

Application of Dantzig-Wolfe Reformulation to Binary Quadratic Problems

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- 1 Introduction
- 2 DWR with a quadratic master problem
- 3 DWR with a quadratic pricing problem
- 4 DWR applied to (kQKP)
- 5 Numerical Results
- 6 Conclusion and perspectives

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Context

A generic BQP reads as follows :

Binary Quadratic Problem (BQP)

$$(BQP) \quad \max\{f(x) = x^\top Qx + L^\top x \mid Ax \leq b, x \in \{0, 1\}\} .$$

- $Q \in \mathbb{Q}^{n \times n}$ and not restricted to be convex.
- $L \in \mathbb{Q}^n$.
- $A \in \mathbb{Q}^{m \times n}$. and $b \in \mathbb{Q}^n$.

Dantzig-Wolfe Reformulation, basic idea

$$\begin{aligned} \max \quad & f(x) \\ & A \leq b \\ & x \in \{0, 1\} \end{aligned}$$

- Let A' , A'' and b' , b'' be a generic row partition of the constraint matrix A and of the rhs vector b .

Dantzig-Wolfe Reformulation, basic idea

$$\begin{aligned} \max \quad & f(x) \\ & A' \leq b' \\ & A' \leq b'' \\ & x \in \{0, 1\} \end{aligned}$$

- Let A' , A'' and b' , b'' be a generic row partition of the constraint matrix A and of the rhs vector b .

Dantzig-Wolfe Reformulation, basic idea

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- Let A' , A'' and b' , b'' be a generic row partition of the constraint matrix A and of the rhs vector b .

Dantzig-Wolfe Reformulation, basic idea

$$\begin{aligned} \max \quad & f(x) \\ & A' \leq b' \\ & x \in \text{conv}\{x | A''x \leq b'', x \in \{0, 1\}\} \end{aligned}$$

- Let A' , A'' and b' , b'' be a generic row partition of the constraint matrix A and of the rhs vector b .
- The continuous relaxation of (BQP) is **strengthened** by **convexifying the constraints $A''x \leq b''$** .

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DWR with quadratic master problem

DWR of constraints A''

(BQP) can be reformulated as follows :

$$(BQP_{DWR(A'')}) \quad \max f(x) \quad (1)$$

$$\text{s.t. } A'x \leq b' \quad (2)$$

$$x \in \text{conv}\{x | A''x \leq b'', x \in \{0, 1\}\} \quad (3)$$

$$x_j \in \{0, 1\} \quad j = 1, \dots, n \quad (4)$$

DWR with quadratic master problem

DWR of constraints A''

(BQP) can be reformulated as follows :

$$(BQP_{DWR(A'')}) \quad \max \quad f(x) \quad (5)$$

$$\text{s.t.} \quad A'x \leq b' \quad [\alpha] \quad (6)$$

$$x_j = \sum_{p \in \mathcal{P}_{DWR(A'')}} x_j^p \lambda^p \quad j = 1, \dots, n \quad [\tau_j] \quad (7)$$

$$\sum_{p \in \mathcal{P}_{DWR(A'')}} \lambda^p = 1 \quad [\beta] \quad (8)$$

$$x_j \in \{0, 1\} \quad j = 1, \dots, n \quad (9)$$

$$\lambda^p \geq 0 \quad p \in \mathcal{P}_{DWR(A'')} \quad (10)$$

$\mathcal{P}_{DWR(A'')}$ = the set of extreme points of $\text{conv}\{x | A''x \leq b'', x \in \{0, 1\}\}$.

DWR with quadratic master problem

Convexification of the objective function

If the objective function is non convex, we need to replace $f(x)$ by an equivalent convex objective function $f'(x)$.

$$(BQP_{DWR(A'')}) \quad \max \quad f'(x) \quad (11)$$

$$\text{s.t.} \quad A'x \leq b' \quad [\alpha] \quad (12)$$

$$x_j = \sum_{p \in \mathcal{P}_{DWR(A'')}} x_j^p \lambda^p \quad j = 1, \dots, n \quad [\tau_j] \quad (13)$$

$$\sum_{p \in \mathcal{P}_{DWR(A'')}} \lambda^p = 1 \quad [\beta] \quad (14)$$

$$x_j \in \{0, 1\} \quad j = 1, \dots, n \quad (15)$$

$$\lambda^p \geq 0 \quad p \in \mathcal{P}_{DWR(A'')} \quad (16)$$

DWR with quadratic master problem

dual of the continuous relaxation of $(BQP_{DWR(A'')})$

$$\min \quad x^\top Qx + \beta \quad (17)$$

$$\text{s.t.} \quad 2Qx + A'^\top \alpha - \tau = -L \quad (18)$$

$$x_p^\top \tau + \beta \leq 0 \quad p \in \mathcal{P}_{DWR(A'')} \quad (19)$$

- *Pricing problem* : variables λ are partially enumerated by solving the separation problem of inequalities (19).

