Spatial Decomposition/Coordination Methods for Stochastic Optimal Control Problems

Practical aspects and theoretical questions

J.-C. Alais, P. Carpentier, J-Ph. Chancelier, M. De Lara, V. Leclère

École des Ponts ParisTech

3 November 2014
Large scale storage systems stand as powerful motivation.
To make a long story short

We look after strategies as solutions of large scale stochastic optimal control problems, for example, the optimal management over a given time horizon of a large amount of dynamical production units.

- To obtain decision strategies (closed-loop controls), we use Dynamic Programming or related methods.
  - Assumption: Markovian case
  - Difficulty: curse of dimensionality
- To use decomposition/coordination techniques, we have to deal with the information pattern of the stochastic optimization problem.
A long-term effort in our group


Lecture outline

1. Decomposition and coordination
   - A bird’s eye view of decomposition methods
   - (A brief insight into Progressive Hedging)
   - Spatial decomposition methods in the deterministic case
   - The stochastic case raises specific obstacles

2. Dual approximate dynamic programming (DADP)
   - Problem statement
   - DADP principle and implementation
   - Numerical results on a small size problem

3. Theoretical questions
   - Existence of a saddle point
   - Convergence of the Uzawa algorithm
   - Convergence w.r.t. information

4. Conclusion
Decomposition and coordination

Decomposition-coordination: divide and conquer

- **Spatial decomposition**
  - Multiple players with their local information
  - Scales: local / regional / national / supranational

- **Temporal decomposition**
  - A state is an information summary
  - Time coordination realized through Dynamic Programming, by value functions
  - Hard nonanticipativity constraints

- **Scenario decomposition**
  - Along each scenario, sub-problems are deterministic (powerful algorithms)
  - Scenario coordination realized through Progressive Hedging, by updating nonanticipativity multipliers
  - Soft nonanticipativity constraints
Couplings for stochastic problems

\[
\min \sum_{\omega} \sum_{i} \sum_{t} \pi_{\omega} L_t^i(x^i_t, u^i_t, w_{t+1})
\]

M. De Lara (École des Ponts ParisTech)
Couplings for stochastic problems: in time

\[
\min \sum_{\omega} \sum_{i} \sum_{t} \pi_\omega L_t^i(x_t^i, u_t^i, w_{t+1})
\]

s.t. \[x_{t+1}^i = f_t^i(x_t^i, u_t^i, w_{t+1})\]
Couplings for stochastic problems: in uncertainty

\[ \min \sum_{\omega} \sum_{i} \sum_{t} \pi_{\omega} L_{t}^{i}(x_{t}^{i}, u_{t}^{i}, w_{t+1}) \]

s.t.  \[ x_{t+1}^{i} = f_{t}^{i}(x_{t}^{i}, u_{t}^{i}, w_{t+1}) \]

\[ u_{t}^{i} = \mathbb{E} \left( u_{t}^{i} \ \middle| \ w_{1}, \ldots, w_{t} \right) \]
Decomposition and coordination

A bird’s eye view of decomposition methods

Couplings for stochastic problems: in space

\[
\begin{align*}
\min & \sum_{\omega} \sum_{i} \sum_{t} \pi_{\omega} L_{t}^{i}(x_{t}^{i}, u_{t}^{i}, w_{t+1}) \\
\text{s.t.} & \quad x_{t+1}^{i} = f_{t}^{i}(x_{t}^{i}, u_{t}^{i}, w_{t+1}) \\
& \quad u_{t}^{i} = \mathbb{E}\left( u_{t}^{i} \mid w_{1}, \ldots, w_{t} \right) \\
& \quad \sum_{i} \theta_{t}^{i}(x_{t}^{i}, u_{t}^{i}) = 0
\end{align*}
\]
Can we decouple stochastic problems?

\[
\min \sum_{\omega} \sum_{i} \sum_{t} \pi_{\omega} L_{t}^{i}(x_{t}^{i}, u_{t}^{i}, w_{t+1})
\]

s.t. \( x_{t+1}^{i} = f_{t}^{i}(x_{t}^{i}, u_{t}^{i}, w_{t+1}) \)

\[
u_{t}^{i} = \mathbb{E} \left( u_{t}^{i} \mid w_{1}, \ldots, w_{t} \right)
\]

\[
\sum_{i} \theta_{t}^{i}(x_{t}^{i}, u_{t}^{i}) = 0
\]
Decompositions for stochastic problems: in time

\[
\begin{align*}
\min & \sum_{\omega} \sum_{i} \sum_{t} \pi_{\omega} L_{t}^{i}(x_{t}^{i}, u_{t}^{i}, w_{t+1}) \\
\text{s.t.} & \quad x_{t+1}^{i} = f_{t}^{i}(x_{t}^{i}, u_{t}^{i}, w_{t+1}) \\
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& \quad \sum_{i} \theta_{t}^{i}(x_{t}^{i}, u_{t}^{i}) = 0
\end{align*}
\]

