# Towards mathematical morpho-logics

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**Abstract.** In this paper, we suggest a new way to process information represented in a logical framework based on mathematical morphology. We show how the basic morphological operations can be expressed in a logical setting. We give some properties, show some links with revision and fusion, and ideas illustrate possible use of morpho-logics to approximation, reasoning and decision.

## 1 Introduction

Mathematical morphology is originally based on set theory. It has been introduced in 1964 by Matheron [16,17], in order to study porous media. But this theory evolved rapidly to a general theory of shape and its transformations, and was applied in particular in image processing and pattern recognition [20]. Additionally to its set theoretical foundations, it also relies on topology on sets, on random sets, on topological algebra, on integral geometry, on lattice theory.

In the knowledge representation community, propositional formulas are used to encode either pieces of knowledge (which may be generic – for instance, integrity constraints – or factual) or "preference items" (such as opinions, desires or goals) and are then used for complex reasoning or decision making tasks. These tasks often make use of operations on propositional formulas which are very similar to those considered in mathematical morphology. We give a (non-exhaustive) list of examples:

- *belief revision* (as shown by Katsuno and Mendelzon [11]) consists of the following operation: let $\varphi$ and $\psi$ be two propositional formulas. The models of the revision $\varphi \circ \psi$ of $\varphi$ by $\psi$ are the models of $\psi$ which are closest (with respect to a given distance) to a model of $\varphi$. Intuitively, using the language of mathematical morphology, it means that $\varphi$ has to be dilated enough to intersect with some models of $\psi$. *Belief update* [11] proceeds to the same kind of dilation but on each individual model of $\varphi$ and then takes the union of all obtained sets of models.
- *belief merging* [12] consists in finding the best compromise between a finite set of formulas $\varphi_1$, ... $\varphi_n$, which amounts to selecting the models which minimize the aggregation (using some given operator) of the distances to each of the $\varphi$'s. This amounts intuitively to dilate simultaneously all the $\varphi$'s until they intersect. Similar operations are at work for the aggregation of preferences in group decision making as proposed in [14].

- one of the tasks involved in *similarity-based reasoning* [8,7] consists in determining if a formula $\varphi$ approximatively entails a formula $\psi$ by looking to what extent $\psi$ has to be extended so as to contain all models of $\varphi$, which again corresponds to a dilation.
- *reasoning with supermodels* [9] uses models of a formula $\varphi$ which are robust enough to resist some perturbations. In some cases, obtaining supermodels consists in eroding the formula so as to be far enough from the countermodels of $\varphi$. Again this corresponds to a classical operation of mathematical morphology (erosion). Another close notion, evoked in [14], is the search for the most representative worlds of a formula.
- in *abductive reasoning* [18], preferred explanations of a formula are defined based on a set of axioms, several of which being closed to properties of morphological operators.

Our aim is therefore to investigate how and why mathematical morphology can be applied on logical formulas. First we note that the fact that a propositional formula can be equivalently defined by the set of its models enables us to apply easily all (set-theoretic) definitions of mathematical morphology to logical objects (worlds, formulas). This will lead us not only to rewriting well-known logical operations used for reasoning or decision making, but also to designing new kinds of logical objects or notions by transposing basic morphological operations to propositional logic. There are two ways of doing that: first, one may define modal logics from mathematical morphology, by establishing links between accessibility relations and structuring elements, and constructing modal operators $\Box$ and $\Diamond$ from erosion and dilation for instance [4] (based for instance on links between morphological operators and approximation operators in the theory of rough sets [3]). Second, one may view morphological operators as transformations on formulas. Here we choose to explore the latter way for reasoning or decision making, etc.

In Section 2, we introduce some notations and logical concepts that are used throughout the paper. In Section 3 we propose definitions for dilation and erosion of logical formulas, by extending these two fundamental morphological operations to the logical framework. We also list some properties, and give some applications for fusion, revision, and specifying formulas. In Section 4, we define opening and closing on formulas, and give some properties and interpretations for approximating and filtering formulas. In Section 5 we provide a few simple examples showing the effects of these four operations. In Section 6, we suggest a few other operations acting on a formula.

## 2 Basic logical concepts

Let $PS$ be a finite set of propositional symbols. $PROP_{PS}$ is the propositional language generated by $PS$ and the usual connectives. Well-formed formulas of $PROP_{PS}$ will be denoted by Greek letters $\varphi$, $\psi$... Interpretations will be denoted by $\omega$, $\omega'$ and the set of all interpretations for $PROP_{PS}$ by $\Omega_{PS}$. $Mod(\varphi) = \{\omega \in \Omega_{PS} \mid \omega \models \varphi\}$ is the set of all models of $\varphi$ (i.e. all interpretations for which $\varphi$ is true).