Let α , τ and β being the dual variables associated to the constraints in the continuous relaxation of $(BQP_{DWR(A'')})$.

Pricing problem

$$(\Pi_{BQP_{DWR(A'')}}(\tau^*, \beta^*)) \quad \max \quad \tau^{*\top} x + \beta^* \quad (20)$$

$$\text{s.t.} \quad A''x \leq b'' \quad (21)$$

$$x_j \in \{0, 1\} \quad j = 1, \dots, n \quad (22)$$

If the optimal value of $(\Pi_{BQP_{DWR(A'')}}(\tau^*, \beta^*))$ is greater than zero, then a column with positive reduced cost is found and added to the master.

- $(BQP_{DWR}) \Rightarrow f(x)$ is quadratic, the pricing problem is (binary) linear.

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DWR with quadratic pricing problem

The objective function can be rewritten directly in terms of the λ variables by introducing $f(\lambda) = \sum_{p \in \mathcal{P}_{DWR(A'')}} c_p \lambda_p$ with

$$c_p = f(x_p) = x_p^\top Q x_p + L^\top x_p .$$

$\overline{DWR}$ of constraints A''

$$\begin{aligned} (BQP_{\overline{DWR}(A'')}) \quad & \max \quad \sum_{p \in \mathcal{P}_{DWR(A'')}} c_p \lambda_p \\ & \text{s.t.} \quad (8) - (12) \end{aligned} \tag{23}$$

DWR with quadratic pricing problem

with $f(x) = \sum_{p \in \mathcal{P}_{DWR(A'')}} c_p \lambda_p$, the pricing problem reduces to the following quadratic problem :

Pricing problem

$$(\Pi_{BQP_{\overline{DWR(A'')}}}(\tau^*, \beta^*)) \quad \max \quad x^\top Qx + L^\top x + \tau^{*\top} x + \beta^* \quad (24)$$

$$\text{s.t.} \quad A''x \leq b'' \quad (25)$$

$$x_j \in \{0, 1\} \quad j = 1, \dots, n \quad (26)$$

where Q is not required to be convex.

- $(BQP_{\overline{DWR}}) \Rightarrow f(\lambda)$ is linear, the pricing problem is (binary) quadratic.

DWR with quadratic pricing problem

The objective function can still be modified using the convexified objective function $f'(x)$.

The pricing reduces to the following quadratic problem :

Pricing problem

$$(\Pi_{BQP_{\overline{DWR}(A'')}}(\tau^*, \beta^*)) \quad \max \quad f'(x) + \tau^{*\top} x + \beta^* \quad (27)$$

$$\text{s.t.} \quad A''x \leq b'' \quad (28)$$

$$x_j \in \{0, 1\} \quad j = 1, \dots, n \quad (29)$$

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(kQKP) Formulation

Notations

n : number of items

a_j : weight of item j ($j = 1, \dots, n$)

b : capacity of the knapsack

c_{ij} : profit associated with the selection of items i and j ($i, j = 1, \dots, n$)

k : number of items to be filled in the knapsack

Assumptions

$c_{ij} \in \mathbb{N} \ i, j = 1, \dots, n, a_j \in \mathbb{N} \ j = 1, \dots, n, b \in \mathbb{N}$

$\max_{j=1, \dots, n} a_j \leq b < \sum_{j=1}^n a_j$

$k \in \{1, \dots, k_{max}\}$

(kQKP) Formulation

Mathematical formulation

$$(\text{kQKP}) \left\{ \begin{array}{ll} \max f(x) = \sum_{i=1}^n \sum_{j=1}^n c_{ij} x_i x_j & \\ \text{s.t.} & \sum_{j=1}^n a_j x_j \leq b \quad (1) \\ & \sum_{j=1}^n x_j = k \quad (2) \\ & x_j \in \{0, 1\} \quad j = 1, \dots, n \end{array} \right.$$

- without constraint (2) : the 0-1 quadratic knapsack problem (*QKP*)
- without constraint (1) : the k-cluster problem

Reformulations for (kQKP)

Reformulation of the **objective function** $f(x)$

QCR/MQCR Method

$(f(x))$ can be reformulated by exploiting the property $x^2 = x$



convex problem

Reformulation of the **feasible region** $\{x | Ax \leq b, x \in \{0, 1\}\}$

Danzig-Wolfe Reformulation

(a subset of constraints is substituted by its convex hull)



tighter formulation

Quadratic Convex Reformulation (QCR) : a two-phase method

[Billionnet, Elloumi and M-C. Plateau, Discrete Applied Mathematics, 2009]

- **Phase 1** : Reformulate the objective function $f(x)$ into an equivalent 0-1 program with a **concave** quadratic objective function $f_{u,\alpha}(x)$.
→ an equivalent **convex** 0-1 program
- **Phase 2** : Apply a standard 0-1 convex quadratic solver to this new problem.