Dynamic Programming
Bellman (56)
Decompositions for stochastic problems: in uncertainty

\[ \min \sum_{\omega} \sum_{i} \sum_{t} \pi_{\omega} L_{t}^{i}(x_{t}^{i}, u_{t}^{i}, w_{t+1}) \]

s.t. \( x_{t+1}^{i} = f_{t}^{i}(x_{t}^{i}, u_{t}^{i}, w_{t+1}) \)

\[ u_{t}^{i} = \mathbb{E} \left( u_{t}^{i} \mid w_{1}, \ldots, w_{t} \right) \]

\[ \sum_{i} \theta_{t}^{i}(x_{t}^{i}, u_{t}^{i}) = 0 \]

Progressive Hedging
Rockafellar - Wets (91)
Decompositions for stochastic problems: in space

\[
\min \sum_{\omega} \sum_{i} \sum_{t} \pi_{\omega} L_{t}^{i} (x_{t}^{i}, u_{t}^{i}, w_{t+1})
\]

s.t. \[ x_{t+1}^{i} = f_{t}^{i} (x_{t}^{i}, u_{t}^{i}, w_{t+1}) \]

\[ u_{t}^{i} = \mathbb{E} \left( u_{t}^{i} \mid w_{1}, \ldots, w_{t} \right) \]

\[ \sum_{i} \theta_{t}^{i} (x_{t}^{i}, u_{t}^{i}) = 0 \]

Dual Approximate Dynamic Programming
Outline of the presentation

1. Decomposition and coordination
   - A bird’s eye view of decomposition methods
   - (A brief insight into Progressive Hedging)
   - Spatial decomposition methods in the deterministic case
   - The stochastic case raises specific obstacles

2. Dual approximate dynamic programming (DADP)
   - Problem statement
   - DADP principle and implementation
   - Numerical results on a small size problem

3. Theoretical questions
   - Existence of a saddle point
   - Convergence of the Uzawa algorithm
   - Convergence w.r.t. information

4. Conclusion
Non-anticipativity constraints are linear

- From tree to scenarios (comb)
- Equivalent formulations of the non-anticipativity constraints
  - pairwise equalities
  - all equal to their mathematical expectation
- Linear structure

\[ u_t = \mathbb{E} \left( u_t \mid w_1, \ldots, w_t \right) \]
Progressive Hedging stands as a scenario decomposition method by dualizing the non-anticipativity constraints

- When the criterion is strongly convex, we use an algorithm "à la Uzawa" to obtain a scenario decomposition.
- When the criterion is linear, Rockafellar - Wets (91) propose to use an augmented Lagrangian, and obtain the Progressive Hedging algorithm.
Decomposition and coordination

(A brief insight into Progressive Hedging)

Data: Initial multipliers \( \{ \{ \lambda_t^{(0)}(\omega) \} \}_{t=0}^{T-1} \) and mean control \( \{ \overline{U}_n^{(0)} \} \) \( n \in T \);

Result: optimal feedback;

repeat

\[ \text{forall the scenario } \omega \in \Omega \text{ do} \]

\[ \text{Solves the deterministic minimization problem for scenario } \omega \text{ with} \]

\[ \text{a measurability penalization, and obtain optimal control } u^{(k+1)}; \]

\[ \text{Update the mean controls} \]

\[ \overline{u}^{(k+1)}_n = \frac{\sum_{\omega \in n} u_t^{(k+1)}(\omega)}{|n|} \]

\[ \text{Update the measurability penalization with} \]

\[ \lambda_t^{(k+1)}(\omega) = \lambda_t^{(k)}(\omega) + \rho \left( U_t(\omega)^{(k+1)} - \overline{u}_n^{(k+1)}(\omega) \right) \]

\[ \text{until } u_t - \mathbb{E}(u_t^i \mid w_1, \ldots, w_t) = 0; \]
1 Decomposition and coordination
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   - **Spatial decomposition methods in the deterministic case**
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4 Conclusion
The system to be optimized consists of interconnected subsystems.

We want to use this structure to formulate optimization subproblems of reasonable complexity.

But the presence of interactions requires a level of coordination.

Coordination iteratively provides a local model of the interactions for each subproblem.

We expect to obtain the solution of the overall problem by concatenation of the solutions of the subproblems.
Example: the “flower model”

Unit Commitment Problem

\[
\min_u \sum_{i=1}^{N} J_i(u_i)
\]

s.t. \[ \sum_{i=1}^{N} \theta_i(u_i) = \theta \]
Intuition of spatial decomposition

- Purpose: satisfy a demand with $N$ production units, at minimal cost

- Price decomposition
  - the coordinator sets a price $\lambda_t$
  - the units send their production $u^{(i)}_t$
  - the coordinator compares total production and demand, and then updates the price
  - and so on...

