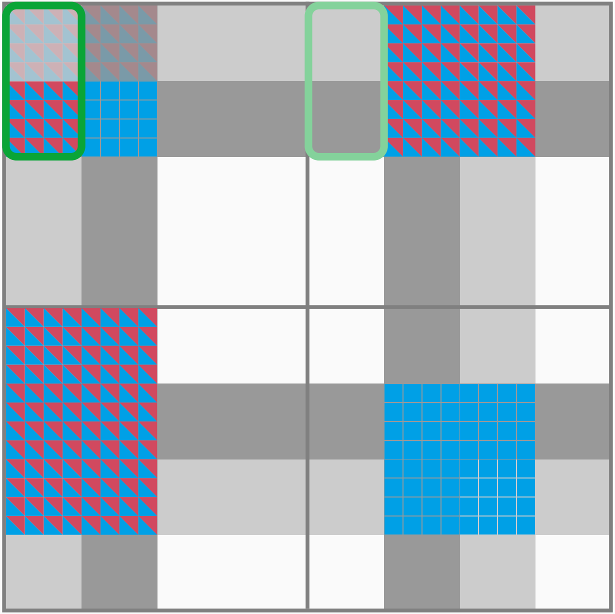

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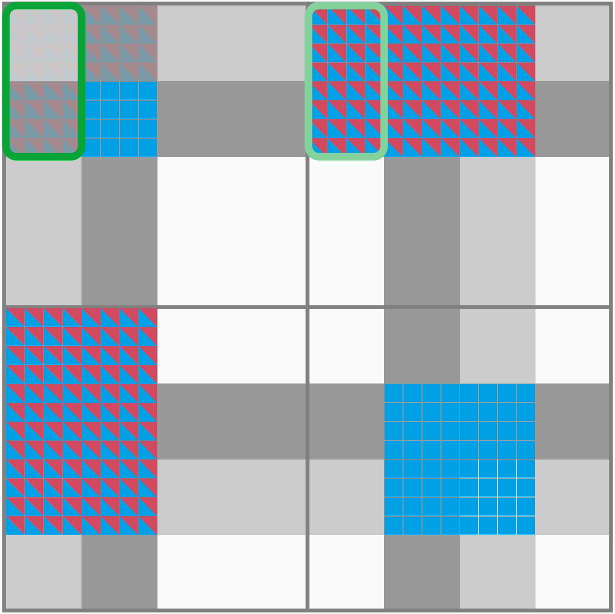

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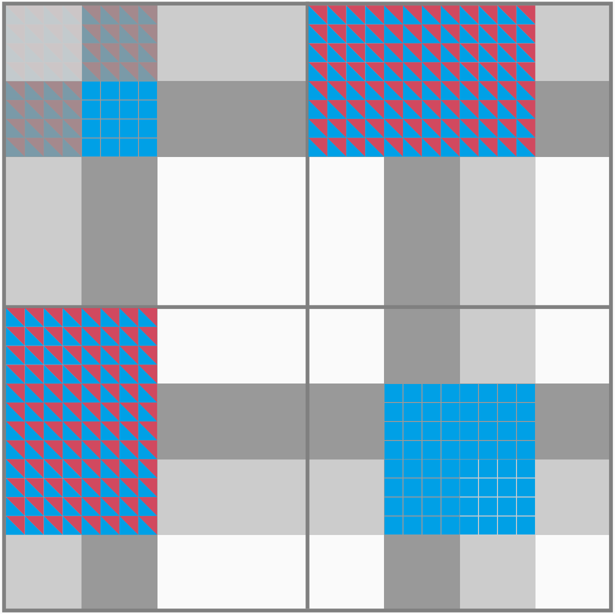

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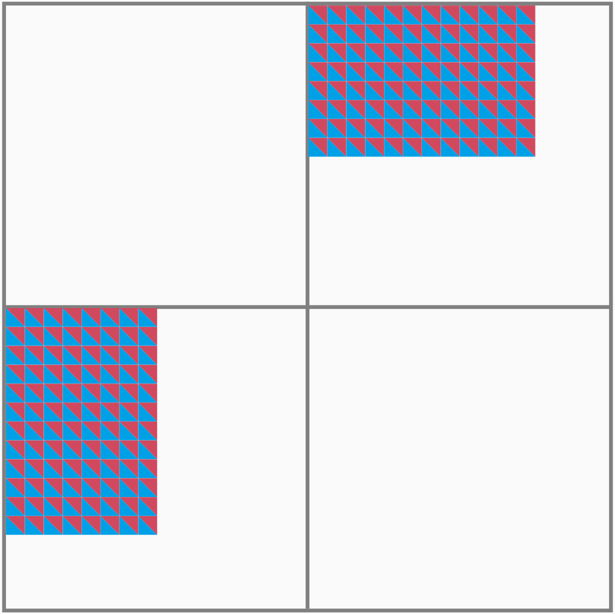

Thanks for your attention.