Conditional and Sequential Approval Voting on Combinatorial Domains

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Abstract
Several methods exist for making collective decisions on a set of variables when voters possibly have preferential dependencies. None is based on approval voting. We define a family of rules for approval-based voting on combinatorial domains, where voters cast conditional approval ballots, allowing them to approve values of a variable conditionally on the values of other variables. We study three such rules. The first two generalize simple multiwinner approval voting and minimax approval voting. The third one is an approval-based version of sequential voting on combinatorial domains. We study some properties of these rules, and compare their outcomes.

1 Introduction
Collective decisions on combinatorial domains cover a large variety of common situations, such as finding a set of dates for a series of meetings, electing a committee of representatives, adopting or rejecting each of a series of yes-no questions about common facilities to be built, or finding a common set of movies for a group to watch together.

Several methods have been proposed and studied for making such collective decisions. An important source of problems is when voters have preferential dependencies between variables: someone may be willing to attend the second meeting on a Friday provided that the first meeting is not already on a Friday; someone may want Ann to be elected at the department council only if both Betty and Charles are not elected too (they all belong to the same research group and would have too much joint power); I may want to watch one comedy and one drama, but not two films of the same genre.

It has been argued many times that making decisions independently on each variable cannot take such preferential dependencies into account. If we want to deal with them, we have to allow voters to express these dependencies. Several classes of methods have been proposed and studied (see [Lang and Xia, 2016] for a recent survey). Direct methods work by asking voters a one-shot report of their preferences, expressed in some language that allows preferential dependencies to be (at least partly) specified; sequential methods proceed by eliciting the voters’ preferences on a first (fixed) variable, decide its value, broadcast it, then eliciting the voters’ preferences on a second variable conditionally on the value chosen for the first one, and so on.

Most of the existing methods for voting in combinatorial domains assume that voters express, implicitly or explicitly, rankings over the values of variables, or else cardinal utility functions. Both may induce a cognitive burden to the voters, as well as an increased complexity in communication and computation. On the other hand, in simple domains, approval voting is a well-studied method for elections, where each voter may approve of any number of candidates, and the candidate with the largest number of approvals wins. Approval voting has nice properties, and is easy to use and understand [Brams and Fishburn, 2005].

Approval voting can be extended to combinatorial domains, and especially to committee elections. The simplest way, if we want to select $k$ candidates, consists in selecting those with the largest $k$ approvals. Other ways of extending approval voting to multiple winner elections have been considered and are surveyed in [Kilgour, 2010]. Similar methods can be used for multiple referendum, as argued in [Amanatidis et al., 2015], due to the structural similarity of committee elections and multiple referenda: in both cases, a decision has to be made over a set of possible subsets (of candidates in one case, of binary issues in the other case) subject to feasibility constraints (such as the number of winners in committee elections, or budget constraints in multiple referenda). However, the methods we know of for multiwinner approval voting do not allow voters to express conditional preferences; they must vote as if they had separable preferences.

We propose a family of methods for reconciliating the simplicity of approval voting with the possibility for voters to express preferential dependencies. In Section 3 we define conditional approval ballots where voters approve some values of variables conditionally on the value of some other variables. In Section 4 we define two first rules based on conditional approval ballots, where the winning alternative minimizes respectively the maximum or the sum, over all voters, of the number of ‘disagreements’, and we show that these rules generalize simple (or ‘minisum’) approval voting and minimax approval voting [Brams et al., 2007]. In Section 5 we define sequential conditional approval voting: given a fixed order of variables, the voters approve some of their values, one variable after the other. We conclude in Section 6.
2 Background and related work

In a committee (or multi-winner) election, there is a set of candidates $C$, a set of voters $N$, each casting a vote (or a ballot), and a number $k$ of winners (or sometimes a more flexible constraint, or even no constraint at all such as in Hall-of-Fame elections). In multiwinner approval balloting, each voter casts an approval ballot where she approves as many candidates as she wants (in some variants, there are also constraints on the number of approvals on a ballot).

In simple multiwinner approval voting, the winners are simply the $k$ candidates approved most often. Simple approval voting can be criticized for failing to guarantee a sufficient level of fairness or representativeness, and a number of other rules that map a collection of approval ballots to a collection of winners (or sometimes a more flexible constraint, or even no constraint at all such as in Hall-of-Fame elections). In multiwinner approval balloting, each voter casts an approval ballot where she approves as many candidates as she wants (in some variants, there are also constraints on the number of approvals on a ballot).

In a designated-post committee election [Benoît and Kornhauser, 2010], $\mathcal{X}$ is a set of different seats to be filled and their domains are the respective sets of candidates for each position (an alternative is feasible only if it does not assign the same candidate to more than one seat). In a standard committee election where the size of the committee is fixed to $k$, $\mathcal{X}$ can be identified with the set of candidates, with binary domains {elected, not elected}, and an alternative is feasible iff $k$ variables have value ‘elected’. In a multiple referendums [Brams et al., 1998], $\mathcal{X}$ is a set of binary issues.