Coordinator

Unit 1

Unit 2

Unit 3
Intuition of spatial decomposition

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Decomposition and coordination
Spatial decomposition methods in the deterministic case

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Intuition of spatial decomposition

- **Purpose**: satisfy a demand with $N$ production units, at minimal cost

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  - the coordinator sets a price $\lambda_t$
  - the units send their production $u^{(i)}_t$
  - the coordinator compares total production and demand, and then updates the price
  - and so on...
Price decomposition relies on dualization

\[
\min_{u_i \in U_i, i=1...N} \sum_{i=1}^{N} J_i(u_i) \quad \text{subject to} \quad \sum_{i=1}^{N} \theta_i(u_i) - \theta = 0
\]

1. Form the **Lagrangian** and assume that a saddle point exists

\[
\max_{\lambda \in \mathcal{V}} \min_{u_i \in U_i, i=1...N} \sum_{i=1}^{N} \left( J_i(u_i) + \langle \lambda, \theta_i(u_i) \rangle \right) - \langle \lambda, \theta \rangle
\]

2. Solve this problem by the **dual gradient algorithm „à la Uzawa”**

\[
u_i^{(k+1)} \in \arg \min_{u_i \in U_i} J_i(u_i) + \langle \lambda^{(k)}, \theta_i(u_i) \rangle, \quad i = 1 \ldots, N
\]

\[
\lambda^{(k+1)} = \lambda^{(k)} + \rho \left( \sum_{i=1}^{N} \theta_i(u_i^{(k+1)}) - \theta \right)
\]
Price decomposition relies on dualization

\[
\min_{u_i \in U_i, i = 1 \ldots N} \sum_{i=1}^{N} J_i(u_i) \quad \text{subject to} \quad \sum_{i=1}^{N} \theta_i(u_i) - \theta = 0
\]

1. Form the Lagrangian and assume that a saddle point exists

\[
\max_{\lambda \in \mathcal{V}} \min_{u_i \in U_i, i = 1 \ldots N} \sum_{i=1}^{N} \left( J_i(u_i) + \langle \lambda, \theta_i(u_i) \rangle \right) - \langle \lambda, \theta \rangle
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\]

\[
\lambda^{(k+1)} = \lambda^{(k)} + \rho \left( \sum_{i=1}^{N} \theta_i(u_i^{(k+1)}) - \theta \right)
\]
Remarks on decomposition methods

- The theory is available for infinite dimensional Hilbert spaces, and thus applies in the stochastic framework, that is, when the $U_i$ are spaces of random variables.

- The minimization algorithm used for solving the subproblems is not specified in the decomposition process.

- New variables $\lambda^{(k)}$ appear in the subproblems arising at iteration $k$ of the optimization process:

$$\min_{u_i \in U_i} J_i(u_i) + \langle \lambda^{(k)} , \theta_i(u_i) \rangle$$

- These variables are fixed when solving the subproblems, and do not cause any difficulty, at least in the deterministic case.
Price decomposition applies to various couplings
Decomposition and coordination

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Conclusion
Decomposition and coordination

Stochastic optimal control (SOC) problem formulation

Consider the following SOC problem

\[
\min_{u, x} \mathbb{E} \left( \sum_{i=1}^{N} \left( \sum_{t=0}^{T-1} L^i_t(x^i_t, u^i_t, w_{t+1}) + K^i(x^i_T) \right) \right)
\]

subject to the constraints

\[
\begin{align*}
x^i_0 &= f^i_{-1}(w_0), \\
x^i_{t+1} &= f^i_t(x^i_t, u^i_t, w_{t+1}), & t = 0 \ldots T-1, \ i = 1 \ldots N \\
u^i_t &\preceq \mathcal{F}_t = \sigma(w_0, \ldots, w_t), & t = 0 \ldots T-1, \ i = 1 \ldots N \\
\sum_{i=1}^{N} \theta^i_t(x^i_t, u^i_t) &= 0, & t = 0 \ldots T-1
\end{align*}
\]
Stochastic optimal control (SOC) problem formulation

Consider the following SOC problem

$$\min_{u,x} \sum_{i=1}^{N} \left( \mathbb{E} \left( \sum_{t=0}^{T-1} L^i_t(x^i_t, u^i_t, w_{t+1}) + K^i(x^i_T) \right) \right)$$

subject to the constraints

$$x^i_0 = f_{-1}^i(w_0), \quad i = 1 \ldots N$$

$$x^i_{t+1} = f_t^i(x^i_t, u^i_t, w_{t+1}), \quad t = 0 \ldots T-1, \quad i = 1 \ldots N$$

$$u^i_t \leq \mathcal{F}_t = \sigma(w_0, \ldots, w_t), \quad t = 0 \ldots T-1, \quad i = 1 \ldots N$$

$$\sum_{i=1}^{N} \theta^i_t(x^i_t, u^i_t) = 0, \quad t = 0 \ldots T-1$$
Dynamic Programming yields centralized controls

- As we want to solve this SOC problem using Dynamic Programming (DP), we suppose to be in the Markovian setting, that is, $w_0, \ldots, w_T$ are a white noise

- The system is made of $N$ interconnected subsystems, with the control $u^i_t$ and the state $x^i_t$ of subsystem $i$ at time $t$

- The optimal control $u^i_t$ of subsystem $i$ is a function of the whole system state $(x^1_t, \ldots, x^N_t)$
  \[ u^i_t = \gamma^i_t(x^1_t, \ldots, x^N_t) \]

---

Naive decomposition should lead to decentralized feedbacks

\[ u^i_t = \hat{\gamma}^i_t(x^i_t) \]

which are, in most cases, far from being optimal…
Decomposition and coordination

The stochastic case raises specific obstacles

The crucial point is that the optimal feedback of a subsystem a priori depends on the state of all other subsystems, so that using a decomposition scheme by subsystems is not obvious...