The underlying idea for constructing morphological operations on logical formulas is to consider set interpretations of formulas and interpretations. Since $PROP_{PS}$ is isomorphic to $2^{\Omega_{PS}}$, i.e., knowing a formula is equivalent to knowing the set of its models, we can identify $\varphi$ with the set of its models $Mod(\varphi)$, and then apply set-theoretic morphological operations. We recall that $Mod(\varphi \vee \psi) = Mod(\varphi) \cup Mod(\psi)$, $Mod(\varphi \wedge \psi) = Mod(\varphi) \cap Mod(\psi)$, and $Mod(\varphi) \subset Mod(\psi)$ iff $\varphi \models \psi$.

# 3 Dilation and erosion

## 3.1 Definitions

Let us first recall the definitions of dilation and erosion of a set $X$ by a structuring element $B$ in $\mathbb{R}^n$, denoted respectively by $D_B(X)$ and $E_B(X)$:

$$D_B(X) = \{x \in \mathbb{R}^n \mid B_x \cap X \neq \emptyset\}, \tag{1}$$

$$E_B(X) = \{x \in \mathbb{R}^n \mid B_x \subset X\}, \tag{2}$$

where $B_x$ denotes the translation of $B$ at $x$,

In these equations, $B$ defines a neighborhood that is considered at each point. It can also be seen as a relationship between points.

Using the previous equivalences, we propose to define dilation and erosion of a formula as follows:

$$Mod(D_B(\varphi)) = \{\omega \in \Omega_{PS} \mid B_\omega \wedge \varphi \text{ } consistent\}, \tag{3}$$

$$Mod(E_B(\varphi)) = \{\omega \in \Omega_{PS} \mid B_\omega \models \varphi\}. \tag{4}$$

In these equations, the structuring element $B$ represents a relationship between worlds, i.e. $\omega' \in B_\omega$ iff $\omega'$ satisfies some relationship with $\omega$. The condition in Equation 3 expresses that the set of worlds in relation to $\omega$ should be consistent with $\varphi$. The condition in Equation 4 is stronger and expresses that all worlds in relation to $\omega$ should be models of $\varphi$.

## 3.2 Structuring element

There are several possible ways to define structuring elements in a context of formulas. We suggest here a few ones. The relationship defined in the previous section can be any relationship between worlds. It defines a "neighborhood" of worlds. If it is symmetrical, it leads to symmetrical structuring elements. If it is reflexive, it leads to structuring elements such that $\omega \in B_\omega$, which leads to interesting properties, as will be seen later. For instance, this relationship can be an accessibility relation as in normal modal logics [10]. An interesting way to choose the relationship is to base it on distances between worlds. This allows to define sequences of increasing structuring elements defined as the balls of a distance. From any distance $\delta$ between worlds, a distance from a world to a formula is derived as a distance from a point to a set: $d(\omega, \varphi) = \min_{\omega' \models \varphi} \delta(\omega, \omega')$.

The most commonly used distance between worlds in knowledge representation (especially in belief revision [6], belief update [11], merging [12] or preference representation [14]) is the Hamming distance $d_H$ where $d_H(\omega, \omega')$ is the number of propositional symbols that are instantiated differently in both worlds. By default, we take $\delta$ to be $d_H$.

Then dilation and erosion of size $n$ are defined from Equations 3 and 4 by using the distance balls of radius $n$ as structuring elements:

$$Mod(D^n(\varphi)) = \{\omega \mid \exists \omega', \omega' \models \varphi \; and \; d_H(\omega, \omega') \leq n\} = \{\omega \mid d(\omega, \varphi) \leq n\}, \tag{5}$$

$$Mod(E^n(\varphi)) = \{\omega \mid \forall \omega', d_H(\omega, \omega') \leq n \Rightarrow \omega' \models \varphi\} = \{\omega \mid d(\omega, \neg\varphi) > n\}. \tag{6}$$

From operations with the unit ball we define the external (respectively internal) boundary of $\varphi$ as $D^1(\varphi) \wedge \neg\varphi$ (respectively $\varphi \wedge \neg E^1(\varphi)$), corresponding to the worlds that are exactly at distance 1 of $\varphi$ (resp. of $\neg\varphi$).

### 3.3 Properties

The main properties of dilation and erosion, which are satisfied in mathematical morphology on sets, hold also in the logical setting proposed here. They are summarized below. The proofs are not given here, but they are straightforward based on set/logic equivalences.