As a running example, a group of friends have to decide on a common menu. The variables are $X_1$ (main dish), with $D_1 = \{m, f, v\}$ (meat, fish, vegetarian dish), and $X_2$ (drink), with $D_2 = \{r, w, b\}$ (red wine, white wine, beer).

The most common way of using approval balloting in such domains consists in asking voters to approve values for each variable separately. This does not require much communication, but it is not possible for voters to report conditional preferences. For instance, a voter may approve $\{m, f\} \subseteq D_1$ and $\{r, w\} \subseteq D_2$ but cannot express that he approves the red wine only if the main dish collectively chosen is meat.

One could also think of asking voters to specify all combinations of values they approve (possibly in a compact way by means of a propositional formula). Here, preferential dependencies can be expressed, but the communication is costly, and moreover, a problem is that once a voter obtains a value he does not like for a variable, his opinion about the other variables does not count. Assume for instance that a voter hates fish and beer, and wants red wine with meat. It is reasonable to expect that he approves the set of menus $\{mr, vw, vr\}$, and disapproves $mv, mb, fr, fw, fb, vb$. However, if the collective decision turns out to give the value fish to $X_1$, our voter won’t have a chance to express that he still does not want to drink beer.

A trade-off between these two (extreme) methods consists in allowing voters to approve sets of values for each variable, conditioned on the value of some other variables.

Definition 1. A conditional approval (CA) ballot over variables $X_1, \ldots, X_p$ with domains $D_1, \ldots, D_p$ is a pair

$$B = \langle G, \{A_i \mid i = 1, \ldots, p\} \rangle$$

where $G$ is a directed graph over $\{X_1, \ldots, X_p\}$, and for each $i$, $A_i$ is a set of conditional approval statements $\{u : a_i(u) \in D_{\text{par}(G,X_i)}\}$, where $a_i(u) \subseteq D_i$, and $\text{par}(G,X_i)$ is the set of parents of $X_i$ in $G$. $B = X_1^{a_i(u)}$ denotes the projection of $B$ over $X_i$ conditioned by $u$.

We say that $B$ is acyclic if $G$ is acyclic. $B$ is a separable CA ballot if $G$ has no edges.

Most often, we will alleviate the notation and only write the CA statements (and not the graph) when we specify a CA ballot. Also, we omit some curly brackets when it cannot lead to any confusion (in particular, for setonets).

Definition 2. Given an CA ballot $B$ and an assignment $d = (d_1, \ldots, d_p) \in D_1 \times \ldots \times D_p$, we say that $d$ disagrees with

A group of voters have to decide of a common outcome in $D^*$. When $X_i$ is binary, we will usually take $D_i = \{x_i, \overline{x}_i\}$. For $J \subseteq \{1, \ldots, p\}$ we note $D_J = x_{j \in J} \overline{x}_{j \in \overline{J}}$, and for $d = (d_1, \ldots, d_p) \in D$, $d_J$ denotes the tuple $d$ projected to the indices in $J$.

Combinatorial domains are sets of alternatives consisting of the Cartesian product (or sometimes, a subset of it) of finite domain values corresponding to issues, variables, attributes, seats, or even individuals (in the case of committee elections). The main difficulty in voting in combinatorial domains is the presence of preferential dependencies. To deal with them, some approaches make use of compact representations, such as [Procaccia et al., 2008] and [Skowron and Faliszewski, 2015]. The computational aspects of strategic behaviour for single- and multi-winner approval voting are addressed in [Meir et al., 2008] and in [Baumeister et al., 2010]. The properties of multiwinner voting rules are studied more generally in [Ellkind et al., 2014].

3 Conditional approval ballots

Let $\mathcal{X} = \{X_1, \ldots, X_p\}$ be a set of variables, each of them associated with a finite value domain $D_i$. Let $D = D_1 \times \ldots \times D_p$, and $D^* \subseteq D$ be a set of feasible alternatives (by default, $D^* = D$). A group of voters have to decide of a
B on variable $X_i$ if $d_i \notin B \uparrow X_i | \{d\}$, where $U = Par_G(X_i)$. The disagreement of $d$ with respect to $B$, denoted by $\delta(d, B)$, is the number of variables $X_i$ on which $d$ disagrees with $B$.