As far as we have to deal with Dynamic Programming, the central concern for decomposition/coordination purpose boils down to

how to decompose a feedback $\gamma_t$ w.r.t. its domain $X_t$ rather than its range $U_t$?

And the answer is

impossible in the general case!
Price decomposition and Dynamic Programming

When applying price decomposition to the problem by dualizing the (almost sure) coupling constraint \( \sum_i \theta^i_t(x^i_t, u^i_t) = 0 \), multipliers \( \Lambda_t^{(k)} \) appear in the subproblems arising at iteration \( k \)

\[
\min_{u^i_t, x^i_t} \mathbb{E} \left( \sum_t L^i_t(x^i_t, u^i_t, w_{t+1}) + \Lambda_t^{(k)} \cdot \theta^i_t(x^i_t, u^i_t) \right)
\]

- The variables \( \Lambda_t^{(k)} \) are fixed random variables, so that the random process \( \Lambda^{(k)} \) acts as an additional input noise in the subproblems.
- But this process may be correlated in time, so that the white noise assumption has no reason to be fulfilled.
- DP cannot be applied in a straightforward manner!

**Question:** how to handle the coordination instruments \( \Lambda_t^{(k)} \) to obtain (an approximation of) the overall optimum?
Price decomposition and Dynamic Programming

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Question: how to handle the coordination instruments \( \Lambda_t^{(k)} \) to obtain (an approximation of) the overall optimum?
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3. Theoretical questions

4. Conclusion
Dual approximate dynamic programming (DADP)

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Optimization problem

The SOC problem under consideration reads

\[
\min_{u,x} \mathbb{E} \left( \sum_{i=1}^{N} \left( \sum_{t=0}^{T-1} L^i_t(x^i_t, u^i_t, w_{t+1}) + K^i(x^i_T) \right) \right) \tag{1a}
\]

subject to dynamics constraints

\[
x^i_0 = f^i_{-1}(w_0) \tag{1b}
\]
\[
x^i_{t+1} = f^i_t(x^i_t, u^i_t, w_{t+1}) \tag{1c}
\]

to measurability constraints:

\[
u^i_t \preceq \sigma(w_0, \ldots, w_t) \tag{1d}
\]
and to instantaneous coupling constraints

\[
\sum_{i=1}^{N} \theta^i_t(x^i_t, u^i_t) = 0 \quad \text{Constraints to be dualized} \tag{1e}
\]
Assumptions

Assumption 1 (White noise)
Noises $w_0, \ldots, w_T$ are independent over time

Hence Dynamic Programming applies: there is no optimality loss to look after the controls $u_t^i$ as functions of the state at time $t$

Assumption 2 (Constraint qualification)
A saddle point of the Lagrangian $\mathcal{L}$ exists

\[
\mathcal{L}(x, u, \Lambda) = \mathbb{E} \left( \sum_{i=1}^N \left( \sum_{t=0}^{T-1} L_t^i(x_t^i, u_t^i, w_{t+1}) + K^i(x_T^i) + \sum_{t=0}^{T-1} \Lambda_t \cdot \theta_t^i(x_t^i, u_t^i) \right) \right)
\]
where the $\Lambda_t$ are $\sigma(w_0, \ldots, w_t)$-measurable random variables

Assumption 3 (Dual gradient algorithm)
Uzawa algorithm applies...
Uzawa algorithm

At iteration $k$ of the algorithm,

1. **Solve** Subproblem $i$, $i = 1, \ldots, N$, with $\Lambda^{(k)}$ fixed

   \[
   \min_{u^i_t, x^i_t} \mathbb{E} \left( \sum_{t=0}^{T-1} \left( L^i_t(x^i_t, u^i_t, w_{t+1}) + \Lambda^{(k)}_t \cdot \theta^i_t(x^i_t, u^i_t) \right) + K^i(x^i_T) \right)
   \]

   subject to

   \[
   x^i_{t+1} = f^i_t(x^i_t, u^i_t, w_{t+1})
   \]
   \[
   u^i_t \preceq \sigma(w_0, \ldots, w_t)
   \]

   whose solution is denoted $(u^i, (k+1), x^i, (k+1))$

2. **Update** the multipliers $\Lambda_t$

   \[
   \Lambda^{(k+1)}_t = \Lambda^{(k)}_t + \rho_t \left( \sum_{i=1}^N \theta^i_t(x^i, (k+1), u^i, (k+1)) \right)
   \]
Structure of a subproblem