*Monotonicity:* Both operators are increasing with respect to $\varphi$, i.e. if $\varphi \models \psi$, then $D_B(\varphi) \models D_B(\psi)$ and $E_B(\varphi) \models E_B(\psi)$, for any structuring element $B$. Dilation is increasing with respect to the structuring element, while erosion is decreasing, i.e. if $\forall \omega \in \Omega_{PS}, B_\omega \subset B'_\omega$, then $D_B(\varphi) \models D_{B'}(\varphi)$ and $E_{B'}(\varphi) \models E_B(\varphi)$.

*Extensivity:* Dilation is extensive ($\varphi \models D_B(\varphi)$) if $B$ is derived from a reflexive relation (as is the case for distance based dilation, since if $\omega \models \varphi$, then $d(\omega, \varphi) = 0$), and erosion is anti-extensive ($E_B(\varphi) \models \varphi$) under the same conditions.

*Iteration:* Dilation and erosion satisfy an iteration property. For instance for distance based operations, we have:

$$D^{n+n'}(\varphi) = D^{n'}[D^n(\varphi)] = D^n[D^{n'}(\varphi)],$$

$$E^{n+n'}(\varphi) = E^{n'}[E^n(\varphi)] = E^n[E^{n'}(\varphi)].$$

This means that the effects of these operations increase with the size of the structuring element, and that the computation can be done either by successive applications of "small" structuring elements or directly by the sum of the structuring elements.

*Commutativity with union or intersection:* Dilation commutes with union or disjunction (this is a fundamental property of dilation[1]): for any family $\varphi_1, ...\varphi_m$ of formulas, we have: $D_B(\vee_{i=1}^m \varphi_i) = \vee_{i=1}^m D_B(\varphi_i)$. Erosion on the other hand commutes with intersection or conjunction.

In general dilation (resp. erosion) does not commute with intersection (resp. union), and only an inclusion relation holds: $D_B(\varphi \wedge \psi) \models D_B(\varphi) \wedge D_B(\psi)$.

*Adjunction relation:* $(E_B, D_B)$ is an adjunction[2], i.e. $D_B(\psi) \models \varphi$ iff $\psi \models E_B(\varphi)$.

*Duality:* Dilation and erosion (respectively opening and closing) are dual operators with respect to the negation: $E_B(\varphi) = \neg D_B(\neg\varphi)$ which allows to deduce properties of an operator from those of its dual operator.

*Relations to distances:* Equation 5 shows how to derive a dilation from a distance. Conversely, we have: $d(\omega, \varphi) = \min\{n \in \mathbb{N} \mid \omega \models D^n(\varphi)\}$, and similarly, we have $d(\omega, \neg\varphi) = \min\{n \in \mathbb{N} \mid \omega \models \neg E^n(\varphi)\}$.

Distances between formulas can also be derived from dilation, as minimum distance and Hausdorff distance. For instance the minimum distance is expressed as: $d_{min}(\varphi, \psi) = \min_{\omega \models \varphi, \omega' \models \psi} d_H(\omega, \omega') = \min\{n \in \mathbb{N} \mid D^n(\varphi) \wedge \psi \neq \emptyset \; and \; D^n(\psi) \wedge \varphi \neq \emptyset\}$. This means that the minimum distance is attained for the minimum size of dilation of both formulas such that they become consistent.

Examples of dilations of erosions are shown in Section 5.

### 3.4 Computational complexity

The commutation of dilation with disjunction, along with the iteration property, allows to recover results of [14]. In particular, when $\varphi$ is a consistent conjunction of literals, i.e., $\varphi = l_1 \wedge l_2 \wedge ... \wedge l_n$, then

$$D^1(\varphi) = \vee_{j=1}^n (\wedge_{i \neq j} l_i).$$

This property, together with the commutation of dilation with disjunction, gives the following result [14]: if $k$ is a fixed integer, then the dilation of

[1] More generally a dilation in a complete lattice is defined as an operator that commutes with union.

[2] Moreover, if two operators form an adjunction there are an erosion and a dilation respectively.

size $k$ $D^k(\varphi)$ of a DNF formula $\varphi$ can be computed in time $\mathcal{O}(n^k)$ – thus in polynomial time. In a similar way, erosion commutes with intersection and can be computed in polynomial time from a CNF formula.