**Example 1.** Consider a voter expressing the CA ballot

$$B = \langle X_1 \rightarrow X_2 \rangle, \{ \{f, v\}; \; m : r; \; f : w; \; v : \{r, w\} \}$$

This voter has a preference for not having meat: $B \uparrow X_1 = \{f, v\}$; but in case the collective decision is to have meat, she still has her word to say about the drink, and approves (only) the red wine: $B \uparrow X_2 | m = \{r\}$. If the collective decision is the vegetarian dish, then she approves both the red and the white wine: $B \uparrow X_2 | v = \{r, w\}$. Now, $d = mw$ disagrees with $B$ on $X_1$ but not $X_2$, therefore $\delta(d, B) = 1$ (she is not satisfied with the choice of the dish, but is satisfied by the choice of the drink given that the dish is meat). Likewise, we have $\delta(fw, B) = \delta(vr, B) = \delta(vw, B) = 0$ (she is satisfied by the choice of the dish and the choice of the drink conditional on the value of the dish), $\delta(fr, B) = \delta(fb, B) = \delta(vb, B) = 1$, and $\delta(mvw, B) = \delta(mwb, B) = 2$.

Equivalently, $p = \delta(\cdot, B)$ can be seen as a utility function which counts the number of conditional preferences satisfied. Thus, even though voters are using approval-like ballots, the preferences they express are not dichotomous.

**Observation 1.** If $B$ is acyclic and $a_i(u) \neq 0$ for all $i \leq p$ and $u \in D_{Par_G(X_p)}$, then $\delta(d, B) = 0$ for some $d \in D$.

This, however, is no longer true if $B$ is not acyclic: let for instance $B = \{x_1 : x_2, \pi_1 : x_2, x_2 : x_1, \pi_2 : x_1\}$, then for every $d \in D$ we have $\delta(d, B) = 1$.

Conditional approval ballots have a clear resemblance with CP-nets [Boutilier et al., 2004]. For the sake of simplicity, assume that variables are binary and let $N(B)$ the CP-net obtained by replacing all conditional approval sets $u : a_i(u)$ by $u : f_i(u) \Rightarrow a_i(u)$, with $\pi_x = x_t$. The semantics of $B$ and $N(B)$ are different: if $\triangleright_B N(B)$ is the partial order over $D$ induced from $N(B)$, and $\triangleright_B$ the order defined by $d \triangleright_B d'$ if $\delta(d, B) \not< \delta(d', B)$, then $\triangleright_B$ and $\triangleright_{N(B)}$ are incomparable.

**Observation 2.** There exists $B$ and $d, d'$ such that $d \triangleright_B d'$ and $d' \not\triangleright_{N(B)} d$.

This can be seen on this example: $X = \{X_1, X_2, X_3\}$; $B = \{x_1 : x_2, \pi_1 : x_2, x_1 x_2 : x_3, x_1 x_2 : \pi_2, \pi_1 x_2 : x_3, x_1 x_2 : x_3\}$. We have $x_1 \pi_2 x_3 \triangleright_{N(B)} \pi_1 x_2 x_3$, because $N(B)$ contains $x_1 \pi_2 x_3$, however $x_1 \pi_2 x_3 \not\triangleright_B \pi_1 x_2 x_3$, because $\delta(\pi_1 x_2 x_3, B) = 1 < \delta(x_1 \pi_2 x_3, B) = 2$.

Note that $\delta(\cdot, B)$ is a function from $D$ to $[0, p]$, but not all functions from $D$ to $[0, p]$ can be expressed by a CA ballot; this can be seen by a simple counting argument. Therefore, assuming that a voter’s preferences are representable by a CA ballot is a domain restriction.

4 Conditional minisum and minimax

**Definition 3.** An $n$-voter conditional approval profile is a collection $P = \{B_1, \ldots, B_n\}$ of CA ballots. It is separable if every $B_i$ is a separable CA ballot. A CA irresolute rule is a function mapping any CA profile to a nonempty subset of $D^*$.

Resolute rules can be obtained by combining an irresolute rule with a tie-breaking mechanism: if $F$ is an irresolute rule and $T$ is a tie-breaking mechanism induced by a fixed priority ranking over $D^*$, then $F^T$ is the composition of $F$ by $T$. (Note that $T$ is a relation over an exponentially large set, and thus should be represented in some compact way; this has no impact in our results.) We now define two specific rules.

4.1 Conditional minisum

**Definition 4.** Given a CA profile $P = \{B_1, \ldots, B_n\}$, the conditional minisum rule outputs the outcomes that minimize the total number of disagreements over all voters:

$$\text{CondMiniSum}(P) = \text{argmin}_{d \in D} \sum_{i=1}^n \delta(d, B_i)$$

**Example 2.** Let $D^* = D = \{m, f, v\} \times \{r, w, b\}$ and let $P$ be the following 19-voter CA profile; the number on the top row denotes the number of voters; for instance, 5 voters express the CA ballot $\{\{f, v\}, m : r; f : w, v : \{r, w\}\}$.

Note that the CA ballots of the second column are separable.