- Subproblem $i$ reads

\[
\min_{u^i, x^i} \mathbb{E} \left( \sum_{t=0}^{T-1} \left( L_t^i(x_t^i, u_t^i, w_{t+1}) + \Lambda_t^{(k)} \cdot \theta_t^i(x_t^i, u_t^i) \right) \right)
\]

subject to

\[
x_{t+1}^i = f_t^i(x_t^i, u_t^i, w_{t+1})
\]

\[
u_t^i \leq \sigma(w_0, \ldots, w_t)
\]

- Without some knowledge of the process $\Lambda^{(k)}$ (we just know that $\Lambda_t^{(k)} \leq (w_0, \ldots, w_t)$), the informational state of this subproblem $i$ at time $t$ cannot be summarized by the physical state $x_t^i$
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4 Conclusion
We outline the main idea in DADP

- To overcome the difficulty induced by the term $\Lambda^{(k)}_t$, we introduce a new adapted information process $y^i = (y^i_0, \ldots, y^i_{T-1})$ for Subsystem $i$
- at each time $t$, the random variable $y^i_t$ is measurable w.r.t. the past noises $(w_0, \ldots, w_t)$
- The core idea is to replace the multiplier $\Lambda^{(k)}_t$ at iteration $k$ by its conditional expectation $\mathbb{E}(\Lambda^{(k)}_t | y^i_t)$
- (More on the interpretation later)

Note that we require that the information process is not influenced by controls
We can now approximate Subproblem $i$

- Using this idea, we replace Subproblem $i$ by

$$\min_{u^i_t, x^i_t} \mathbb{E} \left( \sum_{t=0}^{T-1} \left( L^i_t(x^i_t, u^i_t, w_{t+1}) + \mathbb{E}(\Lambda_t^{(k)} | y^i_t) \cdot \theta^i_t(x^i_t, u^i_t) \right) + K^i(x^i_T) \right)$$

subject to

$$x^i_{t+1} = f^i_t(x^i_t, u^i_t, w_{t+1})$$
$$u^i_t \preceq \sigma(w_0, \ldots, w_t)$$

- The conditional expectation $\mathbb{E}(\Lambda_t^{(k)} | y^i_t)$ is an (updated) function of the variable $y^i_t$,

- so that Subproblem $i$ involves the two noises processes $w$ and $y^i$.

If $y^i$ follows a dynamical equation, DP applies.
We obtain a Dynamic Programming equation by subsystem

Assuming a non-controlled dynamics \( y_{t+1}^i = h_t^i(y_t^i, w_{t+1}) \)
for the information process \( y^i \), the DP equation writes

\[
\begin{align*}
V_T^i(x, y) &= K^i(x) \\
V_t^i(x, y) &= \min_u \mathbb{E}\left(L_t^i(x, u, w_{t+1}) \right) \\
&\quad + \mathbb{E}(\Lambda_t^{(k)} \mid y_t^i = y) \cdot \theta_t^i(x, u) \\
&\quad + V_{t+1}^i(x_{t+1}^i, y_{t+1}^i)
\end{align*}
\]

subject to the dynamics

\[
\begin{align*}
x_{t+1}^i &= f_t^i(x, u, w_{t+1}) \\
y_{t+1}^i &= h_t^i(y, w_{t+1})
\end{align*}
\]
DADP displays three interpretations

- **DADP as an approximation of the optimal multiplier**
  \[ \lambda_t \sim \mathbb{E}(\lambda_t \mid y_t) \]

- **DADP as a decision-rule approach in the dual**
  \[
  \max_{\lambda} \min_u L(\lambda, u) \sim \max_{\lambda} \min_u L(\lambda, u) \left\| \lambda_t \leq y_t \right\|
  \]

- **DADP as a constraint relaxation**
  \[
  \sum_{i=1}^{n} \theta_t(u_t^i) = 0 \sim \mathbb{E}\left( \sum_{i=1}^{n} \theta_t(u_t^i) \mid y_t \right) = 0
  \]
A bunch of practical questions remains open

★ How to choose the information variables $y_t^i$?
  - Perfect memory: $y_t^i = (w_0, \ldots, w_t)$
  - Minimal information: $y_t^i \equiv \text{cste}$
  - Static information: $y_t^i = h_t^i(w_t)$
  - Dynamic information: $y_{t+1}^i = h_t^i(y_t^i, w_{t+1})$

★ How to obtain a feasible solution from the relaxed problem?
  - Use an appropriate heuristic!