QCR - Phase 1 : Addition of two functions null on the feasible set

$$\bullet \quad q_u(x) = \sum_{i=1}^n u_i (x_i^2 - x_i)$$

$$\bullet \quad q_\alpha(x) = \sum_{i=1}^n \alpha_i x_i \left(\sum_{j=1}^n x_j - k \right) \quad \text{or} \quad q_v(x) = v \left(\sum_{j=1}^n x_j - k \right)^2$$

q_α and q_v are implied by the cardinality constraint.

QCR : A two-phase method

QCR - Phase 1 : Reformulation 1

New objective function (concave iff quadratic terms matrix is PSD) :

$$f_{u,\alpha}(x) = f(x) - \sum_{i=1}^n u_i (x_i^2 - x_i) - \sum_{i=1}^n \alpha_i x_i \left(\sum_{j=1}^n x_j - k \right)$$

Reformulation 2 [Faye, Roupin, 4OR, 2007 ; Billionnet, Elloumi and Lambert, Math. Prog., 2012]

$$f_{u,v}(x) = f(x) - \sum_{i=1}^n u_i (x_i^2 - x_i) - v \left(\sum_{j=1}^n x_j - k \right)^2$$

- identical bounds.
- consumes the least amount of computation times.

QCR : A two-phase method

Best values for the parameters $u \in \mathbb{R}^n$ and $v \in \mathbb{R}$

Solve the semi-definite relaxation of the problem ($E - kQKP$) :

$$\left\{ \begin{array}{ll} \max & \sum_{i=1}^n \sum_{j=1}^n c_{ij} X_{ij} \\ \text{s.t.} & X_{ii} = x_i \quad i = 1, \dots, n \quad (u^*) \\ & \sum_{i=1}^n \sum_{j=1}^n X_{ij} - 2k \sum_{j=1}^n x_j = -k^2 \quad (v^*) \\ & \sum_{j=1}^n a_j x_j \leq b \\ & \sum_{j=1}^n x_j = k \\ & \begin{pmatrix} 1 & x^t \\ x & X \end{pmatrix} \succeq 0 \\ & x \in \mathbb{R}^n, X \in \mathbb{R}^{n \times n} \end{array} \right.$$

$E - kQKP_{(SDP1)}$

An improved convex 0-1 quadratic reformulation for $(E - kQKP)$ (1/4)
 [Billionnet, Elloumi and Lambert, Math. Prog., 2012 and Ji, Zheng and Sun, Pacific Journal of Optimization, 2012]

From

$$f(x) = x^t C x$$

Consider the decomposition of $f(x)$

$$f(x) = x^t (C - M) x + x^t M x,$$

where

$$C - M \preceq 0$$

and

$$M = \text{Diag}(u) + P + N$$

with $u \in \mathbb{R}^n$, $P \in \mathbb{R}_+^{n \times n}$ and $N \in \mathbb{R}_-^{n \times n}$

An improved convex 0-1 quadratic reformulation for $(E - kQKP)$ (2/4)

$$\begin{aligned}
 f(x) &= x^t(C - \text{Diag}(u) - P - N)x + x^t(\text{Diag}(u) + P + N)x \\
 &= x^t(C - \text{Diag}(u) - P - N)x + u^t x + \sum_{i,j=1}^n [P_{ij}x_i x_j + N_{ij}x_i x_j] \\
 &= x^t(C - \text{Diag}(u) - P - N)x + u^t x + \sum_{i,j=1}^n [P_{ij}s_{ij} - N_{ij}t_{ij}]
 \end{aligned}$$

where

$$s_{ij} = \min(x_i, x_j),$$

$$t_{ij} = \max(0, x_i + x_j - 1)$$

$$(x_i x_j = \min(x_i, x_j) = \max(0, x_i + x_j - 1) \text{ for any } x_i, x_j \in \{0, 1\})$$

An improved convex 0-1 quadratic reformulation for $(E - kQKP)$ (3/4)

Relax

 $s_{ij} = \min(x_i, x_j)$ to two linear inequalities $s_{ij} \leq x_i$ and $s_{ij} \leq x_j$

and

 $t_{ij} = -\max(0, x_i + x_j - 1)$ to $t_{ij} \leq 0$ and $t_{ij} \leq 1 - x_i - x_j$.

without affecting the optimal solution of the problem.