<table>
<thead>
<tr>
<th>m</th>
<th>f</th>
<th>v</th>
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<tbody>
<tr>
<td>5</td>
<td>4</td>
<td>3</td>
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<tr>
<td>m : r</td>
<td>m : b</td>
<td>m : r</td>
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<td>f : w</td>
<td>f : b</td>
<td>f : w</td>
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<tr>
<td>v : {r, w}</td>
<td>v : {r, w}</td>
<td>v : {r, w, b}</td>
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The table below gives the disagreement values for each group of voters, and the total disagreement value.

<table>
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<tr>
<th>mr</th>
<th>mw</th>
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<th>fr</th>
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<th>fb</th>
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<th>vw</th>
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and we have $\text{CondMiniSum}(P) = \{mr, mw\}$.

If $P$ is separable, then the outcome of $\text{CondMiniSum}(P)$ can be obtained simply by decomposing the vote into a series of standard approval votes, variable by variable. Let $C$ be a set of candidates; a (standard) approval profile is a collection of approval ballots $(A_1, \ldots, A_n)$ where $A_i \subseteq C$ for each $i$; the (standard) approval rule $\text{App}$ maps any approval profile to a winner (or a set of tied winners), and is defined by $\text{App}(A_1, \ldots, A_n) = \text{argmax}_{c \in C} \{\{i : c \in A_i\}\}$. We refer to the winner (or set of winners) obtained by $\text{App}$ as the winner (or set of winners) obtained by the approval profile $(A_1, \ldots, A_n)$.

**Observation 3.** If $P = \{B_1, \ldots, B_n\}$ be a separable CA profile. Then $\text{CondMiniSum}(P) = \text{App}(P \uparrow X_1) \times \ldots \times \text{App}(P \uparrow X_p)$, where $P \uparrow X_i = \{B_1 \uparrow X_i, \ldots, B_n \uparrow X_i\}$.

In particular, if all variables are binary, we are in the context of a multiple referendum without preferential dependencies between issues, and where the decision is made issue by issue according to majority.

On the other hand, classical committee elections correspond to a collection of such separable conditional approval ballots, where binary variables correspond to candidates, together with a constraint on the cardinality of the committee.
Observation 4. Let \( D = \{ \text{yes}, \text{no} \}^p \) (plus possibly a cardinality constraint), and \( P = (B_1, \ldots, B_n) \) a separable CA profile. Then \( \text{CondMiniSum}(P) \) coincides with the output of simple multiwinner approval voting (also called minisum in [Brams et al., 2007]).

4.2 Conditional minimax

Conditional minimax has a utilitarianistic flavor: minimizing the sum of disagreements corresponds to maximizing social welfare. We may instead take an egalitarian point of view and minimize the maximum disagreement:

**Definition 5.** Given a CA profile \( P = (B_1, \ldots, B_n) \), the conditional minimax rule outputs the outcomes that minimize the maximum number of disagreements over all voters:

\[
\text{CondMiniMax}(P) = \arg\min_{d \in D} \max_{i=1}^n \delta(d, B_i)
\]

**Example 2, continued.** The only outcome that has a disagreement at most 1 (in fact, exactly 1) with every conditional ballot is fb, therefore \( \text{CondMiniMax}(P) = \{ fb \} \): every agent is either happy with the fish, or unhappy with the fish, but in that case, happy with the beer given that the dish is fish.

A similar result as Observation 4 holds for conditional minimax:

**Observation 5.** Under the same assumptions as in Observation 4, \( \text{CondMiniMax}(P) \) is the output of minimax approval voting [Brams et al., 2007]).

4.3 Computation

Computing a winning committee for minimax approval voting is \( \text{NP}-\text{hard} \) [Moti and Ami, 1997], which of course carries over to conditional minimax approval voting. On the other hand, the polynomial time computation of winning committees for simple approval voting does not carry over to conditional minimax (even without feasibility constraints):

**Proposition 1.** Given a CA profile \( P \) and an integer \( k \), deciding whether there exists \( d \in D \) such that \( \sum_i \delta(d, B_i) \leq k \) is \( \text{NP}-\text{complete} \), even for binary variables and, for each voter, an acyclic dependency graph with maximal indegree 1.

**Proof.** The problem is clearly in \( \text{NP} \). The hardness proof is based on a reduction from \( \text{MAX}2\text{SAT} \). Consider an instance \( I \) of \( \text{MAX}2\text{SAT} \), defined by a set of variables \( V = \{ x_1, \ldots, x_p \} \), a set of 2-clauses \( C = \{ C_1, \ldots, C_n \} \), and an integer \( t \). We create a set of binary variables \( X = \{ X_1, \ldots, X_p \} \) and, for each clause \( C_i = l_j \lor l_k \), where \( l_j \in \{ x_j, \overline{x_j} \} \) and \( l_k \in \{ x_k, \overline{x_k} \} \), and \( j < k \), a voter \( i \) whose CA ballot \( B_i \) is defined as follows (with the convention \( B_i \supseteq B_j \) if \( l_j = x_j \)):

- the dependency graph has a single edge \( X_j \rightarrow X_k \);
- the CA statements are \( \{ x_j, \overline{x_j} \}, l_j : \{ x_k, \overline{x_k} \}, l_k : \{ l_j \} \), and \( \{ x_q, \overline{x_q} \} \) for each \( q \neq k \).