★ How to accelerate the gradient algorithm?
  - Augmented Lagrangian
  - More sophisticated gradient methods

★ How to handle more complex structures than the flower model?
1. Decomposition and coordination
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4. Conclusion
We consider 3 dams in a row, amenable to DP
Problem specification

- We consider a 3 dam problem, over 12 time steps
- We relax each constraint with a given information process $y^i$
- All random variable are discrete (noise, control, state)

- We test the following information processes
  - **Constant information**: equivalent to replace the a.s. constraint by an expected constraint
  - **Part of noise**: the information process is the inflow of the above dam
    \[ Y_t^i = w_t^{i-1} \]
  - **Phantom state**: the information process mimicks the optimal trajectory of the state of the first dam
    (by statistical regression over the known optimal trajectory in this case)
Numerical results are encouraging

<table>
<thead>
<tr>
<th></th>
<th>DADP - $E$</th>
<th>DADP - $w_t^{-1}$</th>
<th>DADP - dyn.</th>
<th>DP</th>
</tr>
</thead>
<tbody>
<tr>
<td>Nb of it.</td>
<td>165</td>
<td>170</td>
<td>25</td>
<td>1</td>
</tr>
<tr>
<td>Time (min)</td>
<td>2</td>
<td>3</td>
<td>67</td>
<td>41</td>
</tr>
<tr>
<td>Lower Bound</td>
<td>$-1.386 \times 10^6$</td>
<td>$-1.379 \times 10^6$</td>
<td>$-1.373 \times 10^6$</td>
<td></td>
</tr>
<tr>
<td>Final Value</td>
<td>$-1.335 \times 10^6$</td>
<td>$-1.321 \times 10^6$</td>
<td>$-1.344 \times 10^6$</td>
<td>$-1.366 \times 10^6$</td>
</tr>
<tr>
<td>Loss</td>
<td>$-2.3%$</td>
<td>$-3.3%$</td>
<td>$-1.6%$</td>
<td>ref.</td>
</tr>
</tbody>
</table>

|$\Rightarrow$ PhD thesis of J.-C. Alais
Summing up DADP

- Choose an information process $y$ following $y_{t+1} = \tilde{f}_t(y_t, w_{t+1})$

- Relax the almost sure coupling constraint into a conditional expectation

- Then apply a price decomposition scheme to the relaxed problem

- The subproblems can be solved by dynamic programming with the modest state $(x^i_t, y_t)$

- In the theoretical part, we give
  - a consistency result (family of information process)
  - a convergence result (fixed information process)
  - conditions for the existence of multiplier
1. Decomposition and coordination

2. Dual approximate dynamic programming (DADP)

3. Theoretical questions

4. Conclusion
What are the issues to consider?

- We treat the coupling constraints in a stochastic optimization problem by duality methods.

- Uzawa algorithm is a dual method which is naturally described in a Hilbert space, but we cannot guarantee the existence of an optimal multiplier in the space $L^2(\Omega, \mathcal{F}, \mathbb{P}; \mathbb{R}^n)$!

- Consequently, we extend the algorithm to the non-reflexive Banach space $L^\infty(\Omega, \mathcal{F}, \mathbb{P}; \mathbb{R}^n)$, by giving a set of conditions ensuring the existence of a $L^1(\Omega, \mathcal{F}, \mathbb{P}; \mathbb{R}^n)$ optimal multiplier, and by providing a convergence result of the algorithm.

- We also have to deal with the approximation induced by the information variable: we give an epi-convergence result related to such an approximation.

~ PhD thesis of V. Leclère
Abstract formulation of the problem

We consider the following abstract optimization problem

\[
\text{(P)} \quad \min_{u \in U^{\text{ad}}} J(u) \quad \text{s.t.} \quad \Theta(u) \in -C
\]

where $U$ and $V$ are two Banach spaces, and

- $J : U \to \overline{R}$ is the objective function
- $U^{\text{ad}}$ is the admissible set
- $\Theta : U \to V$ is the constraint function to be dualized
- $C \subset V$ is the cone of constraint

Let $U^{\Theta} = \{ u \in U, \Theta(u) \in -C \}$ be the associated constraint set

Here, $U$ is a space of random variables, and $J$ is defined by

\[
J(u) = \mathbb{E}(j(u, w))
\]

The relationship with Problem (1) is almost straightforward...
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Conclusion
Standard duality in $L^2$ spaces

Assume that $\mathcal{U} = L^2(\Omega, \mathcal{F}, \mathbb{P}; \mathbb{R}^n)$ and $\mathcal{V} = L^2(\Omega, \mathcal{F}, \mathbb{P}; \mathbb{R}^m)$

The standard sufficient constraint qualification condition

$$0 \in \text{ri}\left(\Theta(\mathcal{U}^{\text{ad}} \cap \text{dom}(J)) + \mathcal{C}\right)$$

is scarcely satisfied in such a stochastic setting

Proposition 1

*If the $\sigma$-algebra $\mathcal{F}$ is not finite modulo $\mathbb{P}$, then for any subset $\mathcal{U}^{\text{ad}} \subset \mathbb{R}^n$ that is not an affine subspace, the set

$$\mathcal{U}^{\text{ad}} = \left\{ u \in L^p(\Omega, \mathcal{F}, \mathbb{P}; \mathbb{R}^n) \mid u \in \mathcal{U}^{\text{ad}} \quad \mathbb{P} - a.s. \right\}$$

has an empty relative interior in $L^p$, for any $p < +\infty$*

---

*aIf the $\sigma$-algebra is finite modulo $\mathbb{P}$, $\mathcal{U}$ and $\mathcal{V}$ are finite dimensional spaces*
Consider the following optimization problem:

$$\inf_{u_0, u_1} u_0^2 + \mathbb{E}((u_1 + \alpha)^2)$$

s.t. $u_0 \geq a$

$u_1 \geq 0$

$u_0 - u_1 \geq w$ to be dualized

where $w$ is a random variable uniform on $[1, 2]$

For $a < 2$, we can construct a maximizing sequence of multipliers for the dual problem that does not converge in $L^2$.