When $\varphi$ is not under DNF, computing $D^k(\varphi)$ directly from $\varphi$ (without rewriting $\varphi$ under DNF first) is a difficult problem that we leave for further research. Note that, however, inference from the dilation of a formula is (theoretically) not harder than inference from the formula itself. Namely, given any two formulas $\varphi$ and $\psi$ and any integer $k$, determining whether $D^k(\varphi) \models \psi$ is coNP-complete[3]. Obviously, a similar result holds for inference from erosion.

### 3.5 Fusion and revision from dilation

Interestingly, belief revision and belief fusion operators can be recovered from dilations.

We start with some basics about belief revision. A *faithful assignment* [11] is a mapping which associates to each formula $\varphi$ a preorder $\leq_\varphi$ on $\Omega_{PS}$ such that (1) if $\omega \models \varphi$ and $\omega' \models \varphi$ then $\omega \sim_\varphi \omega'$; (2) if $\omega \models \varphi$ and $\omega' \models \neg\varphi$ then $\omega <_\varphi \omega'$; (3) if $\models \varphi_1 \leftrightarrow \varphi_2$ then $\leq_{\varphi_1} = \leq_{\varphi_2}$. As shown by Katsuno and Mendelzon, a revision operator $\circ$ satisfies the "AGM" postulates [1] iff there exists a faithful assignment that maps each formula $\varphi$ to a total preorder $\leq_\varphi$ such that for every propositional formula $\mu$ we have $Mod(\varphi \circ \mu) = \min(Mod(\mu), \leq_\varphi)$[4]. The preorder $\leq_\varphi$ expresses the distance of a world $\omega$ relatively to $\varphi$, i.e., $\omega \leq_\varphi \omega'$ means that $\omega$ is closer to $\varphi$ than $\omega'$. Therefore, a faithful assignment can be defined from a distance $d$ from a world to a formula in the following way: $\omega \leq_\varphi \omega'$ iff $d(\omega, \varphi) \leq d(\omega', \varphi)$. In particular, the revision operator induced by the choice of the distance $d_H$ is known as Dalal's revision operator. Now, it can be easily seen that $\varphi \circ \psi = D^n(\varphi) \wedge \psi$, with $n = \min\{k \in \mathbb{N} \mid D^k(\varphi) \wedge \psi \text{ is consistent}\}$.

We now recall some basics about belief fusion as defined by Konieczny and Pino-Pérez [12]. If $d$ is a distance function from an interpretation to a formula, the $\Delta_{max}$ quasi-fusion of the $m$ formulas $\varphi_1$, ..., $\varphi_m$ is defined by: $Mod(\Delta_{max}(\{\varphi_1, ..., \varphi_m\})) = \min(\Omega_{PS}, \leq_{\{\varphi_1,...,\varphi_m\}})$ where $\omega \leq_{\{\varphi_1,...,\varphi_m\}}$

[3] Here is a sketch of the proof: first, we have that $D^k(\varphi) \not\models \psi$ iff $\exists \omega \models D^k(\varphi)$ s.t. $\omega \models \neg\psi$, and second, $\omega \models D^k(\varphi)$ iff there is a world $\omega' \models \varphi$ such that $d(\omega, \omega') \leq k$. This leads us to the following nondeterministic algorithm: *1. guess two interpretations $\omega$ and $\omega'$; 2. check that $d(\omega, \omega') \leq k$; 3. check that $\omega' \models \varphi$; 4. check that $\omega \models \neg\psi$.* This nondeterministic polynomial algorithm shows the membership of the complementary problem of INFERENCE FROM DILATION to NP, hence, INFERENCE FROM DILATION is in coNP. As to hardness, it is proved by the following polynomial reduction from UNSAT to INFERENCE FROM DILATION: a formula $\varphi$ is in UNSAT iff $D^k(\varphi) \models new$, where $new$ is a propositional variable not appearing in $\varphi$.

[4] The notation $\min(A, \leq)$ where $\leq$ is a total preorder, stands for $\{\omega \in A \mid \forall \omega' \in A, \ \omega \leq \omega'\}$.

$\omega'$ iff $\max_{i=1..m} d(\omega, \varphi_i) \leq \max_{i=1..m} d(\omega', \varphi_i)$. It can be easily seen that $\Delta_{max}(\varphi_1, ..., \varphi_m) = D^n(\varphi_1) \wedge D^n(\varphi_2) \wedge ... \wedge D^n(\varphi_m)$, where $n = \min\{k \in \mathbb{N} \mid D^k(\varphi_1) \wedge ... \wedge D^k(\varphi_n)$ is consistent$\}$.