First, assume that there exists an assignment of the variables \( V \) satisfying at least \( t \) clauses. From that assignment, we define an alternative \( d \) as follows: for all \( q \leq p \), \( d_q = x_q \) if \( x_q \) is assigned to true, and \( d_q = \overline{x_q} \) if \( x_q \) is assigned to false. Then, we study the number of disagreements between \( B_i \) and \( d \). There are two cases to consider:

- If \( C_i \) is satisfied, then either \( l_j \) is true or \( l_k \) is true, and from the definition of \( B_i \), we get \( \delta(d, B_i) = 0 \).
- If \( C_i \) is not satisfied, then both \( l_j \) and \( l_k \) are false, and from the definition of \( B_i \), we get \( \delta(d, B_i) = 1 \).

Thus, since the assignment satisfies at least \( k \) clauses, we have \( \sum_{i=1}^n \delta(d, B_i) \leq p - t \).

Conversely, if there exists a \( d \) such that \( \sum_{i=1}^n \delta(d, B_i) \leq p - t \), then by a similar line of reasoning as above, we can construct an assignment that satisfies at least \( t \) clauses. \( \square \)

On the positive side, there is a model-preserving translation from winner determination for conditional minimax approval voting into maximum satisfiability. We omit the formal details of the general construction but only give an example: the CA ballot \( B = \{ x, x : y, \overline{y} : \{ y, \overline{y} \} \} \) is translated into the set of clauses \( C_B = \{ x, \overline{y} \lor y, x \lor y \lor \overline{y} \} \) (equivalent to \( \{ x, \overline{y} \lor y \} \)); then a CA profile \( P = \{ B_1, \ldots, B_n \} \) is translated into a multi-set of clauses \( C_P = \bigcup_{i=1}^n C_{B_i} \). For all \( d \in D \), we have \( \sum_i \delta(d, B_i) = \alpha \) if and only if the number of clauses that are not satisfied in \( C_P \) is equal to \( \alpha \).

This means that conditional minimax approval voting can be solved by off-the-shelf \( \text{MAX}2\text{SAT} \) solvers. Moreover, in the case of binary variables with acyclic dependency graphs, it is easy to check that this transformation preserves the differential approximation ratio of \( 4.34/(m + 4.34) \) obtained for \( \text{MAX}2\text{SAT} \) in [Escoffier and Paschos, 2007], where \( m \) represents the number of clauses.

4.4 Properties

We now study some of the properties satisfied by conditional minisum and conditional minimax. Because our rules have CA ballots as input, some of the standard properties have to be adapted. Clearly, our rules satisfy anonymity (the outcome is independent from the identity of voters), and their irresolute versions satisfy value neutrality (the outcome is unchanged after any renaming of the values of some variable) and variable neutrality (the outcome is unchanged after any renaming of the variables). Now we define four other important properties. For two profiles \( P = (P_1, \ldots, P_p) \) and \( Q = (Q_1, \ldots, Q_m) \), we note \( P + Q = (P_1, \ldots, P_p, Q_1, \ldots, Q_m) \).

**Definition 6.** Let \( F \) be a CA irresolute rule.

- \( F \) satisfies reinforcement (resp. weak reinforcement) if for any two profiles \( P, Q \), if \( F(P) \cap F(Q) \neq \emptyset \) then \( F(P + Q) = F(P) \cap F(Q) \) (resp. \( \subseteq F(P) \cap F(Q) \)).
- \( F \) satisfies monotonicity if for any profile \( P = (B_1, \ldots, B_n) \), if \( d = (x_1, \ldots, x_p) \in F(P) \) then \( d’ = (x_1, \ldots, x_p) \in F(P) \) and \( B_i[d’] \).

1The translation works also for nonbinary variables, but is more complex, as there is one propositional variable per value \( d_i \in D_i \), and a feasibility constraint ensures that each variable takes exactly one value. More generally, when \( D^* \) is a strict subset of \( D \), the feasibility constraints are expressed as propositional formulas, and are in sufficiently many copies so as to be ‘protected’.

2Given an instance \( I \) of a combinatorial optimization problem, the differential approximation ratio measures “the relative position of the value of an approximated solution in the interval between the value of a worst feasible solution of \( I \) and the value of a best solution of \( I” \) [Escoffier and Paschos, 2007].
is obtained from $B_1$ by adding $x_j$ to the set of values of $X_j$ approved conditioned on $d_P(x_j)$. Then $d \in F(B_1, \ldots, B_{n-1}, B'_1, B_{n+1}, \ldots, B_n)$.