The reformulation is equivalent to the convex 0-1 quadratic program :

$$\left\{ \begin{array}{l} \max \quad f(x) = x^t(C - \text{Diag}(u) - P - N)x + u^t x + \sum_{i,j=1}^n [P_{ij}s_{ij} + N_{ij}t_{ij}] \\ \text{s.t.} \quad \sum_{j=1}^n a_j x_j \leq b \\ \quad \quad \sum_{j=1}^n x_j = k \\ \quad \quad s_{ij} \leq x_i, \quad s_{ij} \leq x_j \quad i, j = 1, \dots, n \\ \quad \quad t_{ij} \leq 0, \quad t_{ij} \leq 1 - x_i - x_j \quad i, j = 1, \dots, n \\ \quad \quad x_j \in \{0, 1\} \quad j = 1, \dots, n \end{array} \right.$$

An improved convex 0-1 quadratic reformulation for $(E - kQKP)$ (4/4)

The optimal parameters (u^*, v^*, P^*, N^*) can be found by solving a SDP problem :

$$\begin{aligned}
 (E - kQKP_{SDP2}) \left\{ \begin{array}{ll} \max & \sum_{i=1}^n \sum_{j=1}^n c_{ij} X_{ij} \\ \text{s.t.} & X_{ii} = x_i \quad i = 1, \dots, n \quad (u^*) \\ & \sum_{i=1}^n \sum_{j=1}^n X_{ij} - 2k \sum_{j=1}^n x_j = -k^2 \quad i = 1, \dots, n \quad (v^*) \\ & \sum_{j=1}^n a_j x_j \leq b \\ & \sum_{j=1}^n x_j = k \\ & X_{ij} \leq x_i, X_{ij} \leq x_j \quad i, j = 1, \dots, n \quad (P^*) \\ & X_{ij} \geq x_i + x_j - 1, X_{ij} \geq 0 \quad i, j = 1, \dots, n \quad (N^*) \\ & \begin{pmatrix} 1 & x^t \\ x & X \end{pmatrix} \succeq 0 \\ & x \in \mathbb{R}^n, X \in S_n \end{array} \right.
 \end{aligned}$$

The application of the QCR method leads to the following reformulation of $(kQKP)$:

$$(kQKP^{u^*, v^*}) \quad \max \quad f_{u^*, v^*}(x) = f(x) - \sum_{i=1}^n u_i^* (x_i^2 - x_i) - v^* \left(\sum_{j=1}^n x_j - k \right)^2 \quad (30)$$

$$\text{s.t.} \quad \sum_{j=1}^n a_j x_j \leq b \quad (31)$$

$$\sum_{j=1}^n x_j = k \quad (32)$$

$$x \in \{0, 1\} \quad (33)$$

The application of the MQCR method leads to the following reformulation of (kQKP) :

$$\begin{aligned}
 (kQKP_M^{u^*, v^*, P^*, N^*}) \quad & \max \quad f_{u^*, v^*, P^*, N^*}(x, s, t) \\
 & = x^T (C - \text{Diag}(u) - P - N)x + u^T x + \sum_{i,j=1}^n [P_{ij}s_{ij} + N_{ij}t_{ij}] \\
 & \quad - \sum_{i=1}^n u_i (x_i^2 - x_i) - v \left(\sum_{j=1}^n x_j - k \right)^2 \\
 \text{s.t.} \quad & \sum_{j=1}^n a_j x_j \leq b \\
 & \sum_{j=1}^n x_j = k \\
 & s_{ij} \leq x_i, \quad s_{ij} \leq x_j \quad i, j = 1, \dots, n \\
 & t_{ij} \leq 0, \quad t_{ij} \leq 1 - x_i - x_j \quad i, j = 1, \dots, n \\
 & x_j \in \{0, 1\} \quad j = 1, \dots, n
 \end{aligned}$$

DWR with a quadratic master problem

Decompositions with a master objective function of the shape
 $f(x) = x^\top Qx + L^\top x$.

- PRO : linear pricing problem.
- CON : the objective function must be convex. $\Rightarrow$ DWR must be applied to $(kQKP^{u^*, v^*})$ or to $(kQKP_M^{u^*, v^*, P^*, N^*})$

DWR with a quadratic master problem

When applied to $(kQKP^{v^*, u^*})$, DWR gives the following model :

$$(kQKP_{DWR(1)}^{u^*, v^*}) \quad \max \quad f_{u^*, v^*}(x) \quad (34)$$

$$\text{s.t.} \quad \sum_{j=1}^n x_j = k \quad (35)$$

$$x_j = \sum_{p \in \mathcal{P}_{(1)}} x_j^p \lambda^p \quad j = 1, \dots, n \quad [\tau_j] \quad (36)$$

$$\sum_{p \in \mathcal{P}_{(1)}} \lambda^p = 1 \quad [\beta] \quad (37)$$

$$x_j \in \{0, 1\} \quad j = 1, \dots, n \quad (38)$$

$$\lambda^p \geq 0 \quad p \in \mathcal{P}_{(1)} \quad (39)$$

$\mathcal{P}_{(1)}$ is the set of extreme points of $\text{conv}\{x \mid \sum_{j=1}^n a_j x_j \leq b, x \in \{0, 1\}\}$