### 3.6 Ultimate erosion

In some problems, it might be interesting to find the most pertinent worlds that are models of a formula. This problem is solved in [14] by taking the absolute maximum of the internal distance function (i.e. the function that associates to each world its distance to $\neg\varphi$). Mathematical morphology offers other tools that could also be interesting. Ultimate erosion is one of them. It consists in eroding iteratively $\varphi$ and, at each step $n$, keeping the connected components of $E^n(\varphi)$ that disappear in $E^{n+1}(\varphi)$. It corresponds exactly to the regional maxima of the internal distance. In the first approach (absolute maximum), the result is the most restrictive. It gives the center of the largest component of $\varphi$, and if $\varphi$ is not connected, the other components are not represented. On the contrary the second approach (ultimate erosion) provides several components, which represent all parts of $\varphi$, belonging to different connected components, or connected by narrow sets of worlds. It could be interesting to relate this approach to the notions of supermodels [9] and of preferred explanations [18]. For instance, it is easy to prove that $\omega \models E^k(\varphi)$ iff $\omega$ is a $(k, 0)$-supermodel of $\varphi$.

Another solution to this problem is provided by the morphological skeleton, as shown in Section 6.

## 4 Opening and closing

### 4.1 Definitions

Let $B$ be any structuring element, and $\check{B}$ its symmetrical (they are different only in case of non symmetrical relation). Opening and closing of a formula $\varphi$ by $B$ are defined respectively as:

$$O_B(\varphi) = D_{\check{B}}(E_B(\varphi)), \tag{7}$$

$$C_B(\varphi) = E_{\check{B}}(D_B(\varphi)). \tag{8}$$

### 4.2 Properties

*Monotonicity:* Opening and closing are both increasing with respect to $\varphi$, for any structuring element. Opening is decreasing with respect to the structuring element, while closing is increasing.

*Extensivity and anti-extensivity:* Closing is extensive and opening is anti-extensive for any structuring element. This is proved by the fact that opening can be written as[5]:

$$Mod(O_B(\varphi)) = \bigvee \{B_\omega \mid \omega \in \Omega_{PS},\ B_\omega \models \varphi\}.$$

*Idempotence:* Contrary to dilation and erosion which satisfy iterativity properties, opening and closing are idempotent operators, i.e. $O_B[O_B(\varphi)] = O_B(\varphi)$ and $C_B[C_B(\varphi)] = C_B(\varphi)$[6]. Moreover, for distance based operations we have:

$$O^n[O^{n'}(\varphi)] = O^{n'}[O^n(\varphi)] = O^{\max(n,n')}(\varphi),$$
$$C^n[C^{n'}(\varphi)] = C^{n'}[C^n(\varphi)] = C^{\max(n,n')}(\varphi).$$

*Duality:* Opening and closing are dual operators with respect to the negation: $O_B(\varphi) = \neg C_B(\neg\varphi)$.

### 4.3 Approximation from opening and closing

The two basic morphological filters[7] are opening and closing. They can be seen as approximation operators, since they "simplify" formulas by either suppressing some irregularities for opening, or adding some parts of $\neg\varphi$ for closing (see examples in Section 5). Families of filters can be built from these two ones. For instance, granulometry [20] consists in applying successively openings with structuring elements of increasing size, such decomposing a formula in parts of different characteristic sizes. Another example is alternate sequential filters [21], which consist in building sequences of opening/closing (or closing/opening), with structuring elements of increasing size. Such transformations are increasing and idempotent, and allow to filter progressively parts of $\varphi$ and $\neg\varphi$.

## 5 Examples

In this Section, we show a few simple examples illustrating the four basic morphological operations. We make use of a graph representation of worlds, where each node represents a world and a link represents an elementary connection between two worlds, i.e. being at distance 1 from each other. We also introduce the notion of connectedness of a formula.

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[5] Indeed, we have $\omega \models O_B(\varphi)$ iff $\check{B}_\omega \wedge E_B(\varphi)$ is consistent. Let $\omega' \models \check{B}_\omega \wedge E_B(\varphi)$. $\omega' \models \check{B}_\omega$ is equivalent to $\omega \models B_{\omega'}$. And $\omega' \models E_B(\varphi)$ is equivalent to $B_{\omega'} \models \varphi$, qed.

[6] Anti-extensivity of opening guarantees that $O_B[O_B(\varphi)] \models O_B(\varphi)$. Moreover, we have (ignoring the subscript for simplicity) $O[O(\varphi)] = D[C[E(\varphi)]]$. Since closing is extensive, we have $E(\varphi) \models C[E(\varphi)]$. Since dilation is increasing, we have $D[E(\varphi)] \models D[C[E(\varphi)]]$, i.e. $O(\varphi) \models O[O(\varphi)]$, qed.