The following properties, participation and strategyproofness, are defined only for irresolute rules.\footnote{For irresolute rules, we would require to first define a preference extension principle (see, e.g., [Brandt and Brill, 2011] for strategyproofness and [Brandl et al., 2015] for participation).}

Moreover, they require a voters' preference $\succeq$ to be induced from her CA ballot. We say that a voter with CA ballot $B_1$ has $\delta$-induced preferences if for all $x, y \in D^*$, $x \succeq y$ if and only if $d(x, B) \leq \delta(y, B)$. $\delta$-induced preferences generalize Hamming-induced preferences [Cuhadaroglu and Lainã, 2012].

**Definition 7.** Let $F^T$ the irresolute version of some irresolute rule $F$, for some tie-breaking mechanism $T$. $F^T$ satisfies

- $F^T$ satisfies $\delta$-participation if for any CA profile $P = (B_1, \ldots, B_n)$ and CA ballot $B_{n+1}$, we have $\delta(F^T(P + \{B_{n+1}\}), B_{n+1}) \leq \delta(F^T(P), B_{n+1})$.
- $F^T$ satisfies $\delta$-strategyproofness if for any CA profile $P = (B_1, \ldots, B_n)$ and CA ballot $B'_1$, $\delta(F^T(B_1, \ldots, B_{n-1}, B'_1, B_{n+1}, \ldots, B_n), B'_1) \geq \delta(F^T(P), B_1)$.

We have the following results for conditional minsum and minmax. All proofs are simple (we omit them).

**Proposition 2.** $\text{CondMinSum}^T$ satisfies reinforcement.

- $\text{CondMinMax}^T$ does not satisfy reinforcement, but satisfies weak reinforcement.
- $\text{CondMinSum}$ and $\text{CondMinMax}$ satisfy monotonicity.
- for any $T$, $\text{CondMinSum}^T$ and $\text{CondMinMax}^T$ satisfy $\delta$-participation.

Since minimax approval voting [Caragiannis et al., 2010] is manipulable, this is a fortiori the case for $\text{CondMinMax}^T$. On the other hand, simple approval voting (or ‘minsum’) is strategyproof; however, as soon as preferences are conditional, strategyproofness is lost:

**Proposition 3.** $\text{CondMinSum}^T$ is $\delta$-manipulable, even for two binary variables.

**Proof.** Without loss of generality, assume $T$ favors $x_1 \cdot x_2$ over $x_1 \cdot x_2$. Consider the two-voter CA profile consisting of one ballot $\{x_1, x_2, x_1 \cdot x_2\}$ and one ballot $\{x_1, x_2, x_1 \cdot x_2\}$. The outcome is $\overline{x_1 \cdot x_2}$. Now, if the second voter casts the ballot $\{x_1, x_1 \cdot x_2, x_1 \cdot x_2\}$ instead, then the outcome is $x_1 \cdot x_2$, which is her preferred alternative under the assumption that her preferences are $\delta$-induced.

\section{Sequential conditional approval voting}

For the sake of simplicity, in this section we assume that $D^* = D$ (handling feasibility constraints is possible but makes the definitions more complicated).

Sequential conditional approval voting (SCAV) is an approval-based version of sequential voting in combinatorial domains [Lang and Xia, 2009]. Its intuitive principle is that decisions are taken variable by variable, following a fixed order; for each variable, the winning value is the one that has the maximal approval score given the values of the variables that come before it. The SCAV rule is defined only for CA profiles whose dependency graph is compatible with a given order $O$ of the variables. In the rest of this section, $O = X_1 > \ldots > X_p$ is a fixed order over $X$.

**Definition 8 (O-legal conditional approval profile).** A CA ballot is $O$-legal if its dependency graph contains no edge from $X_j$ to $X_i$ with $i < j$. A CA profile is $O$-legal if it is composed of $O$-legal CA ballots.

**Definition 9.** The sequential conditional approval voting rule (SCAV) is the rule defined on $O$-legal CA profiles as follows. Let $\text{App}^T$ be the standard approval voting rule on $D^T$ together with a tie-breaking mechanism $T_i$, and $T = (T_1, \ldots, T_p)$. For each $O$-legal profile $P = (B_1, \ldots, B_n)$, let $P_{\text{SCAV}}(x_1^*, \ldots, x_{p-1}^*) = (B_{1}^{x_1^*}, x_{1}^*, \ldots, x_{p-1}^*)$; then $\text{SCAV}^T(P) = (x_1^*, \ldots, x_p^*)$, where

- $x_1^* = \text{App}^T_1(P_{\text{SCAV}})$;
- for each $i = 2, \ldots, p$, $x_i^* = \text{App}^T_i(P_{\text{SCAV}}(x_1^*, \ldots, x_{i-1}^*))$.