(We are in the so-called non relatively complete recourse case, that is, the case where the constraints on $u_1$ induce a stronger constraint on $u_0$)

An optimal multiplier is available in $(L^\infty)^*$...
Consider the following optimization problem:

\[
\inf_{u_0,u_1} u_0^2 + \mathbb{E}((u_1 + \alpha)^2)
\]

s.t.

\[
\begin{align*}
  & u_0 \geq a \\
  & u_1 \geq 0 \\
  & u_0 - u_1 \geq w
\end{align*}
\]

to be dualized

where \(w\) is a random variable uniform on \([1, 2]\).

For \(a < 2\), we can construct a maximizing sequence of multipliers for the dual problem that does not converge in \(L^2\).

(We are in the so-called non relatively complete recourse case, that is, the case where the constraints on \(u_1\) induce a stronger constraint on \(u_0\).)

An optimal multiplier is available in \((L^\infty)^*\)…
Constraint qualification in \((L^\infty, L^1)\)

From now on, we assume that

\[
\mathcal{U} = L^\infty(\Omega, \mathcal{F}, \mathbb{P}; \mathbb{R}^n) \\
\mathcal{V} = L^\infty(\Omega, \mathcal{F}, \mathbb{P}; \mathbb{R}^m) \\
C = \{0\}
\]

where the \(\sigma\)-algebra \(\mathcal{F}\) is not finite modulo \(\mathbb{P}\)

We consider the pairing \((L^\infty, L^1)\) with the following topologies:

- \(\sigma(L^\infty, L^1)\): weak* topology on \(L^\infty\) (coarest topology such that all the \(L^1\)-linear forms are continuous),

- \(\tau(L^\infty, L^1)\): Mackey-topology on \(L^\infty\) (finest topology such that the continuous linear forms are only the \(L^1\)-linear forms)
Weak* closedness of linear subspaces of $L^\infty$

Proposition 2

Let $\Theta : L^\infty(\Omega, \mathcal{F}, \mathbb{P}; \mathbb{R}^n) \to L^\infty(\Omega, \mathcal{F}, \mathbb{P}; \mathbb{R}^m)$ be a linear operator, and assume that there exists a linear operator $\Theta^\dagger : L^1(\Omega, \mathcal{F}, \mathbb{P}; \mathbb{R}^m) \to L^1(\Omega, \mathcal{F}, \mathbb{P}; \mathbb{R}^n)$ such that:

$$\langle v, \Theta(u) \rangle = \langle \Theta^\dagger(v), u \rangle, \quad \forall u, \forall v$$

Then the linear operator $\Theta$ is weak* continuous

Applications

- $\Theta(u) = u - \mathbb{E}(u | \mathcal{B})$: non-anticipativity constraints,
- $\Theta(u) = Au$ with $A \in \mathcal{M}_{m,n}(\mathbb{R})$: finite number of constraints
A duality theorem

\[
\begin{align*}
(P) & \quad \min_{u \in U} J(u) \quad \text{s.t.} \quad \Theta(u) = 0 \\
\text{with} & \quad J(u) = \mathbb{E}(j(u, w))
\end{align*}
\]

Theorem 1

Assume that \( j \) is a convex normal integrand, that \( J \) is continuous in the Mackey topology at some point \( u_0 \) such that \( \Theta(u_0) = 0 \), and that \( \Theta \) is weak* continuous on \( L^{\infty}(\Omega, \mathcal{F}, P; \mathbb{R}^n) \). Then, \( u^* \in \mathcal{U} \) is an optimal solution of Problem \( (P) \) if and only if there exists \( \lambda^* \in L^1(\Omega, \mathcal{F}, \mathbb{P}; \mathbb{R}^m) \) such that

\[
\begin{align*}
\triangleright & \quad u^* \in \arg \min_{u \in \mathcal{U}} \mathbb{E}(j(u, w) + \lambda^* \cdot \Theta(u)) \\
\triangleright & \quad \Theta(u^*) = 0
\end{align*}
\]

Extension of a result given by R. Wets for non-anticipativity constraints
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   - Convergence w.r.t. information

4. Conclusion
Uzawa algorithm

\[
(P) \quad \min_{u \in U} J(u) \quad \text{s.t.} \quad \Theta(u) = 0
\]

with \( J(u) = \mathbb{E}(j(u, w)) \)