[7] A morphological filter is an increasing and idempotent operator.

*Connectedness:* A formula $\varphi$ is said to be connected iff: $\forall(\omega,\omega') \in \Omega^2_{PS}$ such that $\omega \models \varphi$ and $\omega' \models \varphi$, there exists a sequence of $\omega_i$, $i = 0...m$, such that $\forall i, \omega_i \models \varphi$, $\omega_0 = \omega$, $\omega_m = \omega'$, and $\forall i < m, d_H(\omega_i, \omega_{i+1}) = 1$. This means that there is a path from any model of $\varphi$ to any other model that is obtained by changing one propositional symbol at each step, while remaining in $Mod(\varphi)$.

Using the graph representation, a connected formula is a connected component of the graph, in the usual graph theoretical sense. A ball of radius 1 centered at $\omega$ is constituted by $\omega$ and the extremities of the arcs originating in $\omega$. This allows for an easy visualization of the effects of transformations, but is not tractable from a computational point of view if $|\Omega_{PS}|$ is high. Typically, if $|PS| = n$, then the graph has $|\Omega_{PS}| = 2^n$ nodes, and each node has $n$ arcs (if all worlds are considered to be admissible or physically possible).

Let us consider an example with three propositional symbols $a, b, c$. The possible worlds are represented in Figure 1.

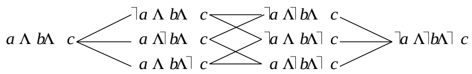

**Fig. 1.** Graph representation of possible worlds with 3 symbols. An arc between two nodes $\omega$ and $\omega'$ means that $d_H(\omega,\omega') = 1$.

Let us consider $\varphi = \neg a \wedge b \wedge c$. Then we have:

$$D^1(\varphi) = (\neg a \wedge b) \vee (\neg a \wedge c) \vee (b \wedge c),$$

$$D^2(\varphi) = \neg a \vee b \vee c = \neg(a \wedge \neg b \wedge \neg c).$$

These results are illustrated in Figure 2. They show the extensivity of dilation, and the iteration property ($D^1(D^1(\varphi)) = D^2(\varphi)$). On this example, the external boundary of $\varphi$ is not connected:

$$D^1(\varphi) \wedge \neg\varphi = (a \wedge b \wedge c) \vee (\neg a \wedge \neg b \wedge c) \vee (\neg a \wedge b \wedge \neg c).$$

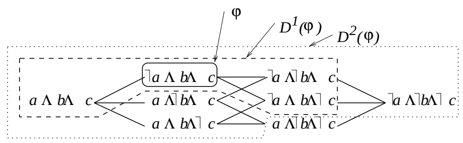


**Fig. 2.** An example of $\varphi$ and two successive dilations.

One of the interesting effects of dilation is that it connects formulas if their components are separated by a size less than the size of the structuring element. Let us take for example the following non connected formula:

$$\varphi = (a \wedge b \wedge c) \vee (\neg a \wedge \neg b \wedge c) \vee (\neg a \wedge b \wedge \neg c).$$

Then we have:

$$D^1(\varphi) = \neg a \vee b \vee c = \neg(a \wedge \neg b \wedge \neg c),$$

which is connected. For instance, $\neg a \wedge b \wedge c$ is at distance 1 of both $a \wedge b \wedge c$ and $\neg a \wedge \neg b \wedge c$, and the addition of this world in the dilation connects these two parts of $\varphi$. In a similar way, it connects $\neg a \wedge \neg b \wedge c$ and $\neg a \wedge b \wedge \neg c$ (see Figures 1 and 2).

In a similar way, dilation fills holes of $\varphi$ (connected components of $\neg\varphi$) which are smaller than the structuring element.

By duality, we have the opposite effects for erosion: it reduces the number of worlds that are models of $\varphi$, it may disconnect a formula, or even suppress small components (smaller than the structuring element).

Let us now consider opening. Opening is a morphological filter (defined as an increasing and idempotent operator). To show the filtering effect, we consider the example in Figure 3, where:

$$\varphi = (\neg a \wedge b) \vee (\neg a \wedge c) \vee (b \wedge c) \vee (\neg a \wedge \neg b \wedge \neg c),$$

$$E^1(\varphi) = \neg a \wedge b \wedge c,$$

$$O^1(\varphi) = D^1[E^2(\varphi)] = (\neg a \wedge b) \vee (\neg a \wedge c) \vee (b \wedge c).$$

This illustrates the anti-extensivity of opening. The part $\neg a \wedge \neg b \wedge \neg c$ has been filtered out by the opening, and the result is more regular than $\varphi$ in the sense that it can be covered by distance balls of radius 1 without intersection with its negation (in this example, there is only one such ball, centered at $\neg a \wedge b \wedge c$). Similarly, it may disconnect formulas by suppressing "narrow" parts of it.