Similarly, we define the SCAV irresolute rule: $\text{SCAV}(P)$ is the set of all $(x_1, \ldots, x_p)$ such that $x_1 \in \text{App}(P_{\text{SCAV}})$ and for all $i = 2, \ldots, p, x_i \in \text{App}(P_{\text{SCAV}}(x_1^*, \ldots, x_{i-1}^*))$.

A cheap protocol for $\text{SCAV}^T$ is composed of $p$ rounds: at round $i$, it elicits only the voters’ approval ballots $B_j^{x_i^*}$ for $j = 1, \ldots, p$. Thus, the communication complexity of $\text{SCAV}^T$ is $O(mp \max_i |D_i|)$.

**Example 2 (continued).** Let $O = X_1 > X_2$. The approval scores for $m$, $f$ and $v$ are respectively 13, 12 and 10: the (unique) selected value for $X_1$ is $m$. Given $X_1 = m$, the approval scores for $r$, $w$ and $b$ are respectively 12, 4 and 7: the selected value for $X_2 = r$. The outcome is nrw.

For irresolute SCAV, some winning alternative can be computed in polynomial time, and determining whether a given alternative is a winner is also polynomial-time computable: at each round, it suffices to check that $x_i \in \text{App}(P_{\text{SCAV}})$.

Recall that winner determination for conditional minsum is NP-hard, even for an $O$-legal profile. Moreover, both conditional minsum and SCAV coincide with simple approval voting when restricted to separable profiles. Thus, we may wonder how good an approximation SCAV is to conditional minsum. The approximation ratio is different whether we measure the quality of an alternative by its number of disagreements $\sum_i |\delta(x_i, B_i)|$ (to be minimized) or its number of agreements $\sum_i (p - |\delta(x_i, B_i)|)$ (to be maximized):

\footnote{Note that writing $B_j^{x_i^*} = x_{j-1}^*$ is a slight abuse of notation, since $O$-legality implies only that $\text{Par}_{O}(X_i) \subseteq \{X_1, \ldots, X_{i-1}\}$.}

\footnote{Note that the restriction of SCAV to separable profiles coincides with simple approval voting.}
Proposition 4. Consider a combinatorial domain with $p$ variables $X_1, \ldots, X_p$ with $|D_i| = \alpha_i$ for all $i$.

- The largest possible ratio, over all $O$-legal CA profiles, between the disagreement score of the SCAV winner and the disagreement score of the conditional minisum winner is $\left(1 - \frac{1}{\alpha_i}\right)/(1 - \frac{1}{p})$.
- The largest possible ratio, over all $O$-legal CA profiles, between the agreement score of the conditional minisum winner and the agreement score of the SCAV winner is $\left(\frac{1}{\alpha_i} + p - 1\right)/\left(\sum_{i=1}^{p} \frac{1}{\alpha_i}\right)$.

Proof sketch. Consider $n$ CA ballots, with the dependency graph whose edges are $\{X_i \rightarrow X_j \mid 1 \leq i < j \leq p\}$. First, we look for an upper bound of the disagreement score of an alternative that wins for SCAV. Assume that $(x_1, x_2, \ldots, x_p)$ is the winning alternative for SCAV. Then, $(x_1, x_2, \ldots, x_p)$ has to be approved by at least $n/\alpha_1$ voters, respectively $n/\alpha_2, \ldots, n/\alpha_p$ voters. It leads to a maximal disagreement score of $n \cdot \left(\sum_{i=1}^{p} \frac{1}{\alpha_i}\right) - 1 - 1/\alpha_1$.

Now, we look for a lower bound of the disagreement score of a winning alternative for SCAV. Assume that $(x_1', x_2', \ldots, x_p')$ is a winning alternative for SCAV. One way to avoid this alternative to be a SCAV winner is by eliminating it at the first round of the sequential vote. It implies that $x_1'$ has to be approved by less than $n/p$ voters, and, in this case, $x_1' \rightarrow x_2', \ldots, x_{p-1}' \rightarrow x_p'$ can be approved by any number of voters, and then, the disagreement score for $(x_1', x_2', \ldots, x_p')$ without being a winner for SCAV is at least $n - \frac{n}{\alpha_1}$. There exist other ways to avoid $(x_1', x_2', \ldots, x_p')$ to be a winning alternative for SCAV, but they lead to disagreement scores that are at least as large as that one. The proof for the bound relative to the agreement score is similar.

Thus, we get a worst-case ratio of $(\sum_{i=1}^{p} \frac{1}{\alpha_i})/(1 - 1/\alpha_1)$. This ratio is reached on a CA profile for which there is a perfect split between all values of $X_1$ (with each voter approving only one value), and then, conditionally on the value chosen for $X_1$ (by tie-breaking), there is also a perfect split for each other variable, while for some of the other values of $X_i$, there is a perfect agreement between all voters on all other variables. The same profile shows that the bound relative to the number of agreements is reached too.