The standard Uzawa algorithm

\[
\begin{align*}
    u^{(k+1)} & \in \arg \min_{u \in U^{\text{ad}}} J(u) + \langle \lambda^{(k)}, \Theta(u) \rangle \\
    \lambda^{(k+1)} & = \lambda^{(k)} + \rho \Theta(u^{(k+1)})
\end{align*}
\]

makes sense with in the \( L^\infty \) setting, that is, the minimization problem is well-posed and the update formula is valid one

Note that all the multipliers \( \lambda^{(k)} \) belong to \( L^\infty(\Omega, \mathcal{F}, \mathbb{P}; \mathbb{R}^m) \),

as soon as the initial multiplier \( \lambda^{(0)} \in L^\infty(\Omega, \mathcal{F}, \mathbb{P}; \mathbb{R}^m) \)
**Theoretical questions**

## Convergence of the Uzawa algorithm

### Convergence result

**Theorem 2**

Assume that

1. $J : \mathcal{U} \to \mathbb{R}$ is proper, weak$^*$ l.s.c., differentiable and $a$-convex
2. $\Theta : \mathcal{U} \to \mathcal{V}$ is affine, weak$^*$ continuous and $\kappa$-Lipschitz
3. $\mathcal{U}^{\text{ad}}$ is weak$^*$ closed and convex,
4. an admissible $u_0 \in \text{dom } J \cap \Theta^{-1}(0) \cap \mathcal{U}^{\text{ad}}$ exists
5. an optimal $L^1$-multiplier to the constraint $\Theta(u) = 0$ exists
6. the step $\rho$ is such that $0 < \rho < \frac{2a}{\kappa}$

Then, there exists a subsequence $\{u^{(n_k)}\}_{k \in \mathbb{N}}$ of the sequence generated by the Uzawa algorithm converging in $L^\infty$ toward the optimal solution $u^*$ of the primal problem.
Remarks about these results

- The result is not as good as expected (global convergence)
- Improvements and extensions (inequality constraint) needed
- The Mackey-continuity assumption forbids the use of bounds
  - In order to deal with almost sure bound constraints, we can turn towards the work of R.T. Rockafellar and R. J-B Wets
  - In a series of 4 papers (stochastic convex programming), they have detailed the duality theory on two-stage and multistage problems, with the focus on non-anticipativity constraints
  - These papers require
    - a strict feasibility assumption
    - a relatively complete recourse assumption
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4. Conclusion
Relaxed problems

Following the interpretation of DADP in terms of a relaxation of the original problem, and given a sequence \( \{\mathcal{F}_n\}_{n \in \mathbb{N}} \) of subfields of the \( \sigma \)-field \( \mathcal{F} \), we replace the abstract problem

\[
\begin{align*}
\text{(P)} & \quad \min_{u \in U} J(u) \quad \text{s.t.} \quad \Theta(u) = 0
\end{align*}
\]

by the sequence of approximated problems:

\[
\begin{align*}
\text{(P}_n) & \quad \min_{u \in U} J(u) \quad \text{s.t.} \quad \mathbb{E}(\Theta(u) \mid \mathcal{F}_n) = 0
\end{align*}
\]

We assume the Kudo convergence of \( \{\mathcal{F}_n\}_{n \in \mathbb{N}} \) toward \( \mathcal{F} \):

\[
\mathcal{F}_n \rightarrow \mathcal{F} \quad \iff \quad \forall x \in L^1(\Omega, \mathcal{F}, \mathbb{P} \mid \mathbb{R}), \quad \mathbb{E}(x \mid \mathcal{F}_n) \xrightarrow{L^1} \mathbb{E}(x \mid \mathcal{F})
\]
Theoretical questions

Convergence w.r.t. information

Convergence result

Theorem 3

Assume that

- $\mathcal{U}$ is a topological space
- $\mathcal{V} = L^p(\Omega, \mathcal{F}, \mathbb{P}; \mathbb{R}^m)$ with $p \in [1, +\infty)$
- $J$ and $\Theta$ are continuous operators
- $\{\mathcal{F}_n\}_{n \in \mathbb{N}}$ Kudo converges toward $\mathcal{F}$

Then the sequence $\{\tilde{J}_n\}_{n \in \mathbb{N}}$ epi-converges toward $\tilde{J}$, with

$$\tilde{J}_n = \begin{cases} 
J(u) & \text{if } u \text{ satisfies the constraints of } (\mathcal{P}_n) \\
+\infty & \text{otherwise}
\end{cases}$$
1. Decomposition and coordination

2. Dual approximate dynamic programming (DADP)

3. Theoretical questions

4. Conclusion
DADP method allows to tackle large-scale stochastic optimal control problems, such as those found in energy management.

A host of theoretical and practical questions remains open.

We would like to test DADP on (smart) grids, extending the works on “flower models” (Unit Commitment problem) and on “chained models” (hydraulic valley management) to “network models” (grids).