DWR with a quadratic master problem

The following reformulations can be obtained :

- $(kQKP_{DWR(1)}^{u^*, v^*}), (kQKP_{M, DWR(1)}^{u^*, v^*, P^*, N^*})$: knapsack constraint is convexified
- $(kQKP_{DWR(2)}^{u^*, v^*}), (kQKP_{M, DWR(2)}^{u^*, v^*, P^*, N^*})$: cardinality constraint is convexified
- $(kQKP_{DWR(1-2)}^{u^*, v^*}), (kQKP_{M, DWR(1-2)}^{u^*, v^*, P^*, N^*})$: both constraints are convexified

DWR with a quadratic pricing problem

Decompositions with a master objective function of the shape $f(x) = c^\top \lambda$.

- The pricing problem is quadratic.
- No restriction is imposed on the convexity of the objective function of the starting compact model.

DWR with a quadratic pricing problem

When applied to ($kQKP$), DWR gives the following model :

$$(kQKP_{\overline{DWR}(1)}) \quad \max \quad \sum_{p \in \mathcal{P}_{(1)}} c^p \lambda^p \quad (40)$$

$$\text{s.t. } x_j = \sum_{p \in \mathcal{P}_{(1)}} x_j^p \lambda^p \quad j = 1, \dots, n \quad [\phi_j] \quad (41)$$

$$\sum_{j=1}^n x_j = k \quad [\gamma] \quad (42)$$

$$\sum_{p \in \mathcal{P}_{(1)}} \lambda^p = 1 \quad [\theta] \quad (43)$$

$$x_j \in \{0, 1\} \quad j = 1, \dots, n \quad (44)$$

$$\lambda^p \geq 0 \quad p \in \mathcal{P}_{(1)} \quad (45)$$

with $\mathcal{P}_{(1)}$ being the set of extreme points of $\text{conv}\{x \mid \sum_{j=1}^n a_j x_j \leq b, x \in \{0, 1\}\}$ and with $c^p = \sum_{j=1}^n c_{ij} x_i^p x_j^p$.

DWR with a quadratic pricing problem

The following reformulations can be obtained :

- $(kQKP_{\overline{DWR}(1)}), (kQKP_{\overline{DWR}(1)}^{\mu^*, v^*}), (kQKP_{M, \overline{DWR}(1)}^{\mu^*, v^*, P^*, N^*})$: knapsack constraint is convexified
- $(kQKP_{\overline{DWR}(2)}), (kQKP_{\overline{DWR}(2)}^{\mu^*, v^*}), (kQKP_{M, \overline{DWR}(2)}^{\mu^*, v^*, P^*, N^*})$: cardinality constraint is convexified

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Experimental environment

- Carried out on an Intel i7-2600 quad core 3.4 GHz with 8 GB of RAM, using only one core
- CSDP integrated into COIN-OR for solving SDP programs
- CPLEX 12.6.2 with default settings
- Average values over 10 instances
- $n \in \{50, 60, \dots, 100\}$
- $k \in [1, n/4]$, $b \in [50, 30k]$, $a_j, c_{ij} \in [1, 100]$

n	δ	$(kQKP)$		$(kQKP_M^{\mu^*, v^*, P^*, N^*})$		$(kQKP_{DWR(1)}^{\mu^*, v^*})$		$(kQKP_{M, DWR(1-2)}^{\mu^*, v^*, P^*, N^*})$	
		Gap	Time	Gap	Time	Gap	Time	Gap	Time
50	25	102,65	0,02	30,89	1,05	38,36	1,97		
	50	150,56	0,06	25,25	0,94	31,05	3,04		
	75	230,29	0,12	105,16	1,09	114,26	1,55		
	100	356,48	0,24	63,81	0,99	69,45	1,56		
60	25	60,76	0,04	130,92	0,04	149,25	1,55		
	50	93,73	0,11	15,08	2,61	19,48	3,86		
	75	212,67	0,25	141,08	2,09	151,22	1,67		
	100	284,47	0,44	62,30	2,08	69,06	2,39		
70	25	130,23	0,06	38,03	5,11	46,84	4,82		
	50	177,07	0,19	72,81	4,27	80,44	6,83		
	75	382,36	0,44	56,26	3,45	63,77	3,25		
	100	252,21	0,77	60,23	3,97	66,08	3,91		
80	25	111,24	0,08	34,05	7,98	41,90	5,64		
	50	271,64	0,26	55,44	9,59	64,09	4,67		
	75	313,33	0,66	83,58	7,42	92,31	4,64		
	100	424,40	1,22	43,25	8,88	49,18	5,32		
90	25	118,45	0,13	112,80	13,75	129,31	4,74		
	50	248,57	0,48	83,15	12,38	92,19	4,52		
	75	388,68	1,06	37,90	5,63	42,13	6,95		
	100	390,02	1,85	26,54	11,27	29,94	5,58		
100	25	169,43	0,16	73,90	23,49	82,78	6,80		
	50	145,72	0,49	17,38	28,06	21,83	8,58		
	75	260,26	1,25	21,67	18,37	27,22	6,23		
	100	472,97	2,44	98,60	23,14	106,58	6,37		
Avg		239,51	0,53	62,09	8,24	69,95	4,43		