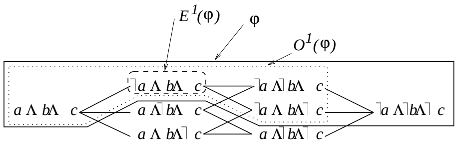


**Fig. 3.** An example of opening.

If we apply again the opening operation, it does not change anything (idempotence property). This can be interpreted as follows: once parts of $\varphi$ smaller than the structuring element have been removed, the other remaining unchanged, there are no more such parts, and a second application of the same transformation (with the same structuring element) has no effect.

By duality, closing is extensive, and connects parts of $\varphi$ if the separation between them is smaller than the structuring element. This is illustrated in Figure 4, where:

$$\varphi = (a \wedge b \wedge c) \vee (\neg a \wedge \neg b \wedge c) \vee (\neg a \wedge b \wedge \neg c),$$

$$D^1(\varphi) = \neg a \vee b \vee c = \neg(a \wedge \neg b \wedge \neg c),$$

$$C^1(\varphi) = E^1[D^1(\varphi)] = (\neg a \wedge b) \vee (\neg a \wedge c) \vee (b \wedge c).$$

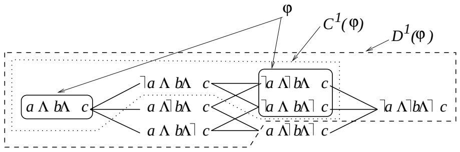


**Fig. 4.** An example of closing.

It is easy to check on this example the idempotence property of closing.

## 6 Some derived operators

In this Section, we give some ideas on other operators. These are just a few examples, that need to be completed.

*Conditioning and reconstruction:* First, we consider problems for which transformations have to be done subject to some constraints (e.g. integrity constraints). Mathematical morphology provides useful tools for this, which are geodesic or conditional operations [20,21,19,15]. Let us take the example of dilation. A geodesic dilation is defined in classical morphology by replacing the distance ball by a geodesic distance ball. In the discrete case, it amounts to taking the intersection of the dilation by the conditioning set. Translating this property to the logical framework, we define a geodesic dilation of $\varphi$ conditionally to $\psi$ by: $D^n_\psi(\varphi) = [D^1(\varphi) \wedge \psi]^n$, where the supscript $n$ means that the operation of dilation/conjunction is iterated $n$ times. This constraints the result to satisfy $\psi$. For instance $\psi$ can represent integrity constraints (as in [13]), or a restriction of allowed worlds, etc.

If this conditional operation is iterated until convergence (achieved in a finite number of steps if $\Omega_{PS}$ is finite), this defines a morphological reconstruction $D^\infty_\psi(\varphi)$. The result is exactly equal to the connected components of $\varphi$ that are consistent with $\psi$. This allows to find all models of $\varphi$ which are connected to models of $\psi$. For instance, it allows to restrict preferences expressed by $\varphi$ in order to make them consistent with $\psi$, the result being maximum for this property. $\psi$ can represent other preferences, or constraints (but less strict than integrity constraints). While conditional operations lead to complete satisfaction of constraints, reconstruction leads only to consistency with constraints.

*Skeleton:* Another morphological approach for simplifying a formula and retaining only its most significant parts is the skeleton. In a continuous space, the skeleton of a set is defined as the centers of maximal balls included in the set [20,21]. The centers of maximal balls of radius $n$ can be computed by taking the difference between the erosion of size $n$ and its opening by a ball of radius going towards 0. In the discrete case, this opening is performed with a ball of radius 1, at the cost of lost of homotopy. We define a maximal ball included in $\varphi$ as follows: a ball $B^n(\omega)$ of radius $n$ centered at $\omega$ is maximal iff $B^n(\omega) \models \varphi$, and there do not exist $\omega', n'$ with $(\omega, n) \neq (\omega', n')$ such that $B^{n'}(\omega') \models \varphi$ and $B^n(\omega) \models B^{n'}(\omega')$. The skeleton of $\varphi$ is then defined as:

$$Mod(Skel(\varphi)) = \{\omega \in \Omega_{PS} \mid \exists n \in \mathbb{N},\ B^n(\omega)\ maximal\ in\ \varphi\}. \tag{9}$$

One interesting property of this definition is that it is invertible, and $\varphi$ can be reconstructed from its skeleton: $\varphi = \vee_n D^n(\psi_n)$ where $\psi_n$ is the set of centers of maximal balls of radius $n$ ($Skel(\varphi) = \vee_n \psi_n$). $\psi_n$ is computed by: $\psi_n = E^n(\varphi) \wedge \neg O^1[E^n(\varphi)]$.