In particular, when all domains have the same cardinality $\alpha$, then these ratios become $p$ and $(1 + (p - 1)\alpha)/p$, and in the case of binary domains, $p = 2 - 1/p$.

Corollary 1. If the quality of a solution is measured by the number of disagreements, then SCAV is a $\sum_{i=1}^{p} \frac{1}{\alpha_i}/(1 - \frac{1}{p})$-approximation of CondMiniSum. If it is measured by the number of disagreements, then it is a $\left(\frac{1}{\alpha_i} + p - 1\right)/\sum_{i=1}^{p} \frac{1}{\alpha_i}$-approximation of CondMiniSum.

Proposition 5. SCAV satisfies anonymity, neutrality, reinforcement and monotonicity. For any tie-breaking mechanism $T$, SCAV does not satisfy $\delta$-participation.

Proof. Anonymity and neutrality are obvious. Reinforcement and monotonicity are easily proven by induction on the variables, and using the fact that standard approval voting satisfies them. For participation, consider the two-voter CA profile $P = (B_1, B_2)$ with $B_1 = \{T_1, T_2, T_3\}$ and $B_2 = \{T_1, T_2, T_7\}$. Assume without loss of generality that $T$ favors $T_1, T_2, T_7$ over $x_1, T_2, T_7$. Thus, $T^P = x_1, T_2, T_7$. Now, consider a third voter with preferences $\delta$-induced by $B_3 = \{x_2, x_3, T_2, T_2, x_1, T_2, T_7\}$. If she votes, the outcome is $x_1, T_2, T_7$, which has only one agreement with her ballot, while $x_1, T_2, T_7$ had two.

Proposition 6. For any tie-breaking mechanism $T$, SCAV is not $\delta$-strategyproof for $p \geq 3$ (even for binary variables), and is strategyproof for two variables.

Proof. For $p = 3$, consider the profile in the proof of Proposition 5. If the third voter expresses his sincere ballot $\{x_1, x_1, x_2, T_1, T_2, x_1, x_2, T_2, x_1, x_3, T_1, T_2, x_1, T_2, T_7\}$, the outcome is $x_1, T_2, T_7$. If he expresses instead the ballot $\{T_1, x_1, x_2, T_2, x_1, x_3, T_1, T_2, x_1, T_2, T_7, x_2, T_2, T_7\}$ the the outcome is $T_1, x_2, T_2, T_7$, which he prefers.

For $p = 2$, let $X = \{X_1, X_2\}$; suppose that voter $i$ with sincere ballot $B$ has a manipulation $B'$; let $x^*$ be the outcome for her sincere vote and $x^+$ the outcome after manipulation. Since $B'$ is a manipulation and $i$ has $\delta$-induced preferences, we have $\delta(x^+, B') < \delta(x^*, B)$. This implies $1 \leq \delta(x^+, B') \leq (1 + (p - 1)\alpha)/p$, and $\sum_{i=1}^{p} \frac{1}{\alpha_i}$-induced by $X_1$ and $x^+$ cannot obtain a better outcome than the sincere outcome; therefore, the manipulation has to bear on $X_1$ and has to change the value of $X_1$, from, without loss of generality, $x_1$ to $x_1$, with $i$ initially approving $x_1$ but not $x_1$. But this implies that $\delta(x^+, B') \leq 1$, while $\sum_{i=1}^{p} \frac{1}{\alpha_i}$-induced by $x_1$ and $x_1$ together with (1-2) implies $\delta(x^+, B') = \delta(x^*, B) = 1$, therefore $i$, who has $\delta$-induced preferences, cannot prefer $x^+ \delta x^*$.

6 Conclusion

We have generalized approval voting to combinatorial domains and to nonseparable preferences. Our rules are natural generalizations of simple (or ‘minisum’) and minimax approval voting. They can be applied for all types of voting on combinatorial domains. Conditional minisum and minimax are computationally hard, but conditional minisum can be solved by MaxSAT solvers. Sequential conditional AV is easy to compute. Our rules (especially conditional minisum) satisfy a number of important properties.

There are other rules for multiwinner elections using approval ballots, where the satisfaction of a voter is not simply the Hamming distance to his preferred outcome (see [Kilgour, 2010] for a survey, and [Aziz et al., 2015b] for their computation). These rules could be adapted to CA ballots in the same way as we did for simple and minimax approval voting. The rules defined in [Amanatidis et al., 2016], where the dissatisfaction degrees of the voters is aggregated by ordered weighted averages intermediate between max and sum, could also be generalized to CA ballots.

Acknowledgement

We thank Bernard Ries for helpful discussions. This research has been supported by the project ANR-14-CE24-0007-01 CoCoRICo-CoDec.
References