n	δ	$(kQKP)$		$(kQKP_M^{\mu^*, v^*, P^*, N^*})$		$(kQKP_{DWR(1)}^{\mu^*, v^*})$		$(kQKP_{M, DWR(1-2)}^{\mu^*, v^*, P^*, N^*})$	
		Gap	Time	Gap	Time	Gap	Time	Gap	Time
50	25	102,65	0,02	30,89	1,05	38,36	1,97	29,15	9,30
	50	150,56	0,06	25,25	0,94	31,05	3,04	23,66	9,71
	75	230,29	0,12	105,16	1,09	114,26	1,55	100,88	8,06
	100	356,48	0,24	63,81	0,99	69,45	1,56	61,58	6,19
60	25	60,76	0,04	130,92	0,04	149,25	1,55	126,07	10,89
	50	93,73	0,11	15,08	2,61	19,48	3,86	14,19	19,05
	75	212,67	0,25	141,08	2,09	151,22	1,67	136,22	8,99
	100	284,47	0,44	62,30	2,08	69,06	2,39	60,72	14,82
70	25	130,23	0,06	38,03	5,11	46,84	4,82	36,52	33,25
	50	177,07	0,19	72,81	4,27	80,44	6,83	70,77	54,37
	75	382,36	0,44	56,26	3,45	63,77	3,25	54,57	22,19
	100	252,21	0,77	60,23	3,97	66,08	3,91	58,94	26,50
80	25	111,24	0,08	34,05	7,98	41,90	5,64	32,87	71,19
	50	271,64	0,26	55,44	9,59	64,09	4,67	53,65	43,98
	75	313,33	0,66	83,58	7,42	92,31	4,64	81,47	43,42
	100	424,40	1,22	43,25	8,88	49,18	5,32	42,06	34,95
90	25	118,45	0,13	112,80	13,75	129,31	4,74	109,63	44,66
	50	248,57	0,48	83,15	12,38	92,19	4,52	81,65	66,75
	75	388,68	1,06	37,90	5,63	42,13	6,95	37,12	102,54
	100	390,02	1,85	26,54	11,27	29,94	5,58	26,02	113,84
100	25	169,43	0,16	73,90	23,49	82,78	6,80	72,72	99,90
	50	145,72	0,49	17,38	28,06	21,83	8,58	17,19	219,77
	75	260,26	1,25	21,67	18,37	27,22	6,23	21,50	158,30
	100	472,97	2,44	98,60	23,14	106,58	6,37	96,82	127,40
Avg		239,51	0,53	62,09	8,24	69,95	4,43	60,25	56,25

Even optimizing over the convex hull of the feasible region does not improve significantly the dual bound provided by DWR with a quadratic master.

The reasons for this behaviour are :

- The optimal solution of the continuous relaxation is attained in the interior of the convex hull of the feasible region.

n	50				60				70				80				90				100			
%	25	50	75	100	25	50	75	100	25	50	75	100	25	50	75	100	25	50	75	100	25	50	75	100
	0	2	0	0	0	2	0	0	0	2	0	0	0	1	0	0	0	1	0	0	0	1	0	0

TABLE: ($kQKP_{DWR}$), root node, quadratic master, Number of optimal solution in the interior of the convex hull of the feasible region

- The objective function over the feasible region is “flat” (due to the convexification via u^* and v^*).