Let us consider again the example of Figure 1, and take:

$$\varphi = (\neg a \wedge b) \vee (\neg a \wedge c) \vee (b \wedge c) \vee (\neg a \wedge \neg b \wedge \neg c).$$

We have:

$$\begin{aligned} E^0(\varphi) &= \varphi, \\ O^1[E^0(\varphi)] &= (\neg a \wedge b) \vee (\neg a \wedge c) \vee (b \wedge c), \\ E^1(\varphi) &= \neg a \wedge b \wedge c, \\ Mod(O^1[E^1(\varphi)]) &= \emptyset. \end{aligned}$$

Therefore, $\psi_0 = \neg a \wedge \neg b \wedge \neg c$, $\psi_1 = \neg a \wedge b \wedge c$, and there is no maximal ball of radius greater than 1 in $\varphi$. Finally, we obtain:

$$Skel(\varphi) = (\neg a \wedge \neg b \wedge \neg c) \vee (\neg a \wedge b \wedge c).$$

The first term corresponds to the "irregularity" of $\varphi$ (which had been filtered by opening in Figure 3), and the second one corresponds to the central part of $(\neg a \wedge b) \vee (\neg a \wedge c) \vee (b \wedge c)$ (see Figure 5). This is an example where the resulting skeleton is not connected, although $\varphi$ is.

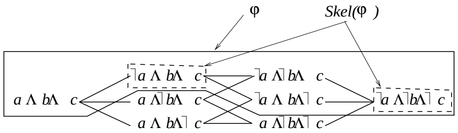


**Fig. 5.** An example of skeleton.

The skeleton obtained by this approach represents all parts of the formula, and may have branches representing irregularities at the boundary of $\varphi$. It can be filtered e.g. by suppressing centers of maximal balls with a small radius (i.e. applying the skeleton on an erosion of $\varphi$), or by first opening the formula to filter out non significant parts before computing the skeleton.

Homotopic skeletons can be obtained by other morphological transformations, called thinning. This is left for future work.

Other operators, acting on set of formulas, can be derived from morphological operators. They can be used in various problems in knowledge representation such as fusion, revision, update, etc. They are not detailed here.

## 7 Conclusion

We have defined in this paper the bases for morpho-logics, at the cross-road between mathematical morphology and formal logic, two domains that to our knowledge were so far completed disconnected. We have shown how to define basic morphological operators on logical formulas, and how these operators can be used in knowledge representation, reasoning and decision making.

Future work is threefold. First, further morphological operators can be investigated in a logical setting. Such operators are connected operators that consider connected components as entities and process them as such, hit-or-miss transformations and derived thinning and thickening (which allow to test a neighborhood of a world not only against $\varphi$, but partly against $\varphi$ and partly against $\neg\varphi$, leading to other types of transformation like homotopic skeleton, convex hull etc.), watershed (for instance to separate the main components of a connected formulas, or finding zones of influence of several connected components), etc. Also the more general lattice framework for mathematical morphology can be exploited on the lattice structure of formulas and interpretations, based on the fundamental adjunction relation.

Second, other ways of logical processing of knowledge and information can be investigated. New operators for merging, fusion, finding compromises, negotiations, etc. can be constructed from morphological operators. Other types of distance can also be used, like geodesic distances or distances based on actions and plans. Another promising extension of our work concerns similarity-based reasoning [8,7]: some approximate entailments have links with morphological operations, either using binary morphology as in this paper, or its extensions to functions [20] or to fuzzy sets [5].

Last, computational aspects of propositional morphological operators deserve some attention. Since the cardinal of $\Omega_{PS}$ is exponentially large in the size of the input, it is generally not reasonable to work explicitely on the interpretations but on the formulas, possibly put in a specific normal form (as for DNF for dilation). Bailleux and Marquis [2] have studied the problem DISTANCE-SAT consisting in finding a model of a formula $\varphi$ close enough to a given partial interpretation. This work is probably a first step towards the practical computation of dilations and erosions induced by distances.

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