n	δ	$(kQKP)$		$(kQKP_{M,DWR(1-2)}^{u^*,v^*,P^*,N^*})$		$(kQKP_{\overline{DWR}(1)})$		$(kQKP_{\overline{DWR}(1)}^{u^*,v^*})$	
		Gap	Time	Gap	Time	Gap	Time	Gap	Time
50	25	102,65	0,02	29,15	9,30	19,43	0,30		
	50	150,56	0,06	23,66	9,71	28,97	0,08		
	75	230,29	0,12	100,88	8,06	119,44	0,05		
	100	356,48	0,24	61,58	6,19	96,34	0,04		
60	25	60,76	0,04	126,07	10,89	79,77	0,14		
	50	93,73	0,11	14,19	19,05	13,13	0,25		
	75	212,67	0,25	136,22	8,99	157,01	0,07		
	100	284,47	0,44	60,72	14,82	77,53	0,07		
70	25	130,23	0,06	36,52	33,25	16,80	1,20		
	50	177,07	0,19	70,77	54,37	47,85	4,30		
	75	382,36	0,44	54,57	22,19	65,16	0,15		
	100	252,21	0,77	58,94	26,50	111,61	0,09		
80	25	111,24	0,08	32,87	71,19	11,67	210,59		
	50	271,64	0,26	53,65	43,98	42,69	0,77		
	75	313,33	0,66	81,47	43,42	95,92	14,00		
	100	424,40	1,22	42,06	34,95	74,78	0,13		
90	25	118,45	0,13	109,63	44,66	57,50	1692,40		
	50	248,57	0,48	81,65	66,75	63,79	190,64		
	75	388,68	1,06	37,12	102,54	59,35	15,16		
	100	390,02	1,85	26,02	113,84	61,68	0,20		
100	25	169,43	0,16	72,72	99,90	41,07	926,10		
	50	145,72	0,49	17,19	219,77	14,75	577,86		
	75	260,26	1,25	21,50	158,30	47,82	3,77		
	100	472,97	2,44	96,82	127,40	154,35	0,36		
Avg		198,20	0,32	60,25	57,02	54,56	202,10		

n	δ	$(kQKP)$		$(kQKP_{M,DWR(1-2)}^{u^*,v^*,P^*,N^*})$		$(kQKP_{DWR(1)})$		$(kQKP_{DWR(1)}^{u^*,v^*})$	
		Gap	Time	Gap	Time	Gap	Time	Gap	Time
50	25	102,65	0,02	29,15	9,30	19,43	0,30	0,00	
	50	150,56	0,06	23,66	9,71	28,97	0,08	0,00	
	75	230,29	0,12	100,88	8,06	119,44	0,05	0,00	
	100	356,48	0,24	61,58	6,19	96,34	0,04	0,00	
60	25	60,76	0,04	126,07	10,89	79,77	0,14	0,00	
	50	93,73	0,11	14,19	19,05	13,13	0,25	0,00	
	75	212,67	0,25	136,22	8,99	157,01	0,07	0,00	
	100	284,47	0,44	60,72	14,82	77,53	0,07	0,00	
70	25	130,23	0,06	36,52	33,25	16,80	1,20	0,00	
	50	177,07	0,19	70,77	54,37	47,85	4,30	0,00	
	75	382,36	0,44	54,57	22,19	65,16	0,15	0,00	
	100	252,21	0,77	58,94	26,50	111,61	0,09	0,00	
80	25	111,24	0,08	32,87	71,19	11,67	210,59	0,00	
	50	271,64	0,26	53,65	43,98	42,69	0,77	0,00	
	75	313,33	0,66	81,47	43,42	95,92	14,00	0,00	
	100	424,40	1,22	42,06	34,95	74,78	0,13	0,00	
90	25	118,45	0,13	109,63	44,66	57,50	1692,40	0,00	
	50	248,57	0,48	81,65	66,75	63,79	190,64	0,00	
	75	388,68	1,06	37,12	102,54	59,35	15,16	0,00	
	100	390,02	1,85	26,02	113,84	61,68	0,20	0,00	
100	25	169,43	0,16	72,72	99,90	41,07	926,10	0,00	
	50	145,72	0,49	17,19	219,77	14,75	577,86	0,00	
	75	260,26	1,25	21,50	158,30	47,82	3,77	0,00	
	100	472,97	2,44	96,82	127,40	154,35	0,36	0,00	
Avg		198,20	0,32	60,25	57,02	54,56	202,10	0,00	

An alternative look at $(kQKP_{\overline{DWR}(1)}^{u^*, v^*})$

Pricing problem

$$(\Pi_{BQP_{\overline{DWR}(A'')}}(\tau^*, \beta^*)) \quad \max \quad f_{u,v}(x) + \tau^{*\top} x + \beta^* \quad (46)$$

$$\text{s.t.} \quad Ax \leq b \quad (47)$$

$$x_j \in \{0, 1\} \quad j = 1, \dots, n \quad (48)$$

$$f_{u,v}(x) = f(x) - \sum_{i=1}^n u_i (x_i^2 - x_i) - v \left(\sum_{j=1}^n x_j - k \right)^2$$

$$f_{u,v}(x) = x^\top \tilde{Q}x + \tilde{L}^\top x - v \left(\sum_{j=1}^n x_j - k \right)^2$$

An alternative look at $(kQKP_{\overline{DWR}(1)}^{u^*, v^*})$

Pricing problem

$$(\Pi_{BQP_{\overline{DWR}(A'')}}(\tau^*, \beta^*)) \quad \max \quad f_{u,v}(x) + \tau^{*\top} x + \beta^* \quad (49)$$

$$\text{s.t.} \quad Ax \leq b \quad (50)$$

$$x_j \in \{0, 1\} \quad j = 1, \dots, n \quad (51)$$

$$f_{u,v}(x) = f(x) - \sum_{i=1}^n u_i (x_i^2 - x_i) - v \left(\sum_{j=1}^n x_j - k \right)^2$$

$$f_{u,v}(x) = x^\top \tilde{Q}x + \tilde{L}^\top x - v \left(\sum_{j=1}^n x_j - k \right)^2$$

The coefficient v can be viewed as **penalty term for the cardinality constraint**.

An alternative look at $(kQKP_{\overline{DWR}(1)}^{u^*, v^*})$

- Solving directly

$$\begin{aligned}
 \max \quad & x^\top \tilde{Q}x + \tilde{L}^\top x - v \left(\sum_{j=1}^n x_j - k \right)^2 \\
 \text{s.t.} \quad & \sum_{j=1}^n a_j x_j \leq b \\
 & x_j \in \{0, 1\} \qquad j = 1, \dots, n
 \end{aligned}$$

is computationally non convenient.

An alternative look at $(kQKP_{\overline{DWR}(1)}^{u^*, v^*})$

$$\begin{aligned} \max \quad & x^\top \tilde{Q}x + \tilde{L}^\top x - v \left(\sum_{j=1}^n x_j - k \right)^2 + \tau^{*\top} x + \beta^* \\ \text{s.t.} \quad & \sum_{j=1}^n a_j x_j \leq b \\ & x_j \in \{0, 1\} \qquad \qquad \qquad j = 1, \dots, n \end{aligned}$$

- The **dual variables “lead” the pricing problem** and make it significantly easier to solve.

An alternative look at $(kQKP_{\overline{DWR}(1)}^{u^*, v^*})$

$$\max \quad x^\top \tilde{Q}x + \tilde{L}^\top x - v \left(\sum_{j=1}^n x_j - k \right)^2 + \tau^{*\top} x + \beta^*$$

$$\text{s.t.} \quad \sum_{j=1}^n a_j x_j \leq b$$

$$x_j \in \{0, 1\} \quad j = 1, \dots, n$$

- The **dual variables** “lead” the pricing problem and make it significantly easier to solve.
- IDEA : **warm-start** with columns heuristically generated $\Rightarrow$ **good dual variables**.

n	δ	Time opt ($kQKP$)	Time opt ($kQKP_M^{\mu^*, v^*, P^*, N^*}$)	Time root ($kQKP_{DWR(1)}^{\mu^*, v^*}$)
50	25	3,7	1,0	40,1
	50	150,8	1,0	34,7
	75	213,1	0,7	14,7
	100	(8) 53,1	1,5	33,2
60	25	3,0	0,9	27,9
	50	282,4	1,4	32,5
	75	(9) 50,9	3,3	16,9
	100	(8) 188,6	3,2	39,3
70	25	23,7	3,4	64,4
	50	(6) 213,8	8,4	72,1
	75	(8) 873,0	16,2	55,1
	100	(4) 60,2	14,6	95,9
80	25	226,6	7,8	78,0
	50	(8) 872,9	26,8	66,7
	75	(5) 278,7	47,8	85,5
	100	(6) 1469,5	96,5	129,5
90	25	(9) 585,9	23,3	89,4
	50	(6) 3708,5	67,8	102,3
	75	(2) 2850,5	735,1	491,4
	100	(3) 146,2	180,4	195,7
100	25	2308,1	65,4	145,8
	50	(6) 1724,3	308,0	289,1
	75	(2) 4243,47	856,8	656,7
	100	(5) 2658,8	649,7	552,3

TABLE: ($kQKP_{DWR(1)}^{\mu^*, v^*}$) root node vs CPLEX

Plan

- 1 Introduction
- 2 DWR with a quadratic master problem
- 3 DWR with a quadratic pricing problem
- 4 DWR applied to (kQKP)
- 5 Numerical Results
- 6 Conclusion and perspectives

Conclusion and perspectives

Conclusion

- Dantzig-Wolfe Reformulation and convexification approaches can be used together.
- DWR is crucial to obtain strong dual bounds.

Perspectives

- Apply Dantzig-Wolfe Reformulation approach to other 0-1 quadratic problems.
- Derive an exact method (Branch & Price) combining Dantzig-Wolfe Reformulation and convexification.